DYNAMICS OF MACHINERY

(AME011)

B.Tech V Semester

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INTRODUCTION

Gyre' is a Greek word, meaning 'circular motion' and Gyration means the whirling motion. A gyroscope is a spatial mechanism which is generally employed for the study of precessional motion of a rotary body. Gyroscope finds applications in gyrocompass, used in aircraft, naval ship, control system of missiles and space shuttle. The gyroscopic effect is also felt on the automotive vehicles while negotiating a turn.

A gyroscope consists of a rotor mounted in the inner gimbal. The inner gimbal is mounted in the outer gimbal which itself is mounted on a fixed frame as shown in Fig.1. When the rotor spins about X-axis with angular velocity ω rad/s and the inner gimbal precesses (rotates) about Y-axis, the spatial mechanism is forced to turn about Z-axis other than its own axis of rotation, and the gyroscopic effect is thus setup. The resistance to this motion is called gyroscopic effect.

Fig. 1 Gyroscope mechanism

ANGULAR MOTION

A rigid body, (Fig.2) spinning at a constant angular velocity ω rad/s about a spin axis through the mass centre. The angular momentum 'H' of the spinning body is represented by a **vector** whose magnitude is 'Iω'. I represents the mass amount of inertia of the rotor about the axis of spin.

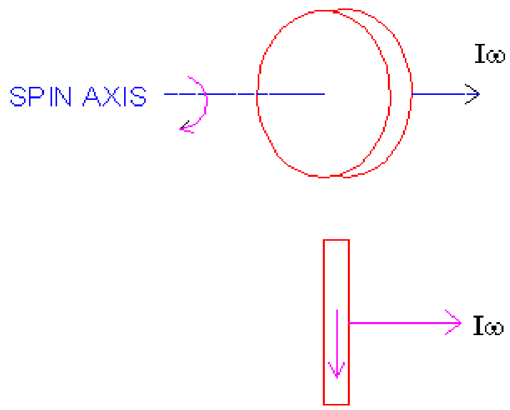

Fig.2 Spinning body

$\because H = I\omega$

The direction of the angular momentum can be found from the right hand screw rule or the right hand thumb rule. Accordingly, if the fingers of the right hand are bent in the direction of rotation of rotor, then the thumb indicates the direction of momentum.

GYROSCOPIC COUPLE

Consider a rotary body of mass *m* having radius of gyration *k* mounted on the shaft supported at two bearings. Let the rotor spins (rotates) about X-axis with constant angular velocity ω rad/s. The X-axis is, therefore, called spin axis, Y-axis, precession axis and Z-axis, the couple or torque axis (Fig.3).

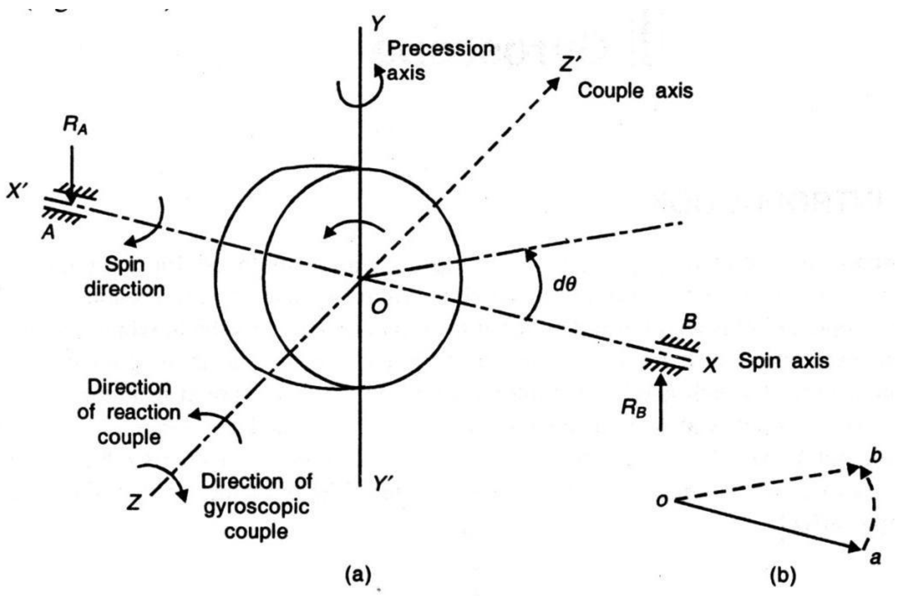

Fig. 3

The angular momentum of the rotating mass is given by,

$$H = mk^2 \omega = I\omega$$

Now, suppose the shaft axis (X-axis) precesses through a small angle δθ about Y-axis in the plane *XOZ,* then the angular momentum varies from *H* to *H* + δ*H,* where δ*H* is the change in the angular momentum, represented by vector ab [Figure 15.2(b)]. For the small value of angle of rotation 5^0, we can write

$$ab = oa \times \delta\theta$$

$$\delta H = H \times \delta\theta$$

$$= I\omega\delta\theta$$

However, the rate of change of angular momentum is:

$$C = \frac{dH}{dt} = \lim_{\delta t \to 0}\left(\frac{I\omega\,\delta\theta}{\delta t}\right)$$

$$= I\omega\,\frac{d\theta}{dt}$$

or $$\mathbf{C = I\omega\omega_p}$$

where C = gyroscopic couple (N-m)
ω = angular velocity of rotary body (rad/s)
ω_p = angular velocity of precession (rad/s)

Direction of Spin vector, Precession vector and Couple/Torque vector with forced precession

To determine the direction of spin, precession and torque/couple vector, right hand screw rule or right hand rule is used. The fingers represent the rotation of the disc and the thumb shows the direction of the spin, precession and torque vector (Fig.4).

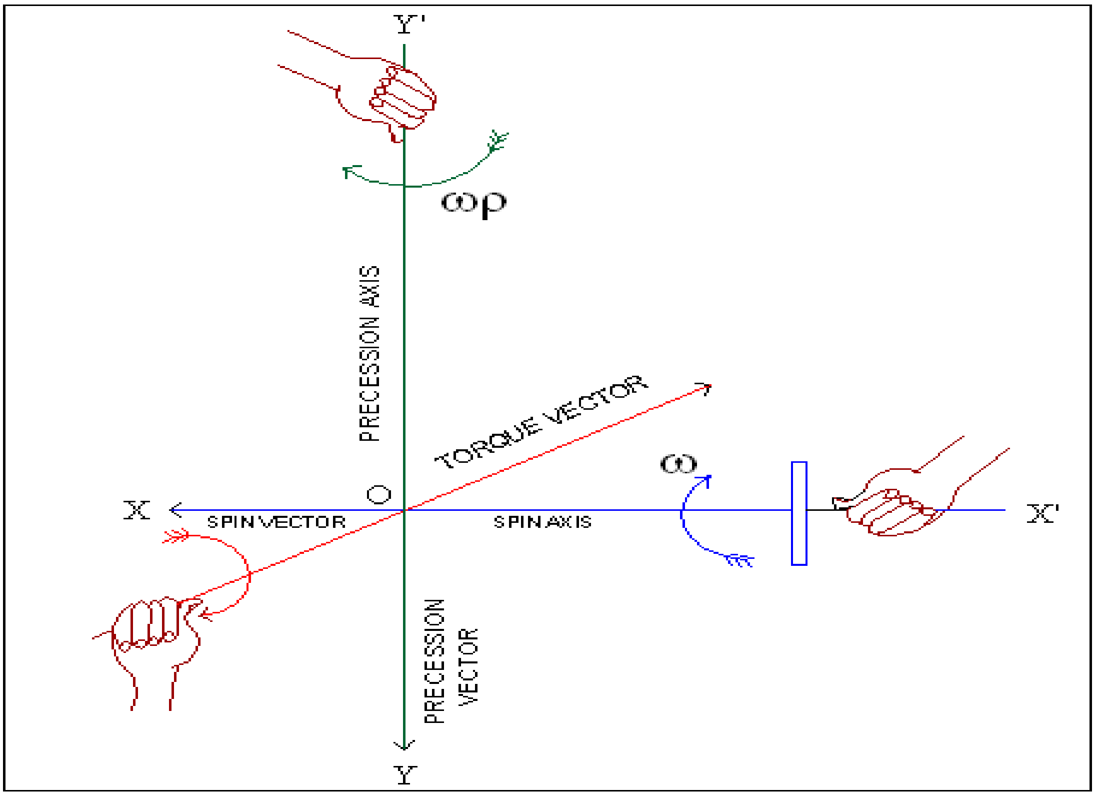

Fig.4. Direction of Spin vector, Precession vector and Couple/Torque vector

The method of determining the direction of couple/torque vector is as follows.

Case (i):

Consider a rotor rotating in anticlockwise direction when seen from the right (Fig.5 and Fig. 6), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector, the following procedure is used.

- Turn the spin vector through 90^0 in the direction of precession on the XOZ plane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

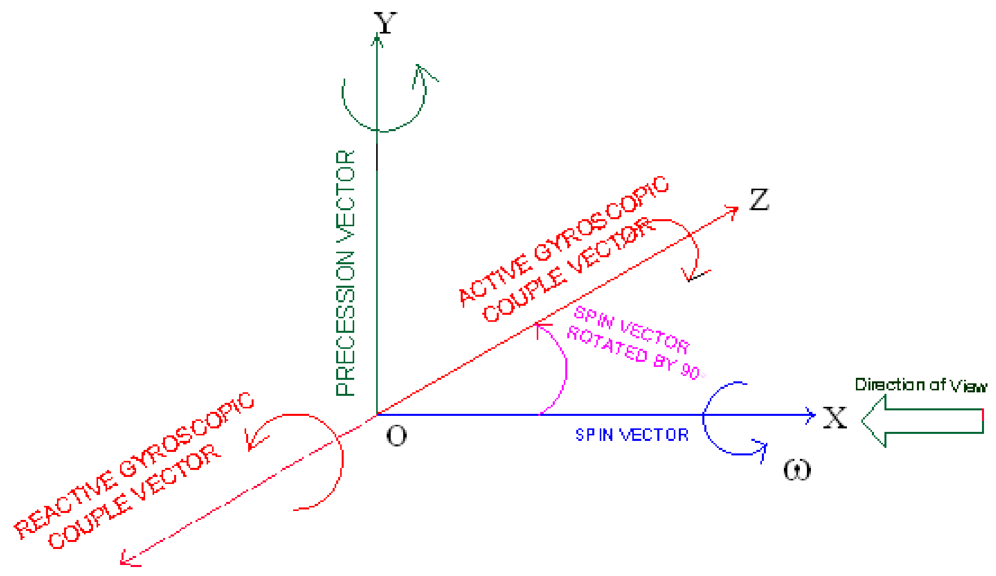

Fig. 5 Direction of active and reactive gyroscopic couple/torque vector

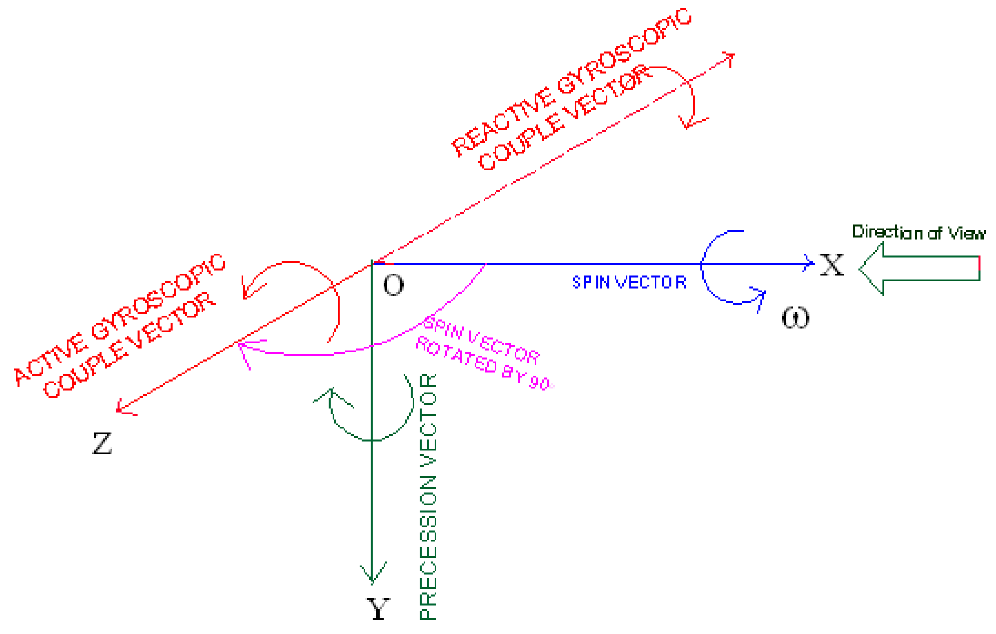

Fig. 6 Direction of active and reactive gyroscopic couple/torque vector

Case (ii):

Consider a rotor rotating in clockwise direction when seen from the right (Fig.7 and Fig. 8), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector,

- Turn the spin vector through 90^0 in the direction of precession on the XOZ plane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

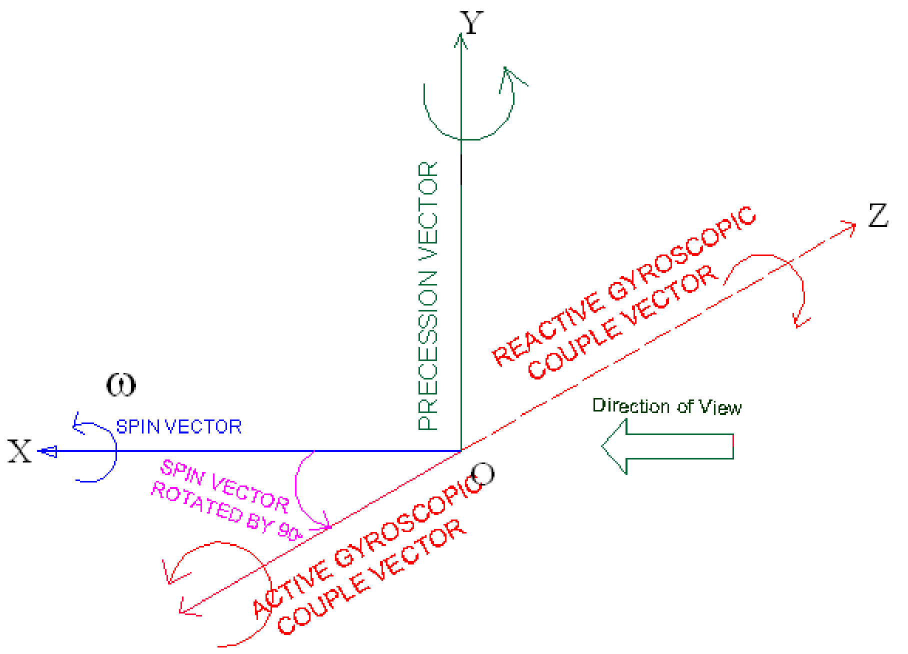

Fig. 7 Direction of active and reactive gyroscopic couple/torque vector

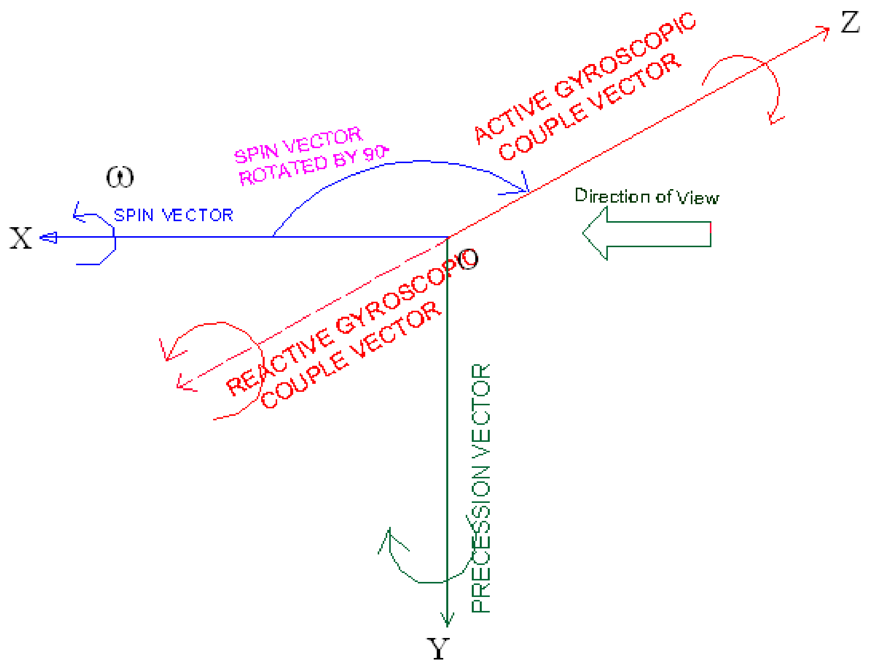

Fig. 8 Direction of active and reactive gyroscopic couple/torque vector

The resisting couple/ reactive couple will act in the direction opposite to that of the gyroscopic couple. This means that, whenever the axis of spin changes its direction, a **gyroscopic couple** is applied to it through the bearing which supports the spinning axis.

Please note that, for analyzing the gyroscopic effect of the body, always reactive gyroscopic couple is considered.

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Problem 1

A disc of 5 kg mass with radius of gyration 70 mm is mounted at span on a horizontal shaft spins at 720 rpm in clockwise direction when viewed from the right hand bearing. If the shaft precesses about the vertical axis at 30 rpm in clockwise direction when viewed from the top, determine the reactions at each bearing due to mass of the disc and gyroscopic effect.

Solution Angular velocity:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 720}{60} = 75.4 \text{ rad/s}$$

Angular velocity of precession:

$$\omega_p = \frac{2\pi N_p}{60} = \frac{2\pi \times 30}{60} = 3.14 \text{ rad/s}$$

Moment of inertia:

$$I = mk^2 = 5 \times 0.07^2 = 0.0245 \text{ kg m}^2$$

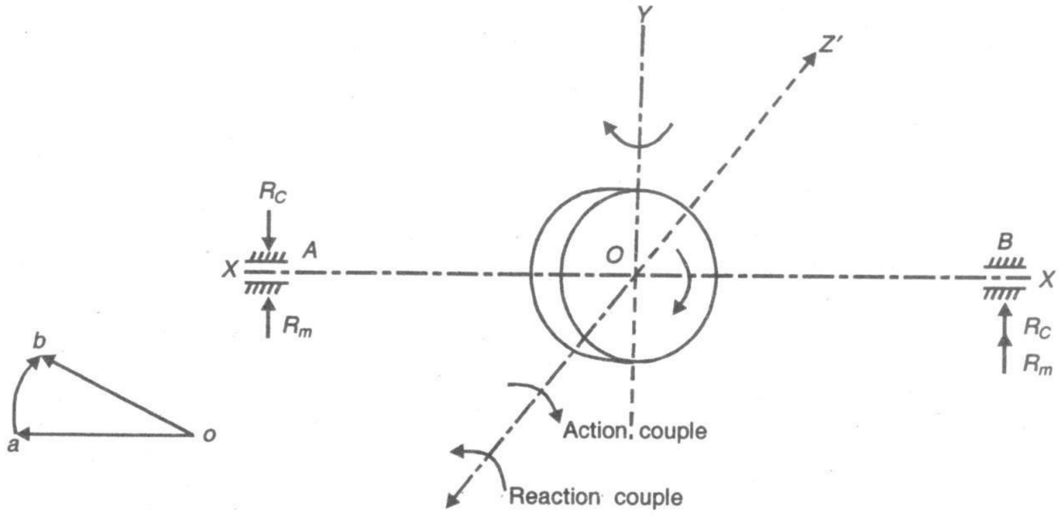


FIG.9a

Gyroscopic couple:

$$C = I\,\omega\omega_p = 0.0245 \times 75.4 \times 3.14 = 5.8 \text{ Nm}$$

This couple induces reaction R_c at the bearing support.

$$R_c \times \frac{120}{1000} = 5.8$$

or $R_c = 48.3$ N

Reaction on the bearings due to weight of the disc, $R_m = mg/2 = 5 \text{x} 9.81/2 = 24.53$ N

The angular momentum vector and induced reactive gyroscopic couple acting in anticlockwise direction is shown in Fig.9b.

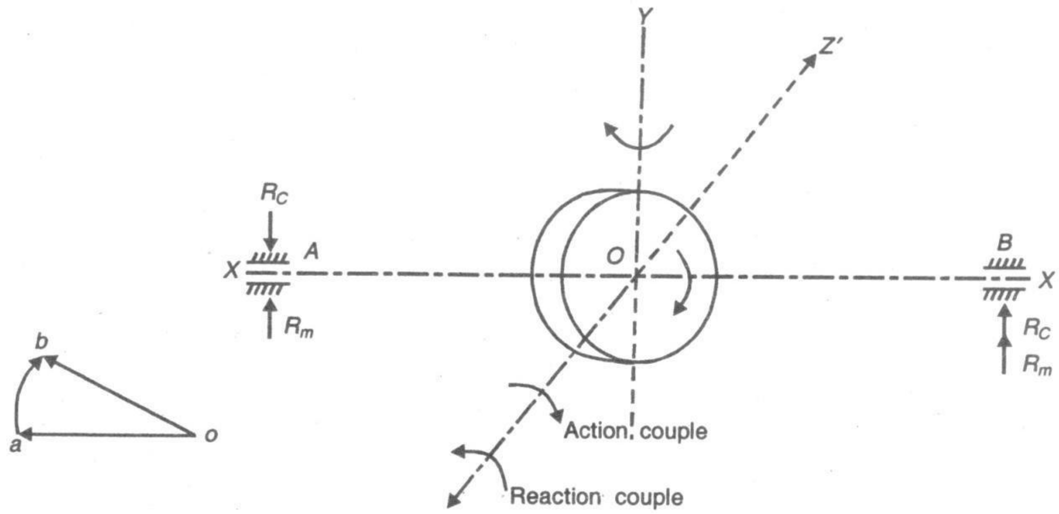


FIG.9b

Gyroscopic couple:

$$C = I\,\omega\omega_p$$
$$= 0.0245 \times 75.4 \times 3.14$$
$$= 5.8 \text{ Nm}$$

This couple induces reaction R_c at the bearing support.

$$R_c \times \frac{120}{1000} = 5.8$$

or

$$R_c = 48.3 \text{ N}$$

The reaction R_c acts in upward direction at right hand bearing and in downward c at left hand bearing.

The reaction due to weight of the disc acts in upward direction. Therefore,

Reaction at bearing A:

$$R_A = R_c - R_m$$
$$= 48.43 - 24.53$$
$$= \mathbf{23.9\ N(\downarrow)}$$

Reaction at bearing B:

$$R_B = R_c + R_m$$
$$= 48.43 + 24.53$$
$$= \mathbf{72.96\ N(\uparrow)}$$

GYROSCOPIC EFFECT ON SHIP

Gyroscope is used for stabilization and directional control of a ship sailing in the rough sea. A ship, while navigating in the rough sea, may experience the following three different types of motion:

(i) Steering—The turning of ship in a curve while moving forward
(ii) Pitching—The movement of the ship up and down from horizontal position in a vertical plane about transverse axis
(iii) Rolling—Sideway motion of the ship about longitudinal axis.

For stabilization of a ship against any of the above motion, the major requirement is that the gyroscope shall be made to precess in such a way that reaction couple exerted by the rotor opposes the disturbing couple which may act on the frame.

Ship Terminology

(i) Bow – It is the fore end of ship
(ii) Stern – It is the rear end of ship
(iii) Starboard – It is the right hand side of the ship looking in the direction of motion
(iv) Port – It is the left hand side of the ship looking in the direction of motion

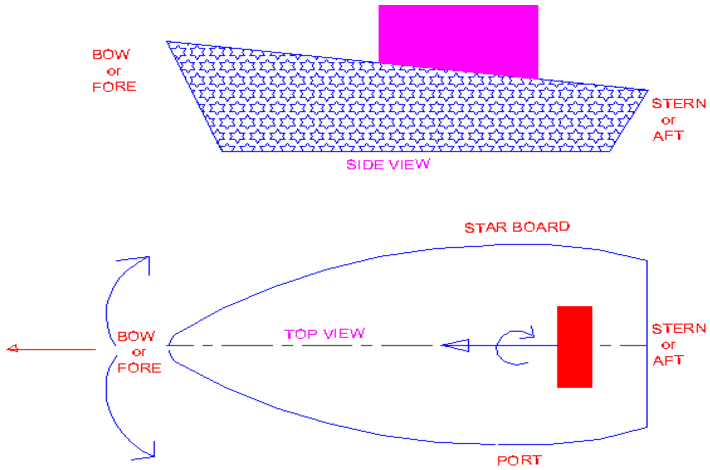


Fig. 10

Consider a gyro-rotor mounted on the ship along longitudinal axis (X-axis) as shown in Fig.10 and rotate in clockwise direction when viewed from rear end of the ship. The angular speed of the rotor is ω rad/s. The direction of angular momentum vector *oa*, based on direction of rotation of rotor, is decided using right hand thumb rule as discussed earlier.The gyroscopic effect during the three types of motion of ship is discussed.

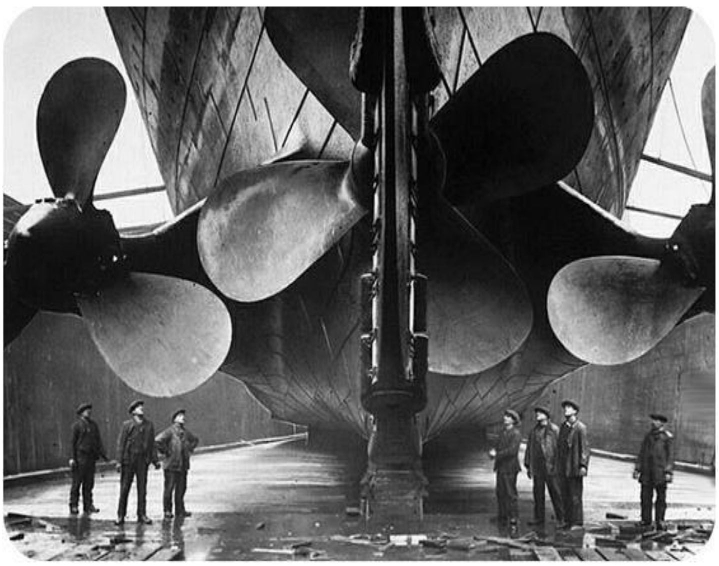
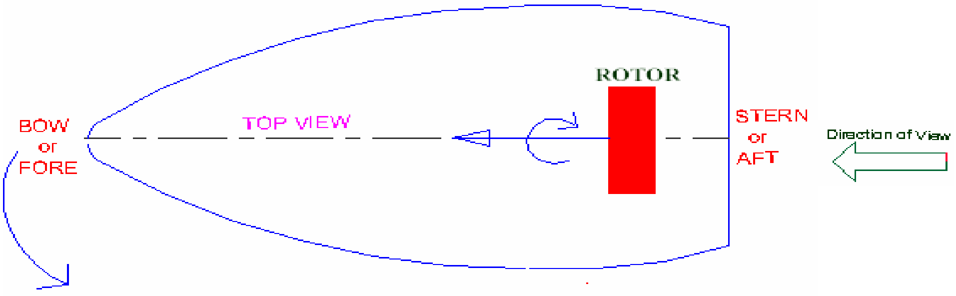


Fig.11

Gyroscopic effect on Steering of ship

(i) Left turn with clockwise rotor

When ship takes a left turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way.



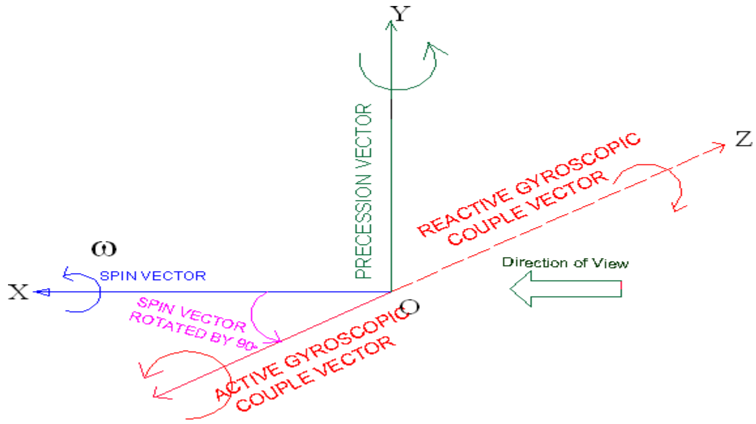


Fig. 12

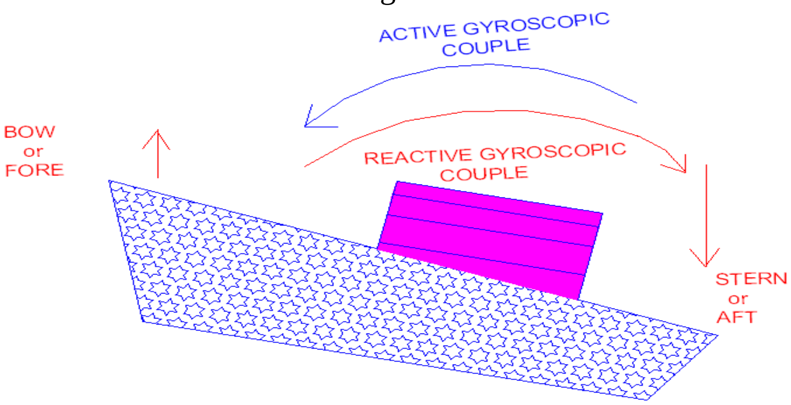


Fig. 13

Note that, always reactive gyroscopic couple is considered for analysis. From the above analysis (Fig.12), the couple acts over the ship between stern and bow. This reaction couple tends to raise the front end (bow) and lower the rear end (stern) of the ship.

(ii) Right turn with clockwise rotor

When ship takes a right turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple acts on the ship is analyzed (Fig 14). Again, the couple acts in vertical plane, means between stern and bow. Now the reaction couple tends to lower the bow of the ship and raise the stern.

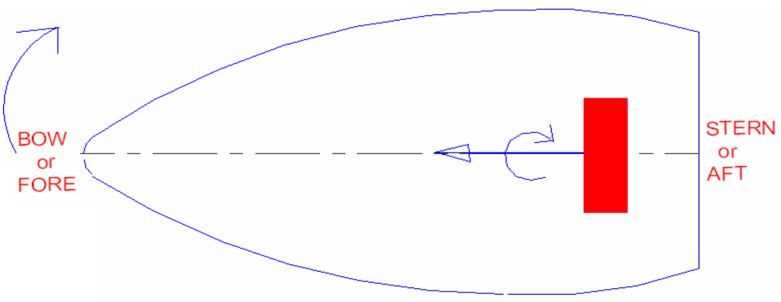


Fig. 14

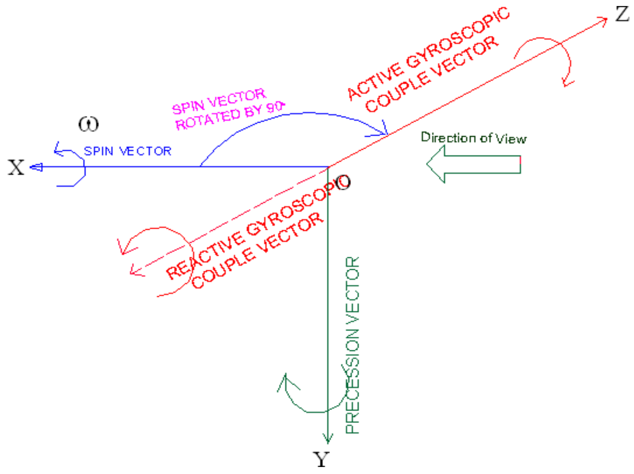


Fig.15

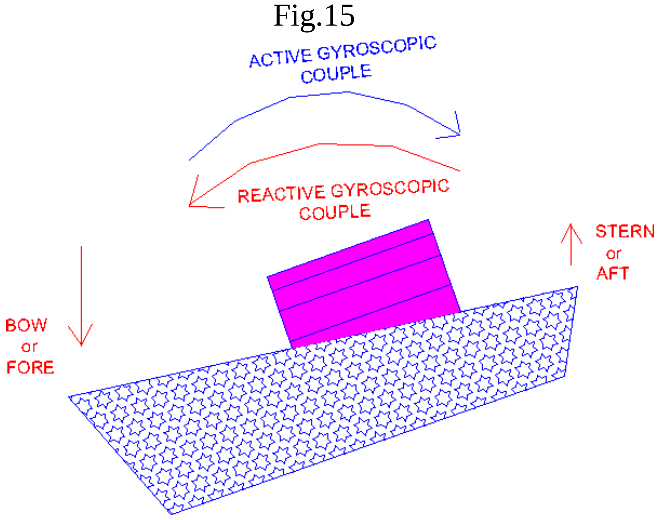


Fig. 16

(iii) Left turn with anticlockwise rotor

When ship takes a left turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way (Fig.18).

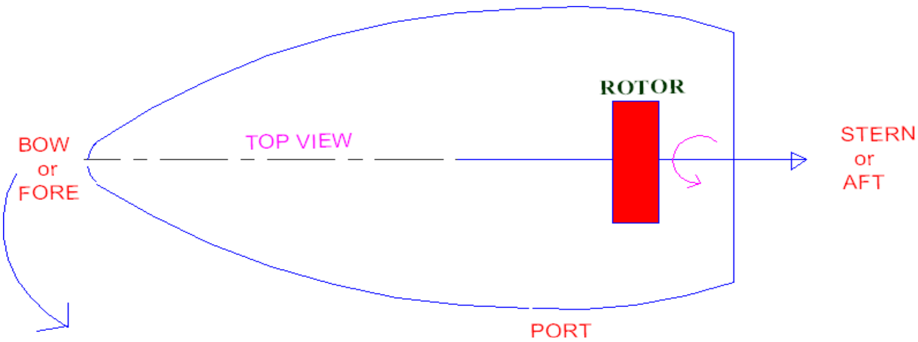


Fig. 17

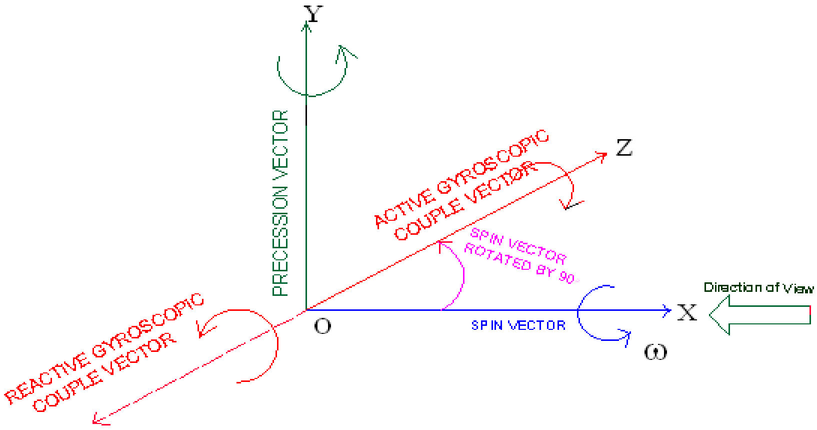


Fig.18

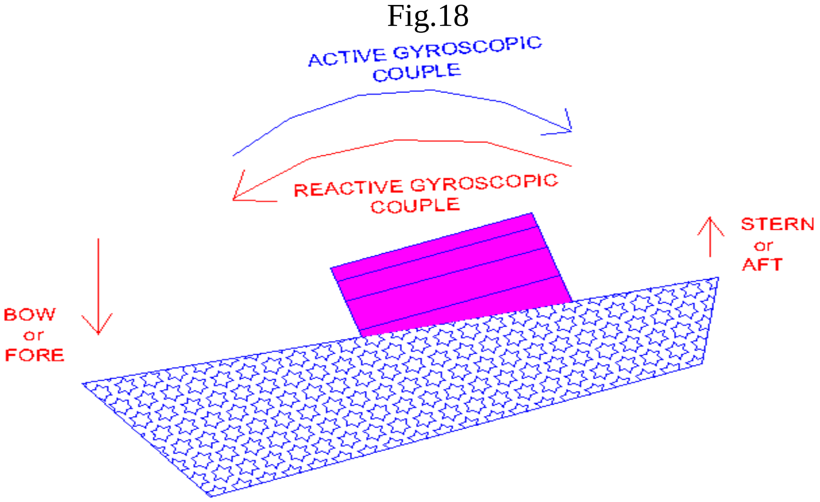


Fig. 19

The couple acts over the ship is between stern and bow. This reaction couple tends to press or dip the front end (bow) and raise the rear end (stern) of the ship.

(iv) Right turn with anticlockwise rotor

When ship takes a right turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is according to Fig 20. Now, the reaction couple tends to raise the bow of the ship and dip the stern.

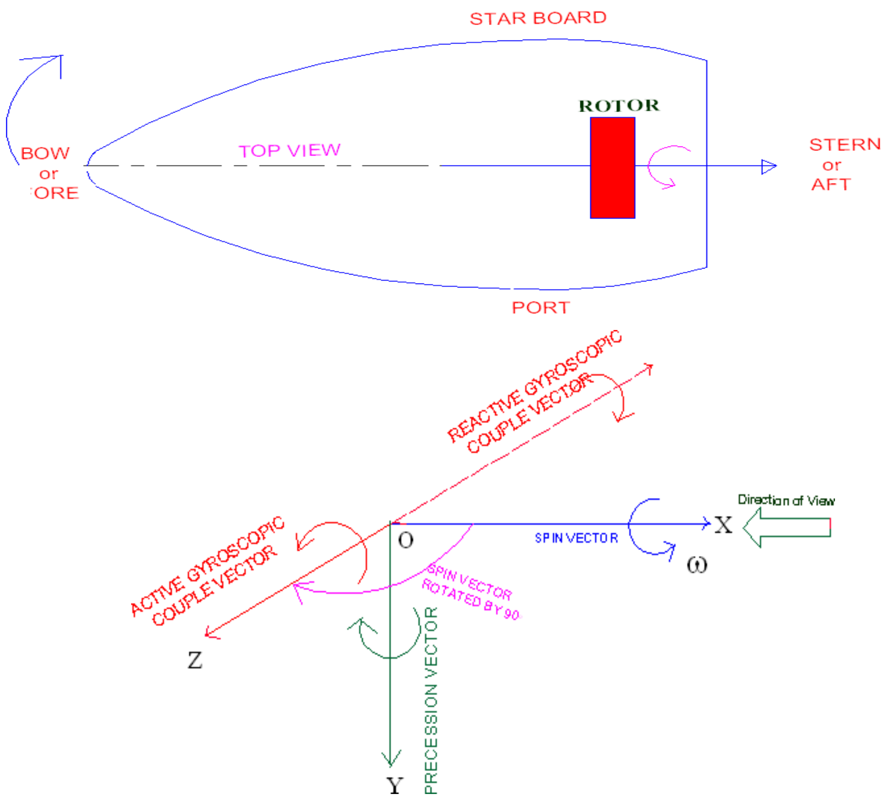


Fig.20

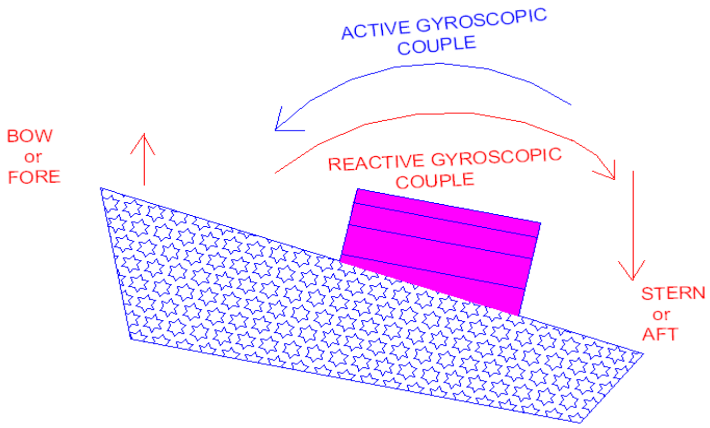


Fig. 21

Gyroscopic effect on Pitching of ship

The pitching motion of a ship generally occurs due to waves which can be approximated as sine wave. During pitching, the ship moves up and down from the horizontal position in vertical plane (Fig.22. & Fig. 23)

Fig.22 Pitching action of ship

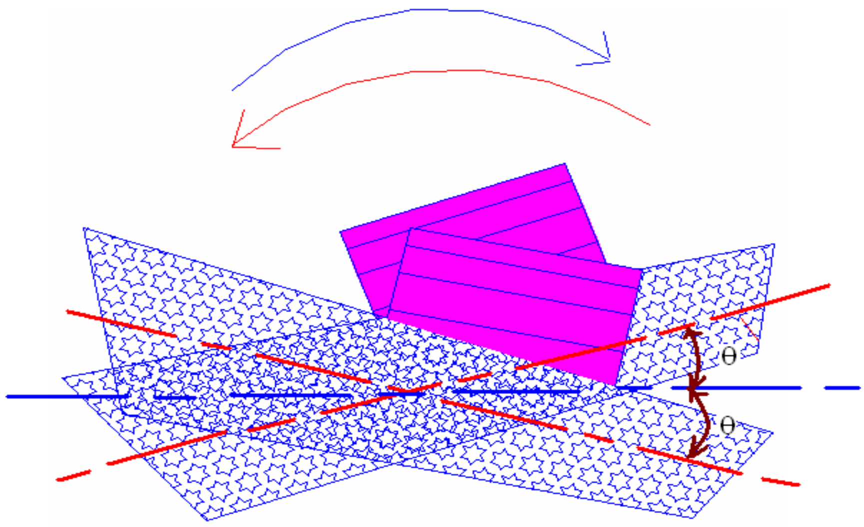


Fig.23 Pitching action of ship

Let θ = angular displacement of spin axis from its mean equilibrium position

A = amplitude of swing

(= angle in degree $\times \frac{2\pi}{360°}$)

and ω_0 = angular velocity of simple hormonic motion $\left(= \frac{2\pi}{\text{time period}}\right)$

The angular motion of the rotor is given as

$$\theta = A \sin \omega_0 t$$

Angular velocity of precess:
$$\omega_p = \frac{d\theta}{dt}$$
$$= \frac{d}{dt}(A \sin \omega_0 t)$$

or
$$\omega_p = A\omega_0 \cos \omega_0 t$$

The angular velocity of precess will be maximum when $\cos \omega_0 t = 1$

or
$$\omega_{p\max} = A\omega_0$$
$$= A \times \frac{2\pi}{t}$$

Thus the gyroscopic couple:
$$C = I\,\omega\,\omega_p$$

Consider a rotor mounted along the longitudinal axis and rotates in clockwise direction when seen from the rear end of the ship. The direction of momentum for this condition is shown by vector *ox* (Fig.24). When the ship moves up the horizontal position in vertical plane by an angle δθ from the axis of spin, the rotor axis (X-axis) processes about Z-axis in XY-plane and for this case Z-axis becomes precession axis. The gyroscopic couple acts in anticlockwise direction about Y-axis and the reaction couple acts in opposite direction, i.e. in clockwise direction, which tends to move towards **right side** (Fig.25). However, when the ship pitches down the axis of spin, the direction of reaction couple is reversed and the ship turns towards **left side** (Fig. 26).

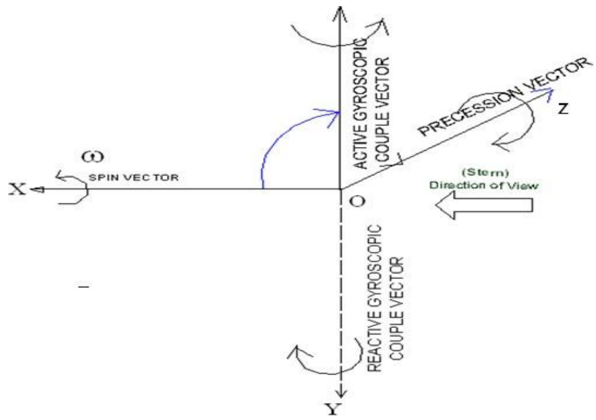


Fig. 24

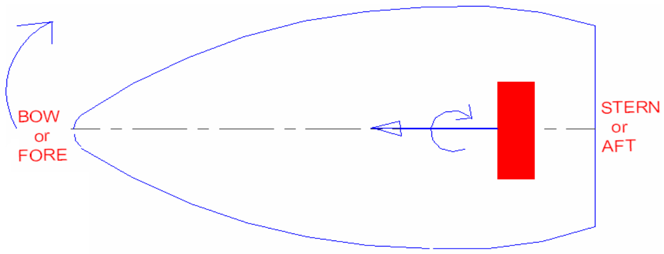


Fig. 25

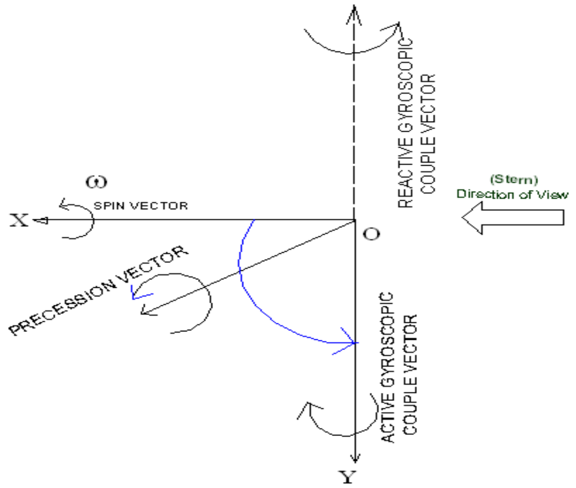


Fig.18

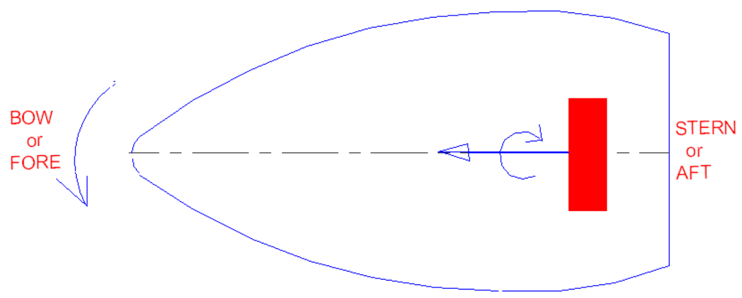


Fig.26

Similarly, for the anticlockwise direction of the rotor viewed from the rear end (Stern) of the ship, the analysis may be done.

Gyroscopic effect on Rolling of ship.

The axis of the rotor of a ship is mounted along the longitudinal axis of ship and therefore, there is **no** precession of this axis. Thus, no effect of gyroscopic couple on the ship frame is formed when the ship rolls.

Fig.27

Problem 2

A turbine rotor of a ship has a mass of 3500 kg and rotates at a speed of 2000 rpm. The rotor has a radius of gyration of 0.5 m and rotates in clockwise direction when viewed from the stern (rear) end. Determine the magnitude of gyroscopic couple and its direction for the following conditions

(i) When the ship runs at a speed of 12 knots and steers to the left in a curve of 70 m radius

(ii) When the ship pitches 6° above and 6° below the horizontal position and the bow (Front) end is lowered. The pitching motion is simple harmonic with periodic time 30 sec.

(iii) When the ship rolls and at a certain instant, it has an angular velocity of 0.05 rad/s clockwise when viewed from the stern

Also find the maximum angular acceleration during pitching.

Solution Given, 1 knot = 1.86 kmph, the linear velocity of the ship:

$$V = 1.86 \times 12 = 22.32 \text{ kmph}$$

$$= \frac{22.32 \times 1000}{3600} = 6.2 \text{ m/s}$$

Angular velocity of the rotor: $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2000}{60}$

$$= 209.44 \text{ rad/s}$$

Precession velocity: $\omega_p = \dfrac{V}{R} = \dfrac{6.2}{70} = 0.08857$ rad/s

Moment of inertia: $I = mk^2 = 3500 \times 0.5^2 = 875$ kg m^2

Gyroscopic couple: $C = I\omega\omega_p$

$$= 875 \times 209.44 \times 0.08857$$
$$= \mathbf{16231.34\ Nm}$$

When ship steers to the left, the reaction gyroscopic couple action is in anticlockwise direction and the bow of the ship is raised and stern is lowered, as shown in Fig.28.

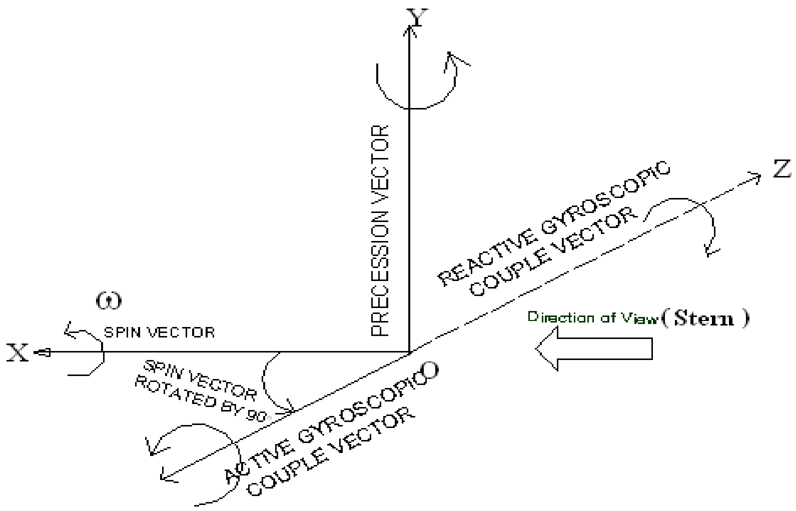


Fig.28

(ii) Amplitude of swing: $A = \dfrac{6° \times 2\pi}{360°} = 0.1047$ rad

Angular displacement: $\theta = A \sin \omega_0 t$

Angular velocity of precession: $\omega_p = \dfrac{d\theta}{dt} = A\omega_0 \cos\omega_0 t$

Maximum angular velocity of precession:

$$\omega_{pmax} = \omega_0 A$$

where $\omega_0 = \dfrac{2\pi}{\text{time period of oscillation}} = \dfrac{2\pi}{30}$

$= 0.2094$ rad/s

$$\omega_{pmax} = 0.2094 \text{ x } 0.1047 = 0.022 \text{ rad/s}$$

Maximum couple for pitching:

$$C_{max} = I\omega\omega_{pmax}$$
$$= 875 \text{ x } 209.44 \text{ x } 0.022$$
$$= 4031.72 \text{ Nm}$$

The effect of gyroscopic couple due to pitching is shown in Fig.29. The reactive gyroscopic couple will act in anticlockwise direction seen from top and it will turn ship towards the left side.

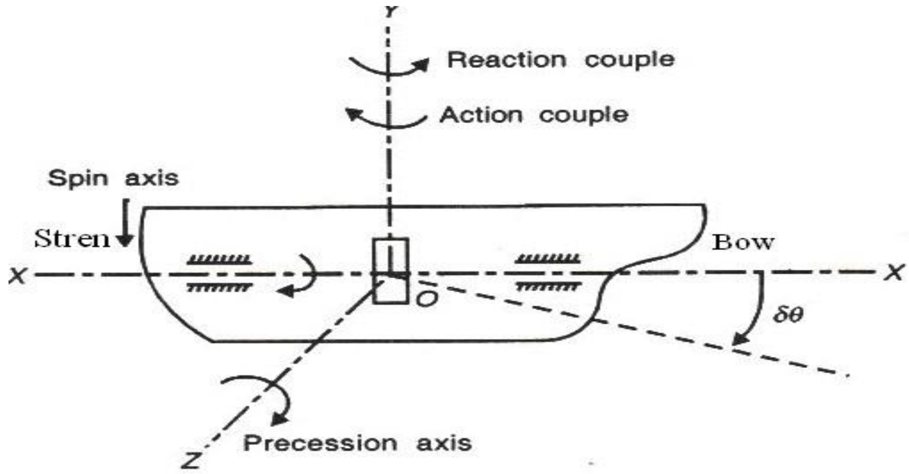


Fig.29

iii) Angular velocity of precession while the ship rolls is:

$$\omega p = 0.05 \text{ rad/s}$$

and gyroscopic couple : C $= 1\omega\omega p$
$= 875 \times 209.44 \times 0.05$
$= 9163$ Nm

Since the ship rolls in the same plane as the plane of spin, there is **no gyroscopic effect**.

Angular velocity of precess during pitching is:

$$\omega_p = \frac{d\theta}{dt} = A\omega_0 \cos \omega_0 t$$

Therefore, angular acceleration:

$$\alpha = \frac{d^2\theta}{dt^2} = -A\omega_0^2 \sin \omega_0 t$$

Maximum angular acceleration:

α_{max} $= -A\omega_0^2$
$= 0.1047 \times 0.2094^2$
$= 0.00459 \text{ rad/s}^2$

Problem 3

A ship is propelled by a rotor of mass of 2000 kg rotates at a speed of 2400 rpm. The radius of gyration of rotor is 0.4 m and spins clockwise direction when viewed from bow (front) end. Find the gyroscopic couple and its effect when;

(i) the ship takes left turn at a radius of 350 m with a speed of 35 kmph
(ii) the ship pitches with the bow rising at an angular velocity of 1 rad/s
(iii)the ship rolls at an angular velocity of 0.15 rad/s

Solution

Angular velocity:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 251.33 \text{ rad/s}$$

Linear velocity: $V = 35 \text{ kmph} = \frac{35 \times 1000}{3600} = 9.72 \text{ m/s}$

Moment of inertia: $I = mk^2 = 2000 \times 0.4^2 = 320 \text{ kg m}^2$

Steering towards left

Angular velocity of precession: $\omega_p = \frac{V}{R} = \frac{9.72}{350} = 0.0278 \text{ rad/s}$

Gyroscopic couple:
$$C = I\omega\omega_p$$
$$= 320 \times 251.33 \times 0.0278$$
$$= 2235.8 \text{ Nm}$$

The reaction gyroscopic couple will act in anticlockwise and will tend to lower the bow as shown in Figure 30.

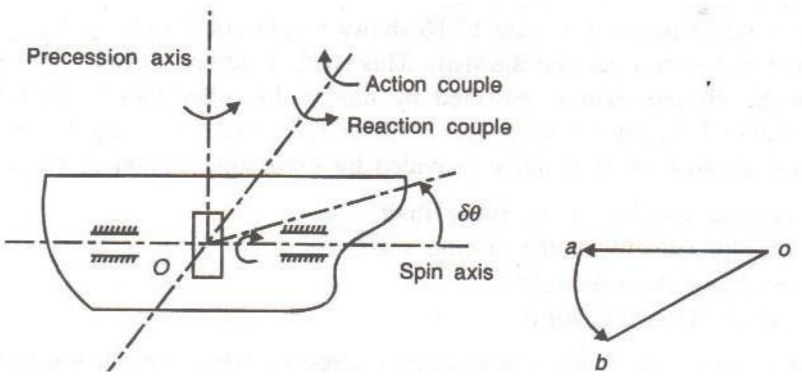


Fig.30

Pitching. Angular velocity of precession during pitching a)$_p$ = 1.0 rad/s
Gyroscopic couple: C = 320 x 251.33 x 1.0
= 80425.6 **Nm Ans.**

The reaction gyroscopic couple acting in anticlockwise direction will tend to turn the bow towards the Right side as shown in Figure 31.

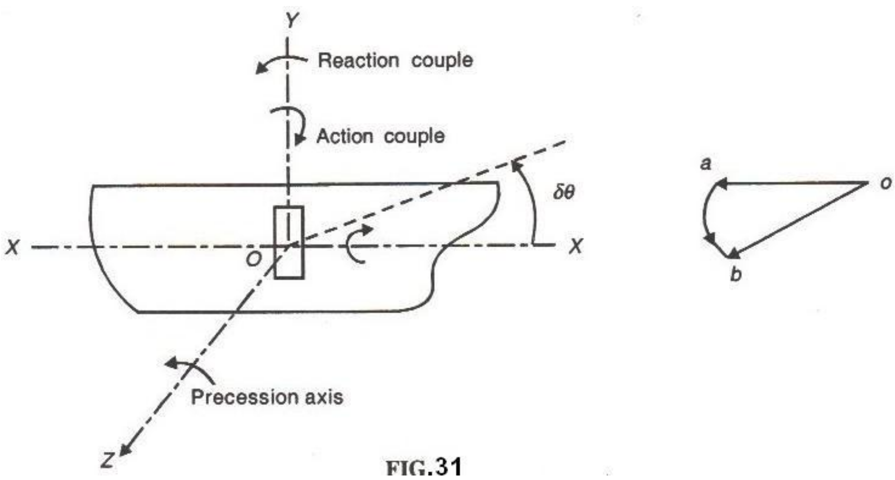


FIG.31

Rolling, Gyroscopic couple: C = l6XQ_p
= 320 x 251.33 x 0.15 = 12063.84 Nm

During rolling, the ship rolls in the same plane as the plane of spin and there will be no gyroscopic effect.

Gyroscopic Effect on Aeroplane

Aeroplanes are subjected to gyroscopic effect when it taking off, landing and negotiating left or right turn in the air.

Let

ω = Angular velocity of the engine rotating parts in rad/s,
m = Mass of the engine and propeller in kg,
r_W = Radius of gyration in m,
I = Mass moment of inertia of engine and propeller in kg m^2,
V = Linear velocity of the aeroplane in m/s,
R = Radius of curvature in m,
ω_p =Angular velocity of precession $= \frac{V}{R}$ rad/s

∴ Gyroscopic couple acting on the aero plane **= C = I ω ω_p**

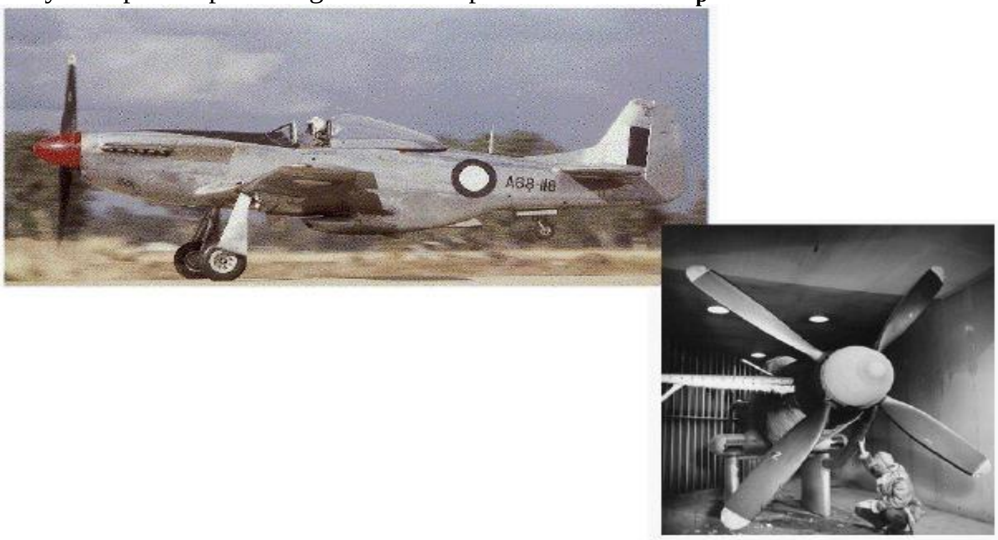


Fig.32

Let us analyze the effect of gyroscopic couple acting on the body of the aero plane for various conditions.

Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

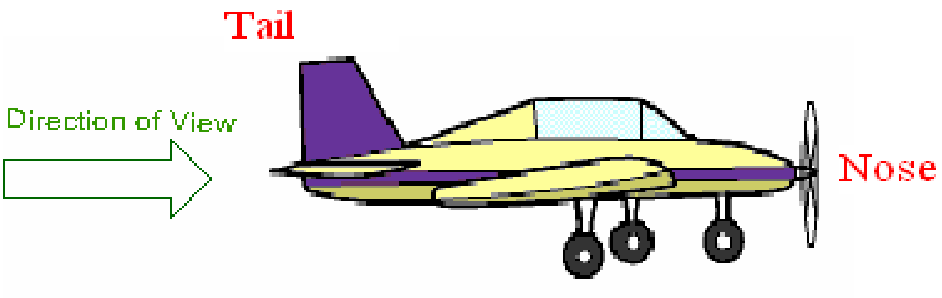


Fig.33

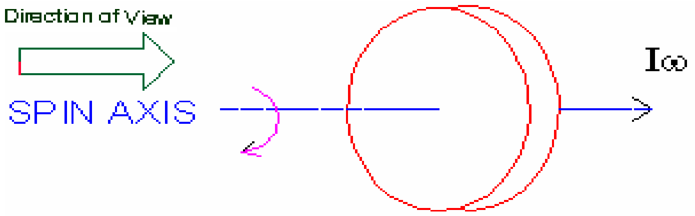


Fig.34

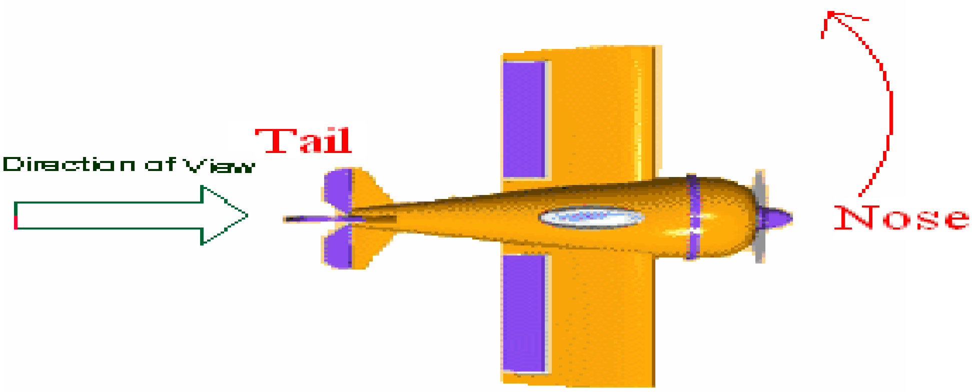


Fig.35

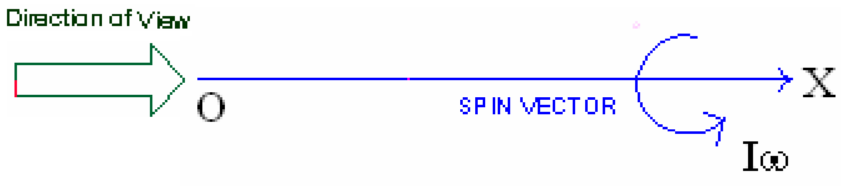


Fig.36

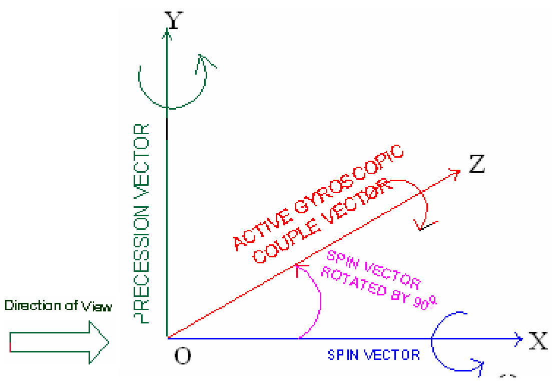


Fig.37

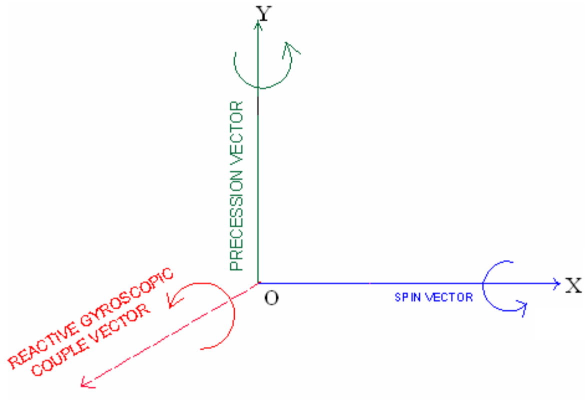


Fig.38

According to the analysis, the reactive gyroscopic couple tends to dip the tail and raise the nose of aeroplane.

Fig.39

Case (ii): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT

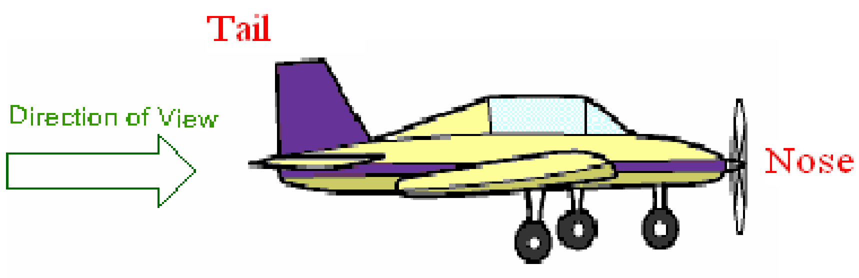


Fig.40

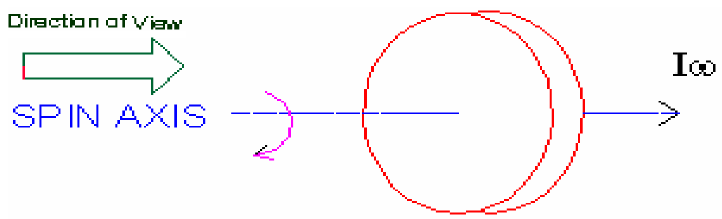


Fig.41

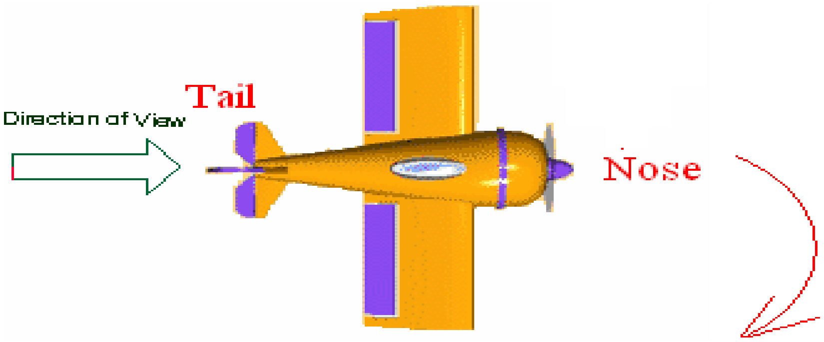


Fig.42

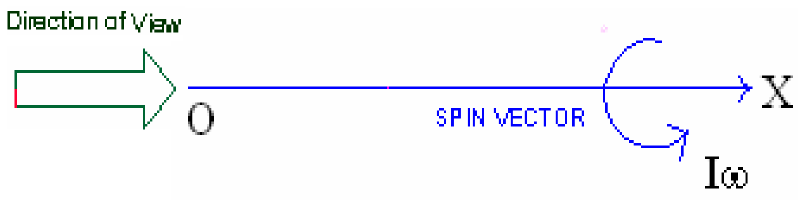


Fig.43

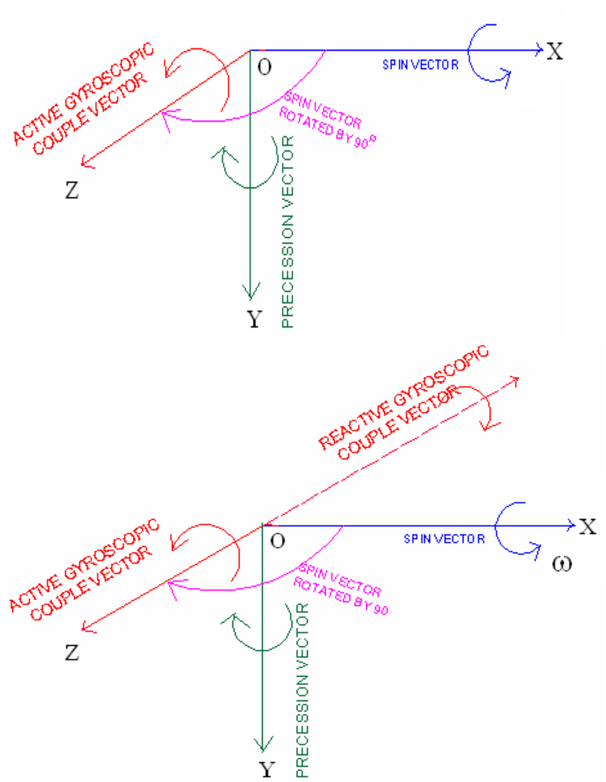


Fig. 44

According to the analysis, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

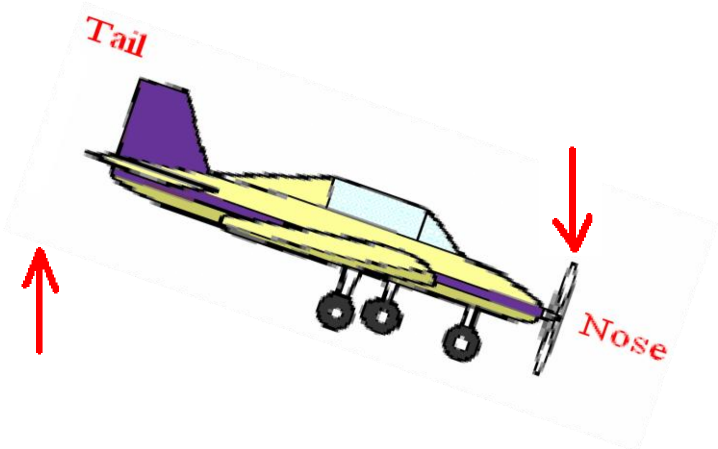


Fig.45

Case (iii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

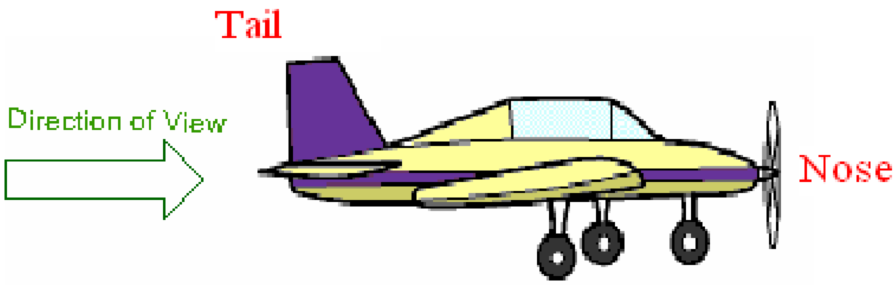


Fig.46

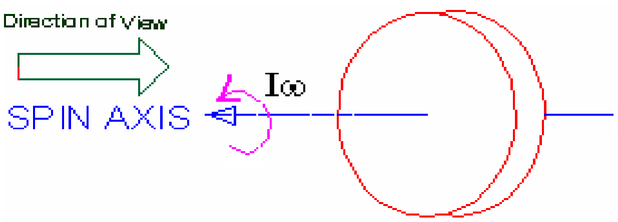


Fig.47

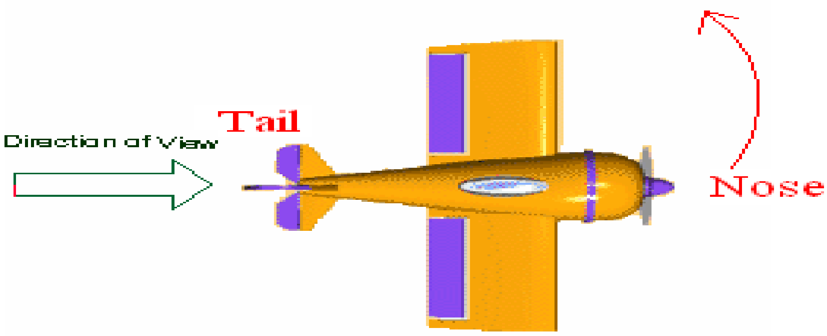


Fig.48

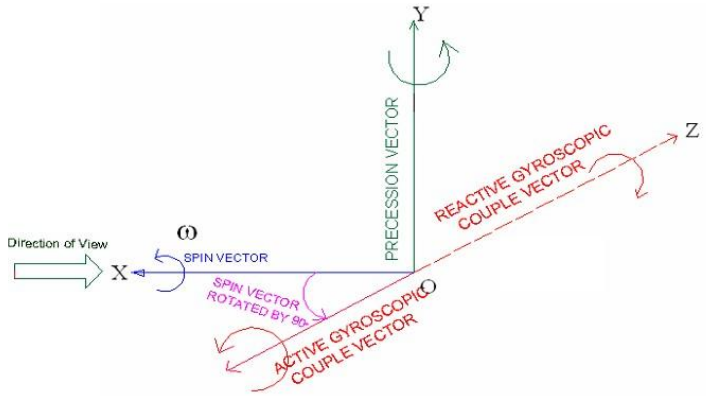


Fig.49

The analysis indicates, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

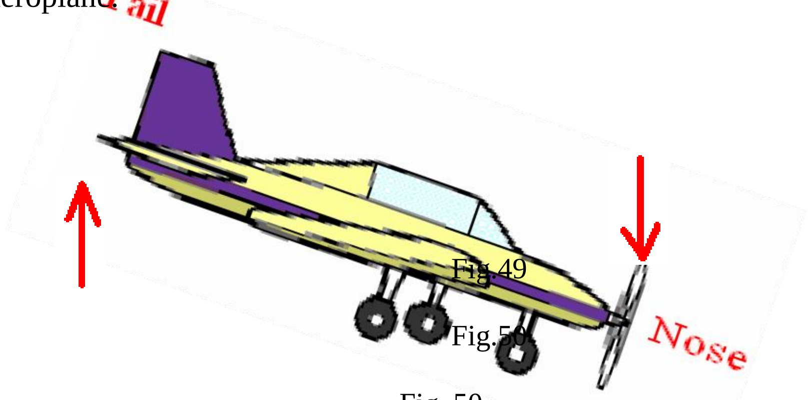


Fig. 50

Case (iv): **PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT**

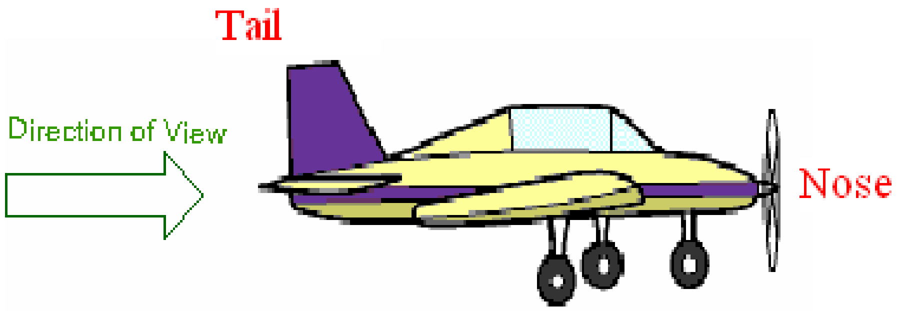


Fig.51

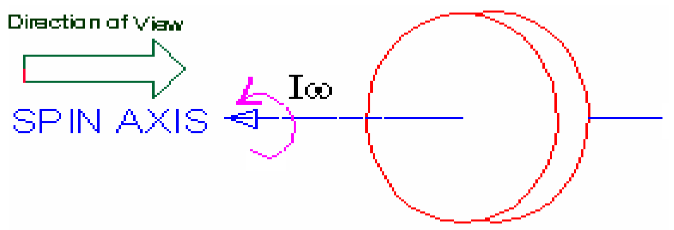


Fig.52

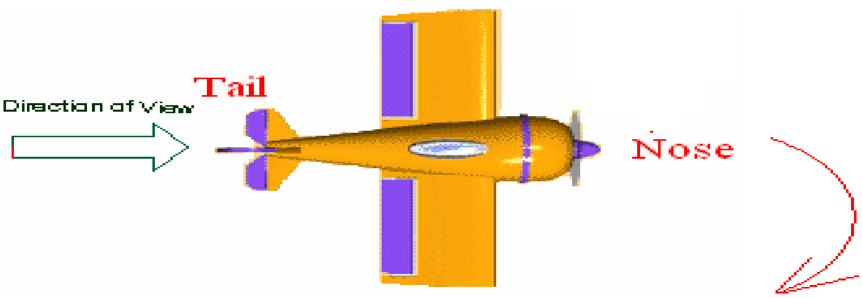


Fig.53

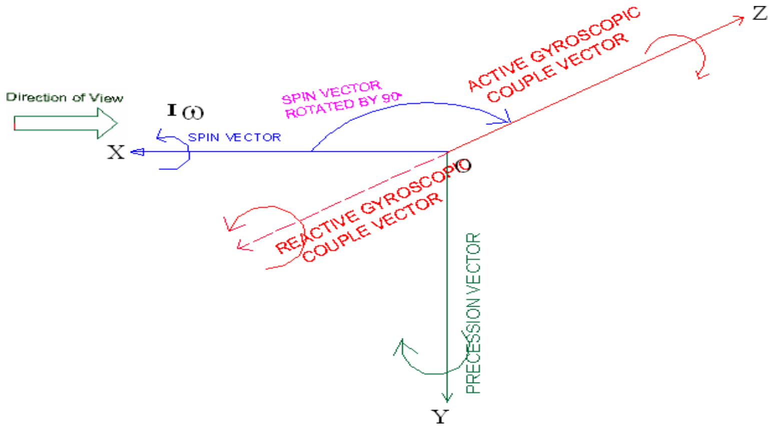


Fig.54

The analysis shows, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

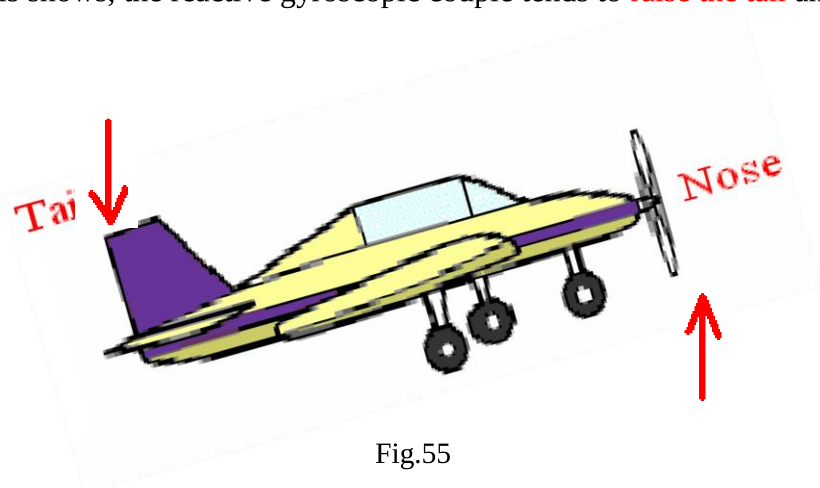


Fig.55

Case (v): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards

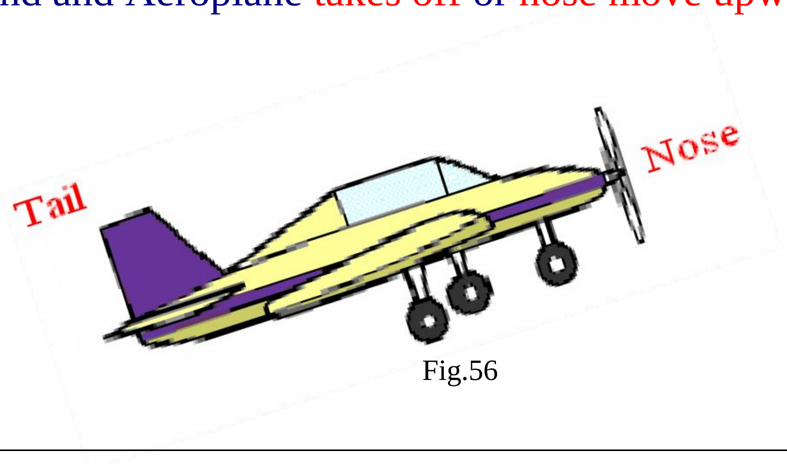


Fig.56

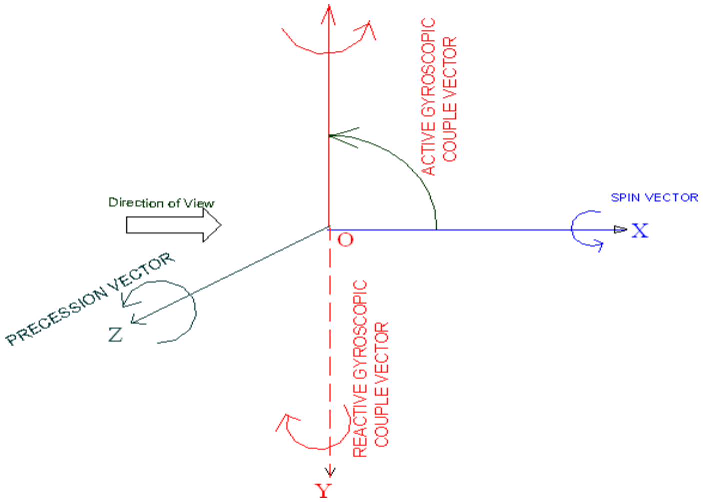


Fig.57

The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right

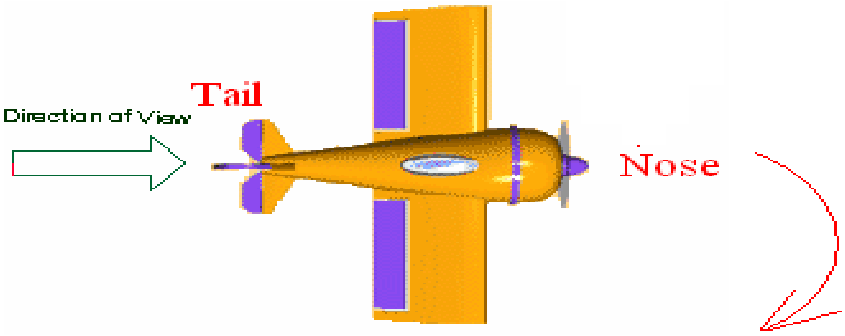


Fig.58

Case (vi): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards

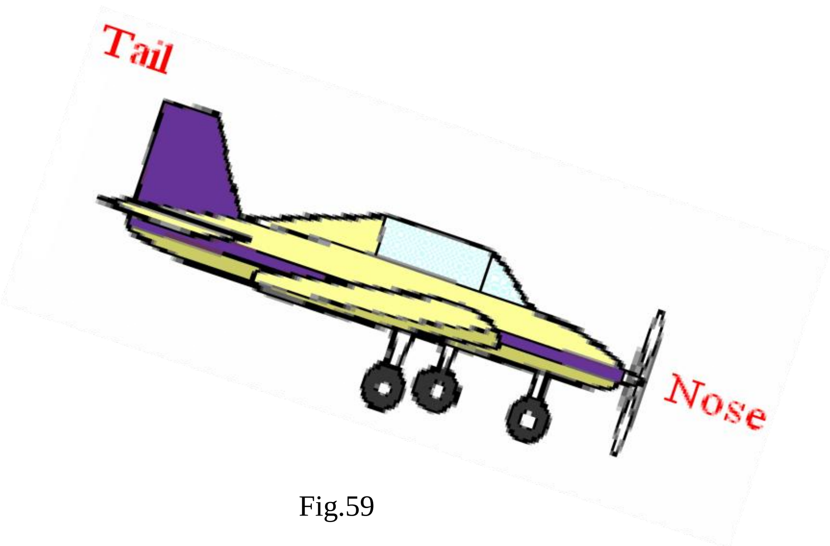


Fig.59

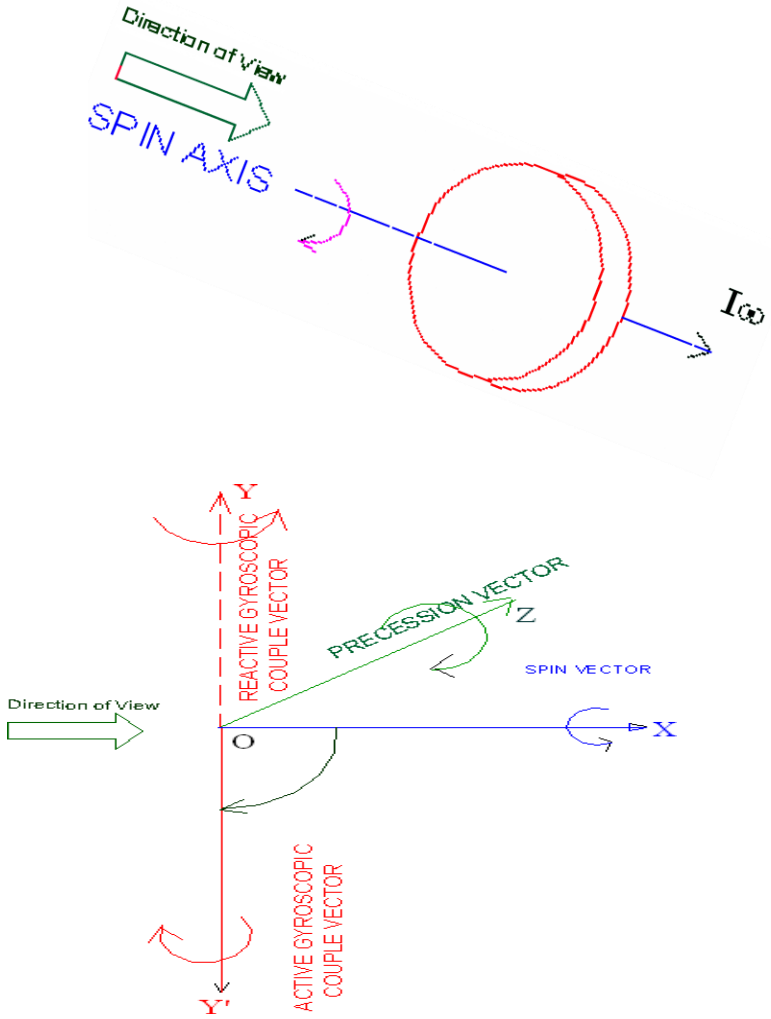


Fig. 61

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left

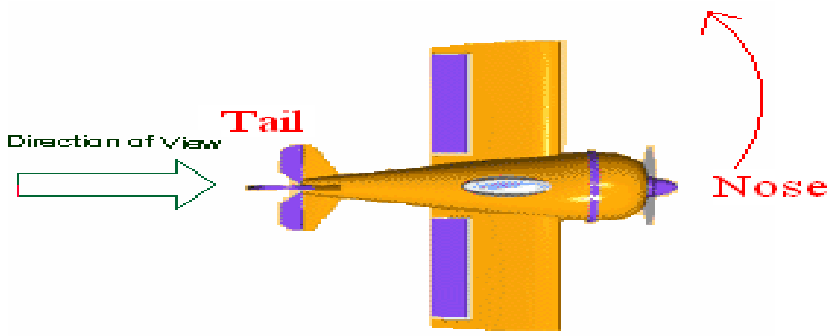


Fig.62

Case (vii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards

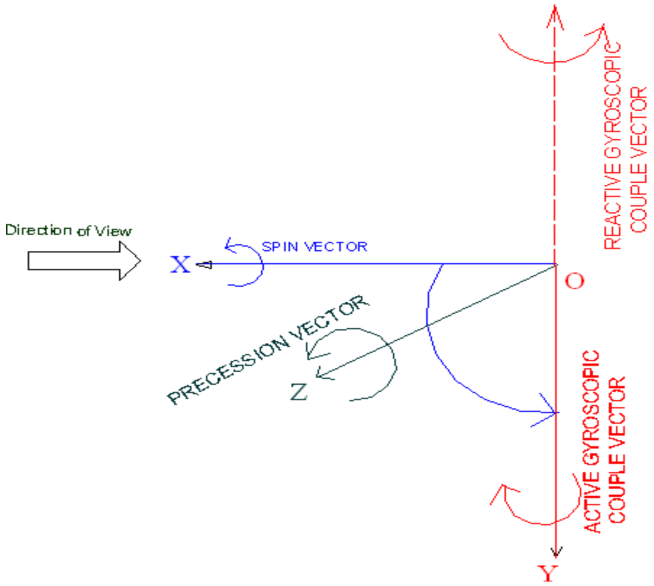


Fig.63

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left

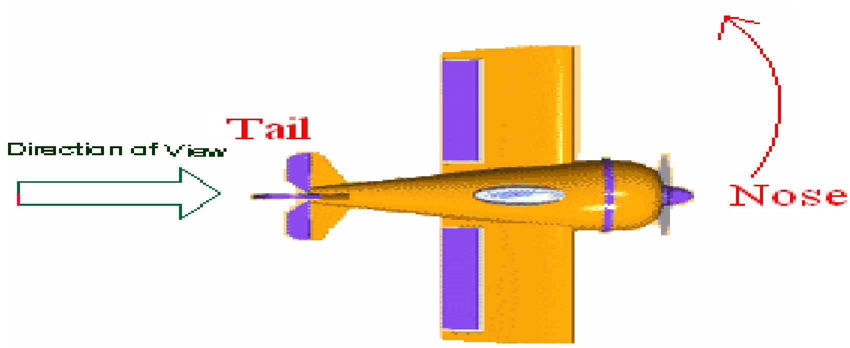


Fig.64

Case (viii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards

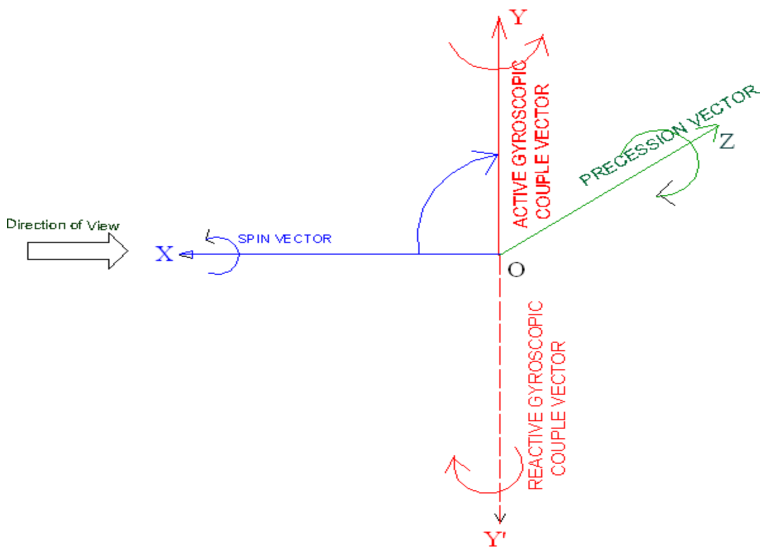


Fig.65

The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right

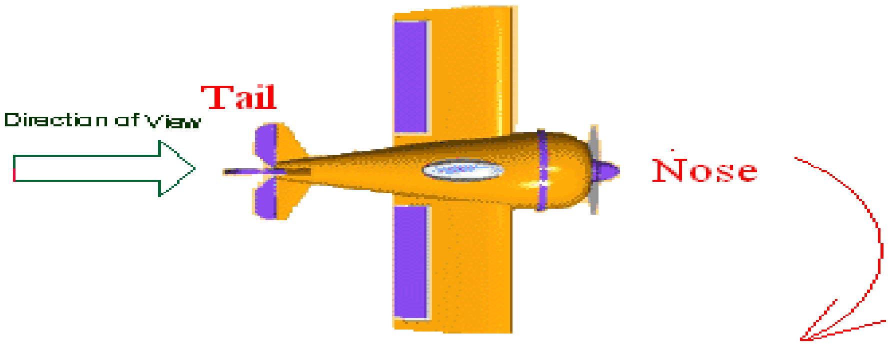


Fig.66

Problem 4

An aeroplane flying at a speed of 300 kmph takes right turn with a radius of 50 m. The mass of engine and propeller is 500 kg and radius of gyration is 400 mm. If the engine runs at 1800 rpm in clockwise direction when viewed from tail end, determine the gyroscopic couple and state its effect on the aeroplane. What will be the effect if the aeroplane turns to its left instead of right?

Solution Angular velocity of aeroplane engine:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1800}{60} = 188.49 \text{ rad/s}$$

Angular velocity of precession: $\omega_p = \frac{V}{R}$

or $$\omega_p = \frac{300 \times 1000}{3600} \times \frac{1}{50}$$

$$= 1.67 \text{ rad/s}$$

Moment of inertia: $$I = mk^2 = 500 \times 0.4^2$$

$$= 80 \text{ kg m}^2$$

Gyroscopic couple: $$c = I\omega\omega_p$$

$$= 80 \times 188.49 \times 1.67$$

$$= \mathbf{25182.26 \text{ Nm}}$$ **Ans.**

Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT

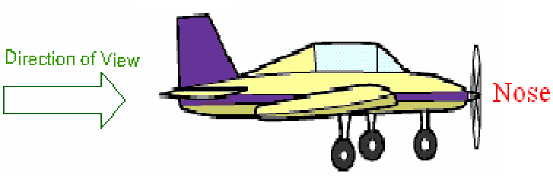


Fig.67

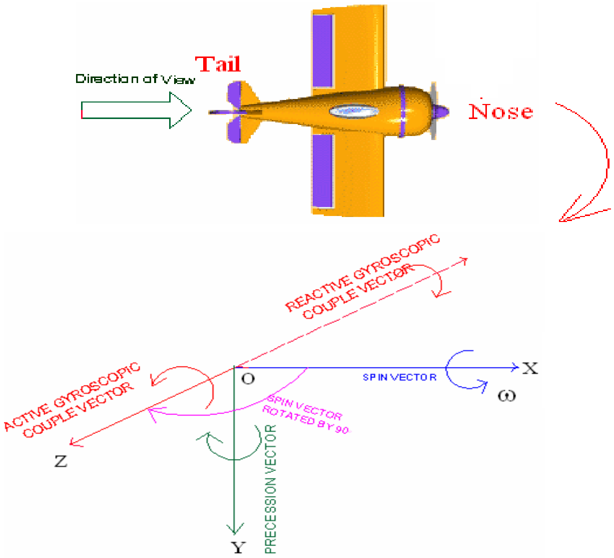


Fig.68

According to the analysis, the reactive gyroscopic couple tends to dip the nose and raise the tail of the aeroplane.

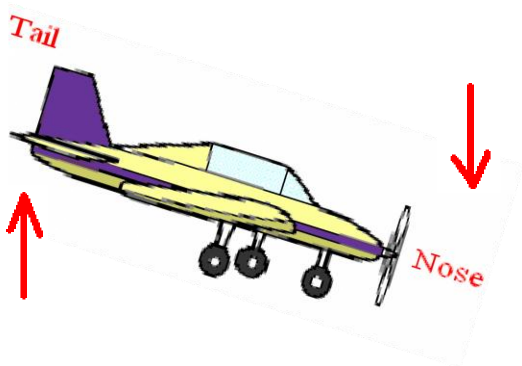


Fig.69

When aeroplane turns to its left, the magnitude of gyrocouple remains the same. However, the direction of reaction couple is reversed and it will raise the nose and dip the tail of the aeroplane.

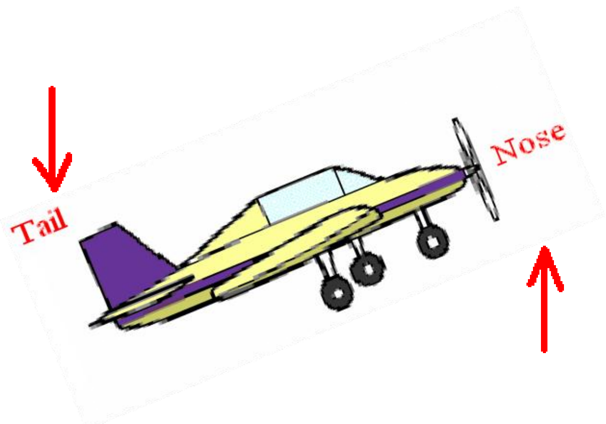


Fig.70

Stability of Automotive Vehicle

A vehicle running on the road is said to be stable when no wheel is supposed to leave the road surface. In other words, the resultant reactions by the road surface on wheels should act in upward direction. For a moving vehicle, one of the reaction is due to gyroscopic couple produced by the rotating wheels and rotating parts of the engine. Let us discuss stability of two and four wheeled vehicles when negotiating a curve/turn.

Stability of Two Wheeler negotiating a turn

Fig.71

Fig. 71 shows a two wheeler vehicle taking **left turn** over a curved path. The vehicle is inclined to the vertical for equilibrium by an angle θ known as angle of heel.

Let

m = *Mass of the vehicle and its* rider in kg,

W = *Weight of the vehicle and* its rider in newtons = $m.g$,

h = *Height of the centre of* gravity of the vehicle and rider,

r_W = *Radius of the wheels,*

R = *Radius of track or* curvature,

I_W = *Mass moment of inertia of each wheel,*

I_E = *Mass moment of inertia of the rotating parts of the engine,*

ω_W = Angular velocity of the wheels,

ω_E = Angular velocity of the engine rotating parts,

G = *Gear ratio* = ω_E / ω_W,

v = *Linear velocity of the vehicle* = $\omega_W \times r_W$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium.

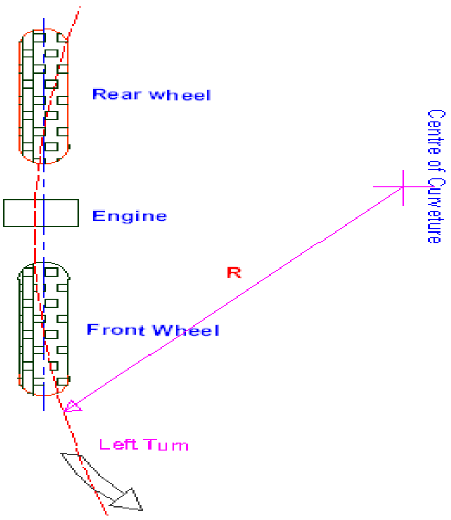


Fig.72

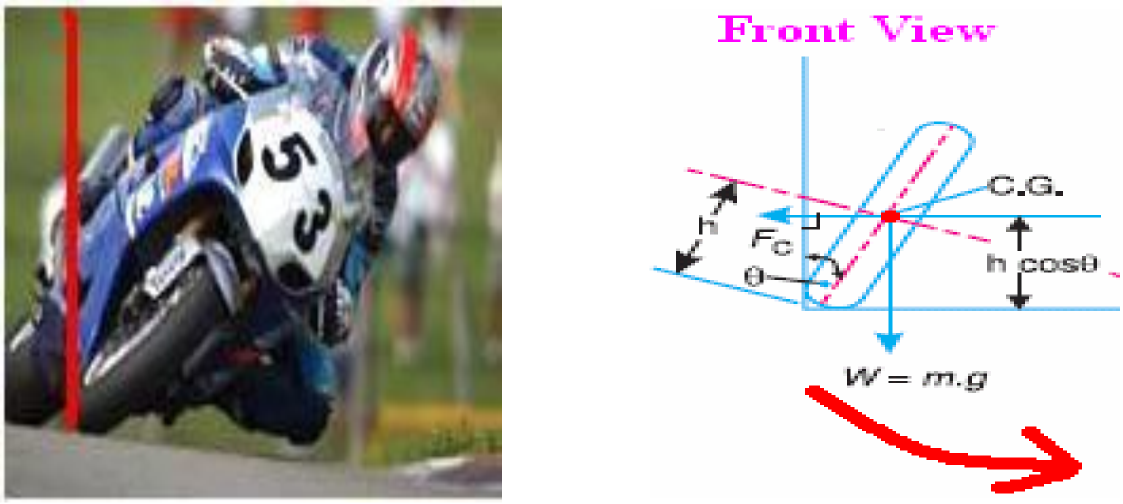


Fig.73

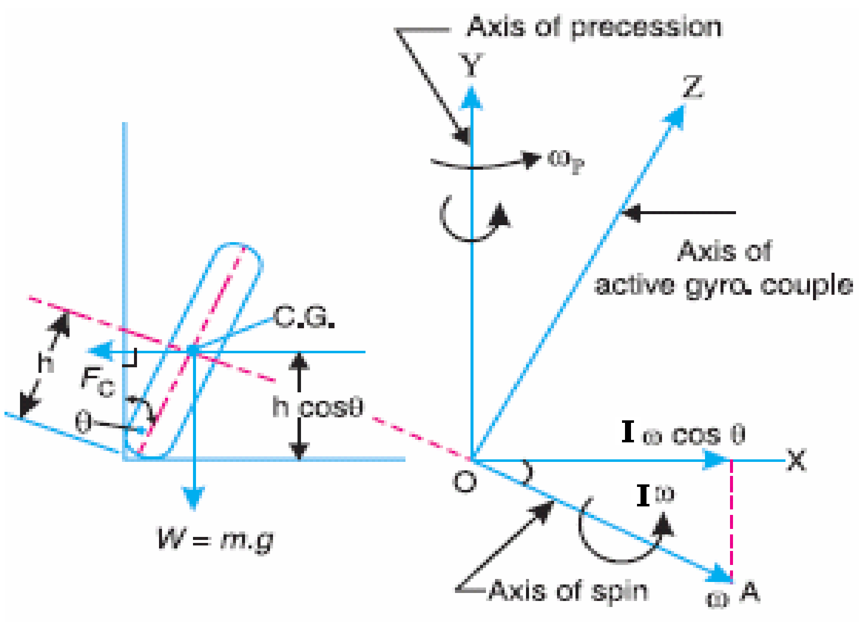


Fig.74

Let us consider the effect of the gyroscopic couple and centrifugal couple on the wheels.

1. Effect of Gyroscopic Couple

We know that, $V = \omega_W \times r_W$

$\omega_E = G.\omega_W$ *or* $\omega_E = G.v/r_W$

Angular momentum due to wheels = $2\, I_w\, \omega_W$

Angular momentum due to engine and transmission = $I_E\, \omega_E$

Total angular momentum ($I\, x\omega$) $= 2\, I_w\, \omega_W \pm I_E\, \omega_E$

$$= \; 2I_w \frac{v}{r_w} \pm I_E G \frac{v}{r_w}$$

$$= \frac{v}{r_w}(2I_w \pm GI_E)$$

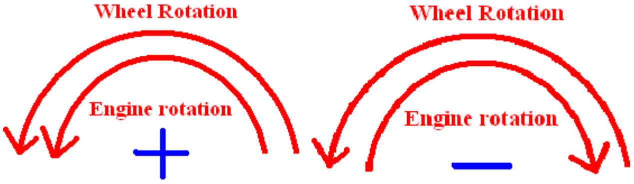
Also, Velocity of precession = $\omega_p = \frac{V}{R}$

It is observed that, when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig… This angle is known as 'angle of heel'. In other words, the axis of spin is inclined to the horizontal at an angle θ , as shown in Fig.73 Thus, the angular momentum vector I ω due to spin is represented by OA inclined to OX at an angle θ. But, the precession axis is in vertical. Therefore, the spin vector is resolved along OX.

Gyroscopic Couple,

$$C_g = (I\omega)\cos\theta \times \omega_p$$
$$C_g = \frac{v^2}{Rr_w}(2I_w \pm GI_E)\cos\theta$$

Note: When the engine is rotating in the same direction as that of wheels, then the positive sign is used in the above equation. However, if the engine rotates in opposite direction to wheels, then negative sign is used.



The gyroscopic couple will act over the vehicle outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig…

Analysis:

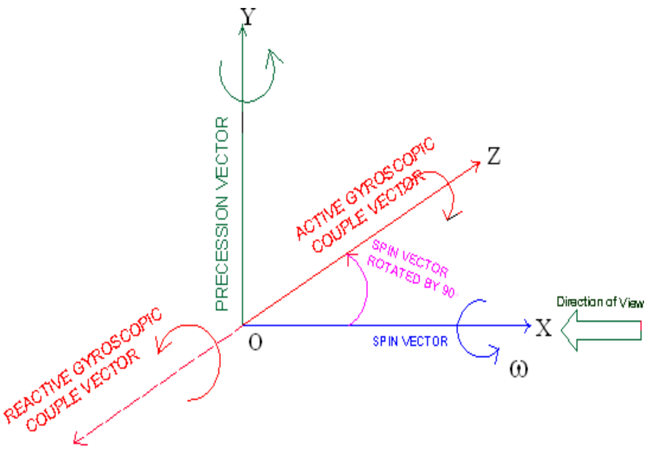


Fig.75

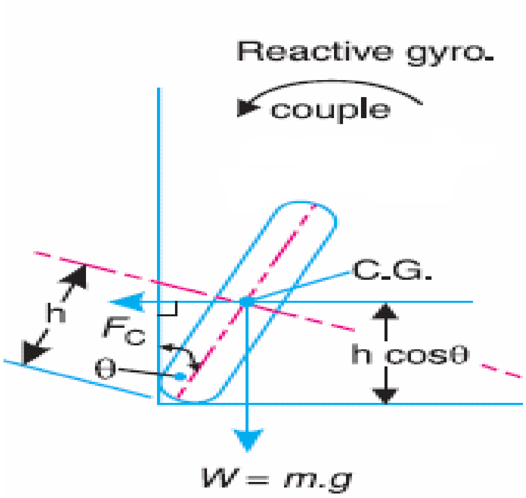


Fig.76

2. Effect of Centrifugal Couple

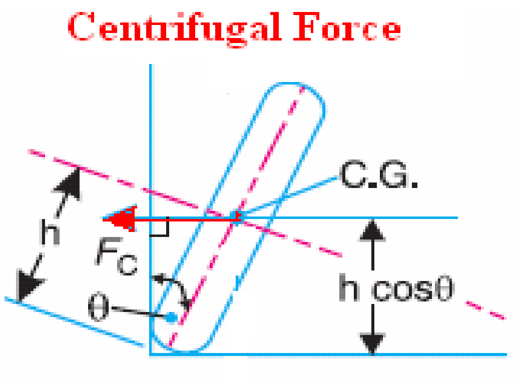


Fig. 77

We have,

Centrifugal force, $F_c = \dfrac{mv^2}{R}$

or

Centrifugal Couple, $C_c = F_c \times h\cos\theta$

$$= \frac{mv^2}{R} h\cos\theta$$

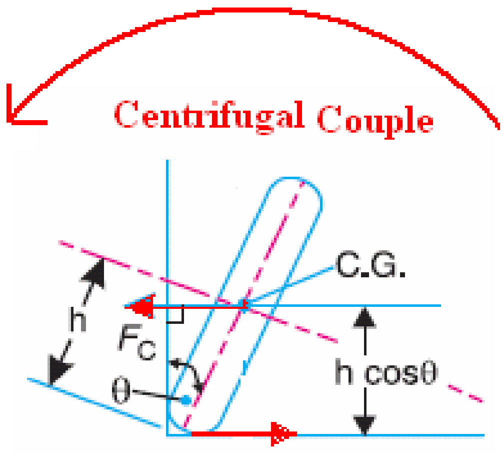


Fig.78

The Centrifugal couple will act over the two wheeler outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig.78

Therefore, the total Over turning couple: $C = C_g + C_c$

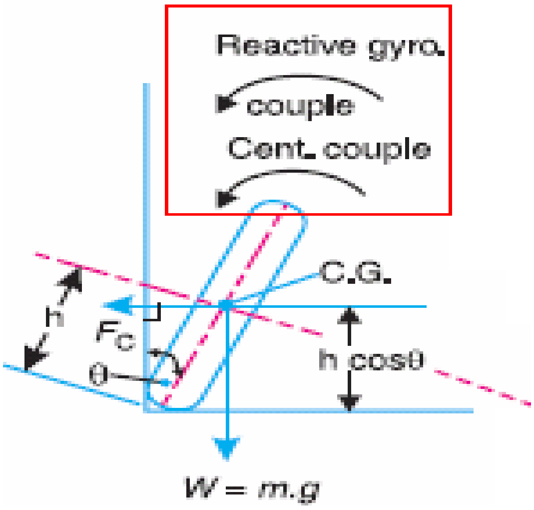


Fig.79

$$C = \frac{v^2}{Rr}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta$$

For the vehicle to be in equilibrium, overturning couple should be equal to balancing couple acting in clockwise direction due to the weight of the vehicle and rider.

∴ $C = mgh \sin\theta$

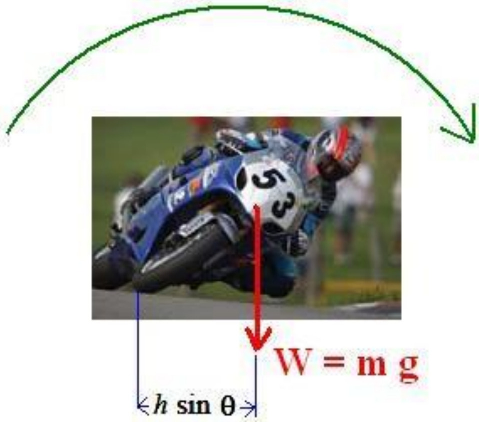


Fig.80

For the stability, overturning couple must be equal to balancing couple,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta = mgh\sin\theta}$$

Therefore, from the above equation, the value of angle of heel (θ) may be determined, so that the vehicle does not skid. Also, for the given value of θ, the maximum vehicle speed in the turn with out skid may be determined.

Problem 5

A motorcycle and its rider together weighs 2000 N and their combined centre of gravity is 550 mm above the road when motorcycle is upright. Each wheel is of 580 mm diameter and has a moment of inertia of 1.0 kgm^2. The moment of inertia of rotating parts of engine is 0.15 kg m^2. The engine rotates at 5 times the speed of the vehicle and the same sense. Determine the angle of heel necessary when motorcycle is taking a turn over a track of 35 m radius at a speed of 60 kmph.

Solution:

Velocity of vehicle : $v = \frac{60 \times 1000}{3600} = 16.67$ m/s

Angular velocity of wheel: $\omega = \frac{2v}{d} = \frac{2 \times 16.67}{0.58} = 57.48$ rad/s

Angular velocity of precession: $\omega_p = \frac{v}{R} = \frac{16.67}{35} = 0.476$ rad/s

(i) Gyroscopic couple due to two wheels:

$C_w = 2I_w \omega\omega_p \cos\theta$
$= 2 \times 1.0 \times 57.48 \times 0.476 \times \cos\theta$
$= 54.72 \cos\theta$ Nm

(ii) Gyroscopic couple due to rotating parts of engine:

$C_E = I_E G\omega\omega_p \cos\theta$
$= 0.15 \times 5 \times 57.48 \times 0.476 \times \cos\theta$
$= 20.52\cos\theta$ Nm

(iii) Centrifugal force due to angular velocity of die wheel:

$$F_c = \frac{mv^2}{R} = \frac{2000 \times 16.67^2}{9.81 \times 35} = 1618.7 \text{ N}$$

Centrifugal couple: $C_c = 1618.7 \times 0.55 \cos\theta$
$= 890.28 \cos\theta$ Nm

Total overturning couple: $C = C_w + C_e + C_c$
$= (54.72 + 20.52 + 890.28) \cos\theta$
$= 965.52 \cos\theta$ Nm

Balancing coupl $= mgh \sin\theta$

$$= \frac{2000}{9.81} \times 9.81 \times 0.55 \sin\theta$$

$= 1100 \sin\theta$ Nm

For the stability of motorcycle, overturning couple should be equal to resisting couple.

$\therefore$ $1100 \sin\theta = 965.52 \cos\theta$

or

$$\tan\theta = \frac{965.52}{1100} = 0.877$$

heel angle: $\theta = \mathbf{41.27°}$

Problem 6

A motor cycle with its rider has a mass of 300 kg. The centre of gravity of the machine and rider combined being 0.6 m above the ground with machine in vertical position. Moment of inertia of each wheel is 0.525 kg m^2 and the rolling diameter of 0.6 m. The engine rotates 6 times the speed of the road wheels and in the same sense. The engine rotating parts have a mass moment of inertia of 0.1686 kg m^2. Find (i) the angle of heel necessary if the vehicle is running at 60 km/hr round a curve of 30 m (ii) If the road and tyre friction allow for the angle of heel not to exceed 50^o, what is the maximum road velocity of the motor cycle.

Solution:

m = 300 kg, h = 0.6 m, I_w = 0.525 kg m^2 , dw=0.6 m; r_w = 0.3 m, G = 6, I_E = 0.1686 m , V= 60km/hr = 16.66 m/s, R = 30 m (i) θ = ? (ii) θ = 50^o V=?

(i) Angle of heel,

We have,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta = mgh\sin\theta}$$

$$\therefore \frac{16.66^2}{30}\left[\frac{2x0.525 + 6x0.1685}{0.3} + 300\ x\ 0.6\right]\cos\theta = 300x9.81x0.6x\sin\theta$$

$$\theta = 45^o$$

(ii) Given, $\theta = 50^o$, V=?,

$$\boxed{\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta = mgh\sin\theta}$$

$$\therefore \frac{V^2}{30}\left[\frac{2\ x\ 0.525 + 6\ x\ 0.1685}{0.3} + 300\ x\ 0.6\right]\cos 50 = 300\ x\ 9.81\ x\ 0.6\ x\sin 50$$

$$\therefore V = 66\ Kmph$$

Stability of Four Wheeled Vehicle negotiating a turn.

Stable condition

Unstable Condition

Fig.81

Consider a four wheels automotive vehicle as shown in Figure 82. The engine is mounted at the rear with its crank shaft parallel to the rear axle. The centre of gravity of the vehicle lies vertically above the ground where total weight of the vehicle is assumed to be acted upon.

Let

m = Mass of the vehicle (kg)

W = Weight of the vehicle (N) = *m.g,*

h = Height of the centre of gravity of the vehicle (m)

r_W = *Radius of the wheels (m)*

R = Radius of track or curvature (m)

I_W = *Mass moment of inertia of each wheel (kg-m^2)*

I_E = *Mass moment of inertia of the rotating parts of the engine (kg-m^2)*

ω_W = Angular velocity of the wheels (rad/s)

ω_E = Angular velocity of the engine (rad/s)

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle (m/s)= $\omega_W \times r_W$,

x = Wheel track (m)

b = Wheel base (m)

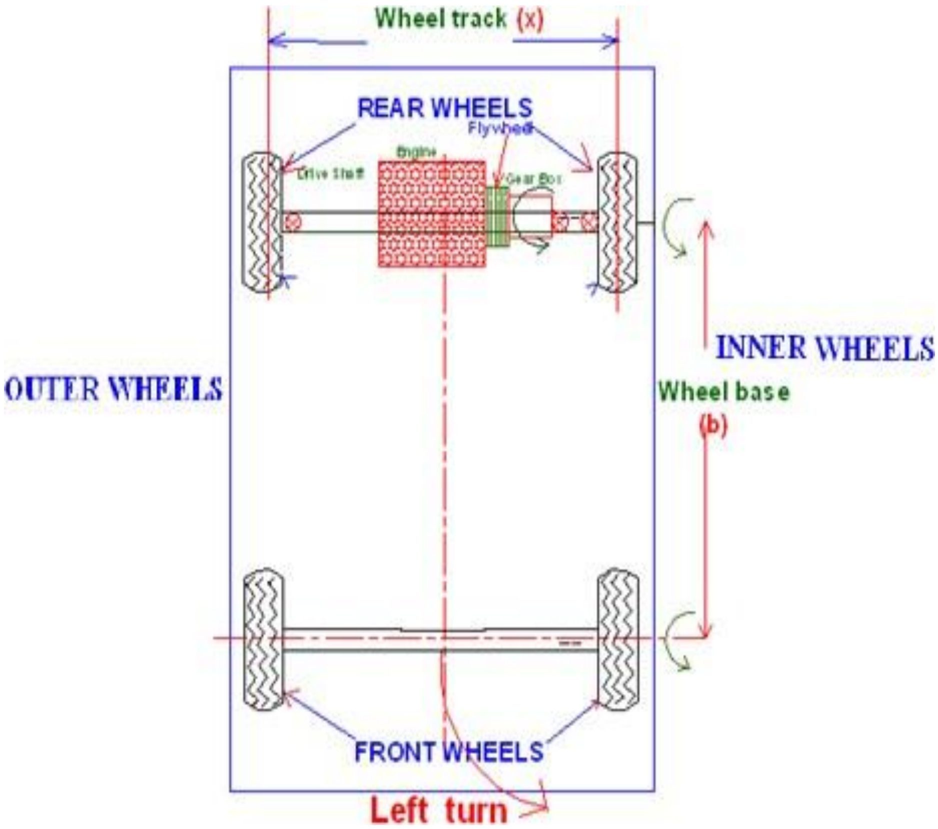


Fig.82

(i) Reaction due to weight of Vehicle

Weight of the vehicle. Assuming that weight of the vehicle *(mg)* is equally distributed over four wheels. Therefore, the force on each wheel acting downward is m*g/4* and the reaction by the road surface on the wheel acts in upward direction.

$$R_w = \frac{mg}{4}$$

(ii) Effect of Gyroscopic couple due to Wheel

Gyroscopic couple due to four wheels is,

$$C_w = 4\, I_w \omega \omega_p$$

(iii) Effect of Gyroscopic Couple due to Engine

Gyroscopic couple due to rotating parts of the engine

$$C_E = I_E\, \omega\, \omega_p = I_E\, G\, \omega\, \omega_p$$

Therefore, Total gyroscopic couple:

$$C_g = C_w + C_E = \omega\, \omega_p\, (4I_W \pm I_E G)$$

When the wheels and rotating parts of the engine rotate in the same direction, then positive sign is used in the above equation. Otherwise negative sign should be considered.

Assuming that the vehicle takes a left turn, the reaction gyroscopic couple on the vehicle acts between outer and inner wheels.

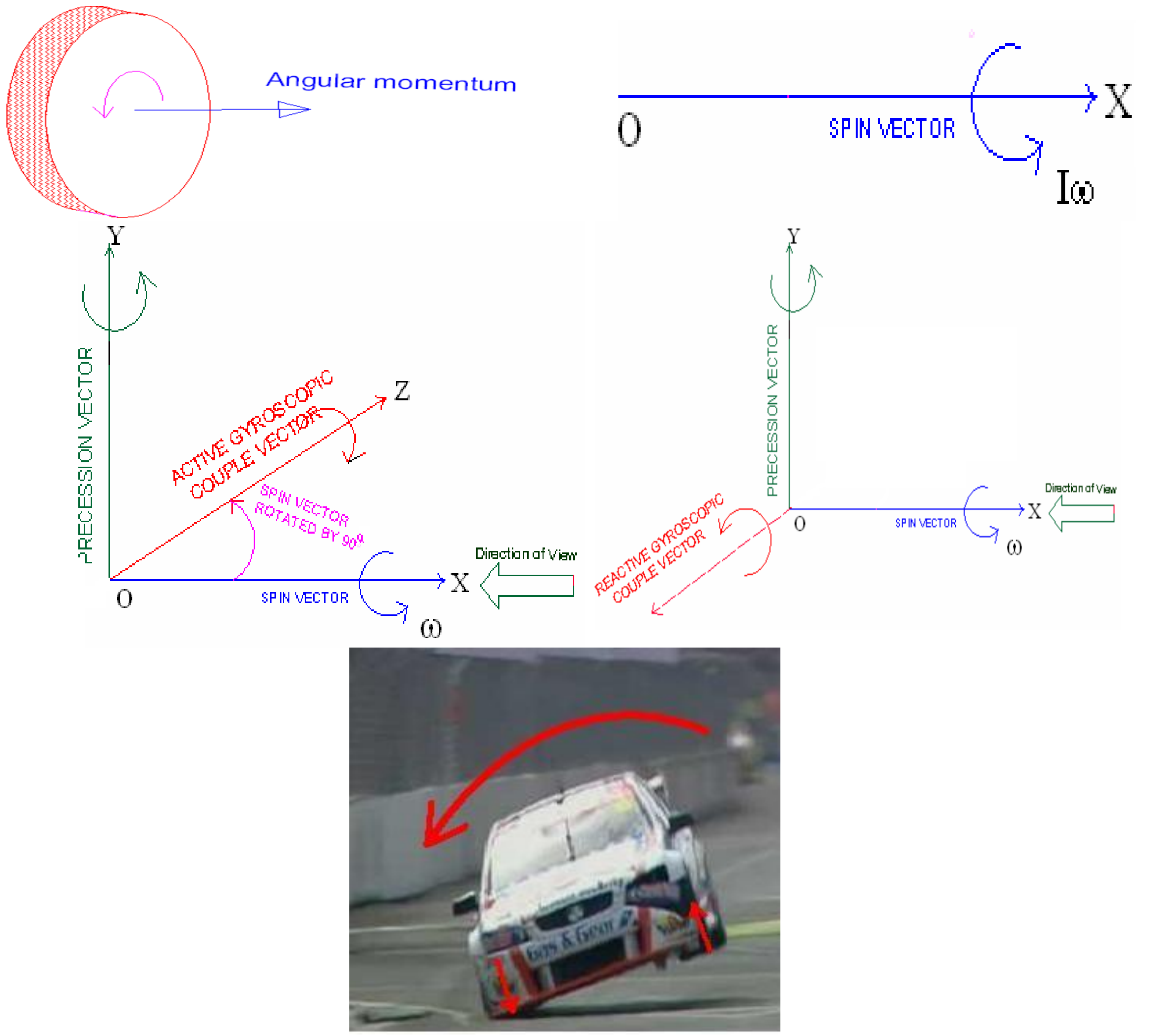


Fig.83

This gyroscopic couple tends to press the outer wheels and lift the inner wheels.

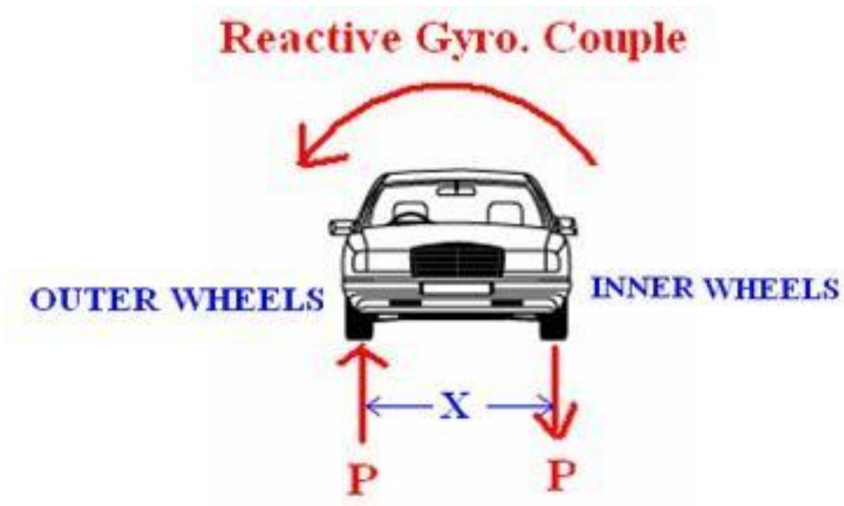


Fig.84

Due to the reactive gyroscopic couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be P Newtons, then,

$$\mathbf{P}\ x\ X = C_g$$

$$\mathbf{P} = \frac{C_g}{x}$$

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_g}{2X}$$

(iii) **Effect of Centrifugal Couple**

When a vehicle moves on a curved path, a centrifugal force acts on the vehicle in outward direction through the centre of gravity of the vehicle(Fig…)

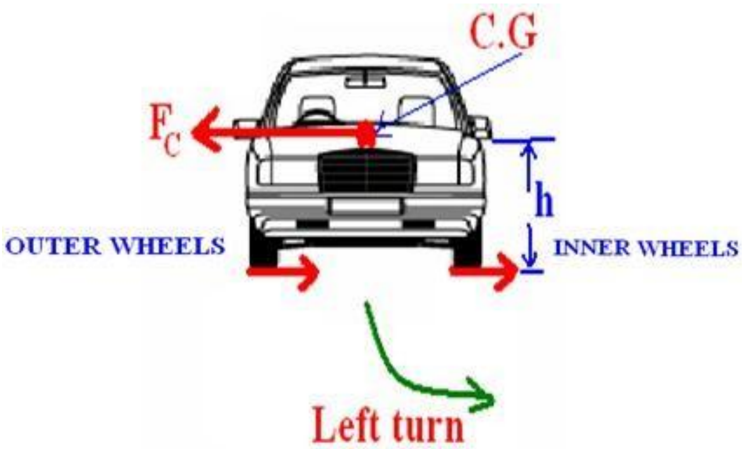


Fig.85

Centrifugal force,

$$F_c = m\omega_p^2 R = \frac{mv^2}{R}$$

This force forms a Centrifugal couple.

$$C_c = \frac{mv^2 h}{R}$$

This centrifugal couple tends to press the outer and lift the inner

Fig.86

Due to the centrifugal couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be F Newtons, then,

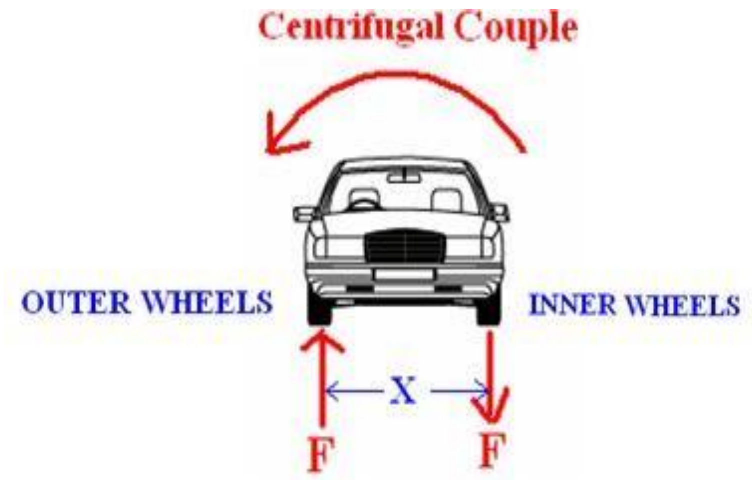


Fig.87

Road reaction on each outer/Inner wheel,

$$\frac{F}{2} = \frac{C_c}{2X}$$

The reactions on the outer/inner wheels are as follows,

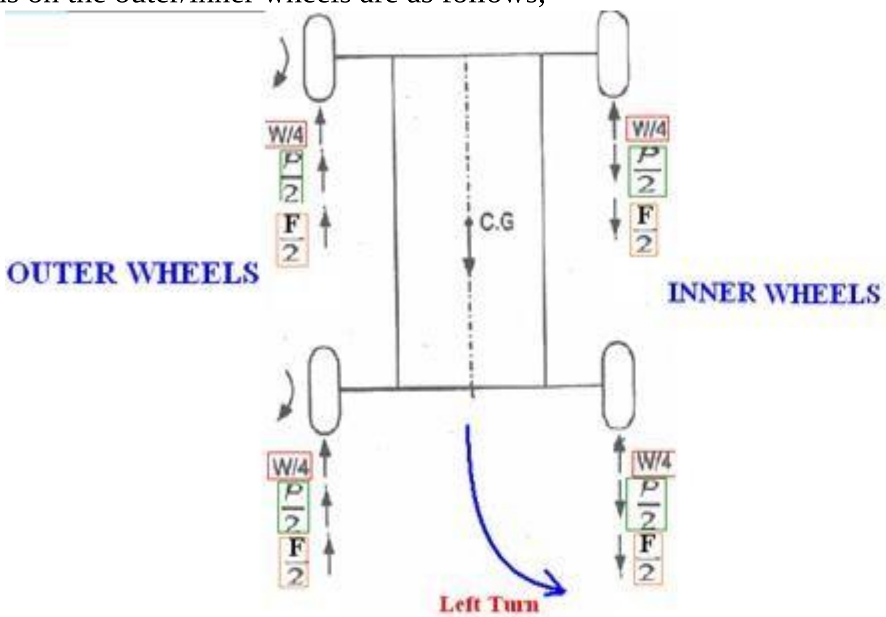


Fig.88

Total vertical reaction at each outer wheels

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

Total vertical reaction at each inner wheels

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

Problem 7

An automobile car is travelling along a track of 100 m mean radius. The moment of inertia of 500 mm diameter wheel is 1.8 kg m^2. The engine axis is parallel to the rear axle and crank shaft rotates in the same sense as the wheel. The moment of inertia of rotating parts of the engine is 1 kg m^2. The gear ratio is 4 and **the mass of** the vehicle is 1500 kg. If the centre of gravity of the vehicle is 450 mm above the road level and width of the track of the vehicle is 1.4 m, determine the limiting speed of the vehicle for condition that all four wheels maintain contact with the road surface.

Solution Let ν = limiting velocity of the vehicle.

Angular velocity: $\omega = \frac{v}{r} = \frac{v}{0.25}$ rad/s

Precession velocity: $\omega_p = \frac{v}{R} = \frac{v}{100}$ rad/s

(i) Reaction due to gyroscopic couple:

(a) Gyroscopic couple due to four wheels:

$$C_w = 4I_w\omega\omega_p$$

$$= 4 \times 2 \times \frac{v}{0.25} \times \frac{v}{100} = 0.32\, v^2 \text{ Nm}$$

(b) Gyroscopic couple due to engine parts:

$$C_e = I_e G\omega\omega_p$$

$$= 1 \times 4 \times \frac{v}{0.25} \times \frac{v}{100} = 0.16\, v^2 \text{ Nm}$$

Total gyroscopic couple:

$$C_g = C_w + C_e$$

$$= 0.32v^2 + 0.16v^2 = 0.48v^2 \text{ Nm}$$

Reaction due to total gyroscopic couple on each outer wheel:

$$R_g = \frac{C_g}{2b} = \frac{0.48v^2}{2 \times 1.5} = 0.16\, v^2 \text{N} (\uparrow)$$

Reaction due to total gyroscopic couple on each inner wheel:

$$C_g = 0.16 \ v^2 N \ (\downarrow)$$

(ii) Reaction due to centrifugal couple:

Centrifugal force: $$F_c = \frac{mv^2}{R} = \frac{1500 \times v^2}{100} = 15v^2 \text{ N}$$

Overturning couple due to centrifugal force:

$$C_c = F_c \text{ x } h$$

$$= 15 \ v^2 \text{x } 0.45 = 6.75 \ v^2 \text{ Nm}$$

Vertical downward reaction on each inner wheel is:

$$R_c = \frac{C_c}{2b} = \frac{6.75v^2}{2 \times 1.5} = 2.25v^2 \text{ N} (\downarrow)$$

(iii) Reaction due to weight of the vehicle:

$$R_w = \frac{mg}{4} = \frac{1500 \times 9.81}{4} = 3678.75 \text{ N} (\uparrow)$$

The limiting condition to avoid lifting of inner wheels from the road surface is:

$$R_i = R_w - R_c - R_g > 0$$

or

$$R_w > R_c + R_g$$

$$3678.75 \geq 2.25v^2 + 0.16 \ v^2$$

$$v = 39.07 \text{ m/s} \quad \text{or} \quad \mathbf{140.65 \ kmph}$$

or

Problem 8

A four wheeled motor vehicle of mass 2000 kg has a wheel base of 2.5 m, track width 1.5m and height of c.g is 500 mm above the ground level and lies 1 m from the front axle. Each wheel has an effective diameter of 0.8m and a moment of inertia of 0.8 kgm2. The drive shaft, engine flywheel rotating at 4 times the speed of road wheel in clockwise direction when viewed from the front and is equivalent to a mass of 75 kg having a radius of gyration of 100mm.If the vehicle is taking a right turn of 60 m radius at 60kmph, determine the load on each wheel.

Solution,

Since the C.G of the vehicle is 1 m from the front,

The percentage of weight on the front wheels = (2.5-1)/2.5 x 100

= 60%

The percentage of weight on the rear wheels = 40 %

Total weight on the front wheels = 11772 N

Total weight on the rear wheels = 7848 N

Weight on each of front wheel = 5886 N = $W_F/2$

Weight on each of rear wheel = 3924 N = $W_R/2$

The road reaction due to weight of the vehicle is always upwards

Effect of Gyroscopic couple due to Wheel,

$$\mathbf{C_W = 4I_W.\ \omega_W.\ \omega_P}$$
$$\mathbf{= 37.1\ Nm}$$

Gyroscopic couple due to wheels acts between outer and inner wheels.

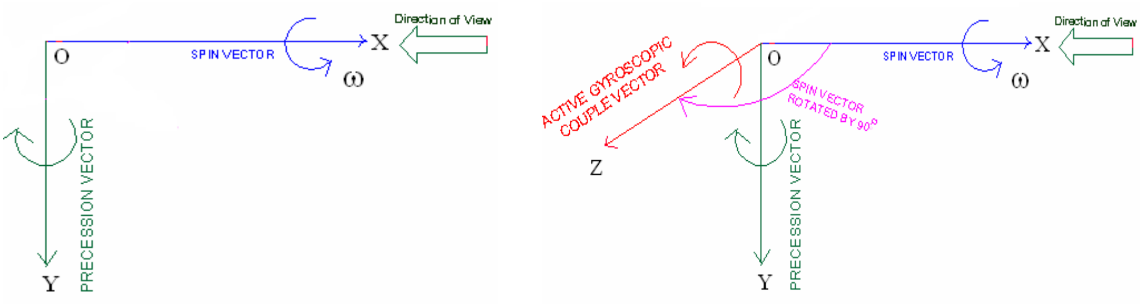


Fig.89

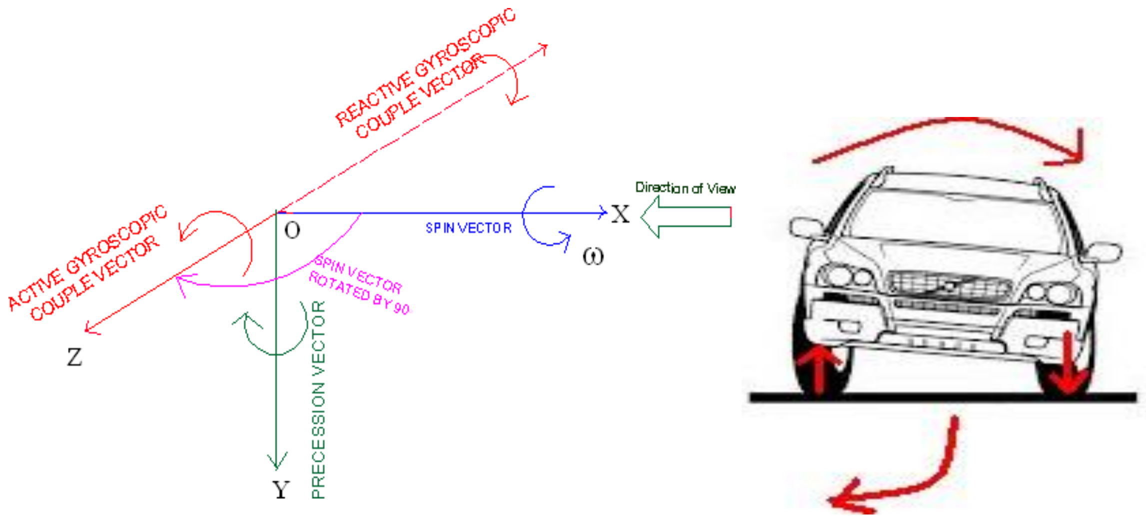


Fig.90

The gyroscopic couple tends to press the outer and lift the inner wheels

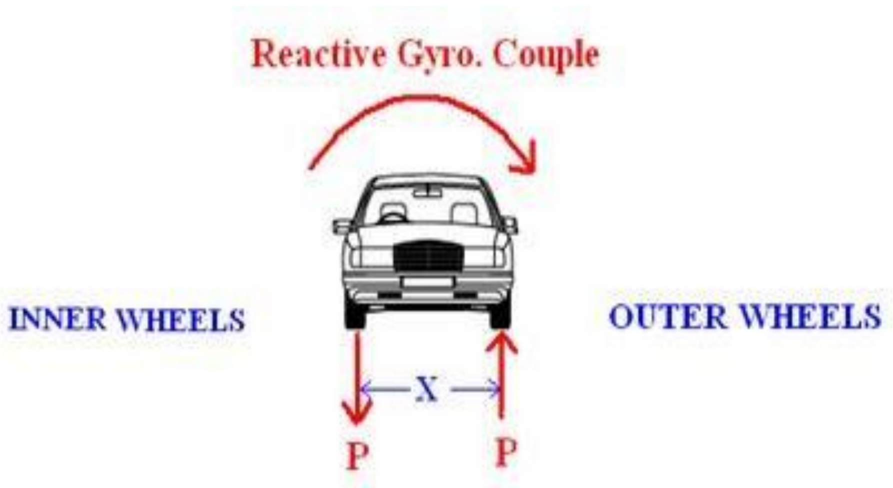


Fig. 91

The road reaction is vertically upward for outer wheels and downward for inner wheels

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_W}{2X} = 12.37 \text{ N}$$

Effect of Gyroscopic Couple due to Engine

Gyroscopic couple due to engine

$\mathbf{C_E = I_E.\ \omega_E.\ \omega_P}$

$\mathbf{C_E = I_E.G.\ \omega_W.\ \omega_P}$

$\mathbf{= 34.7\ N\ m}$

Gyroscopic couple due to engine acts between Front and Rear wheels.

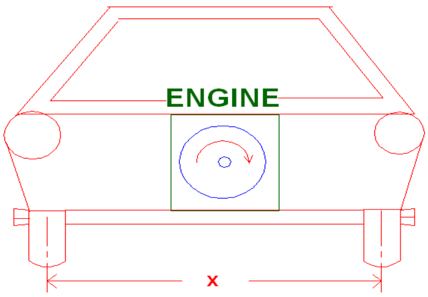


Fig. 92

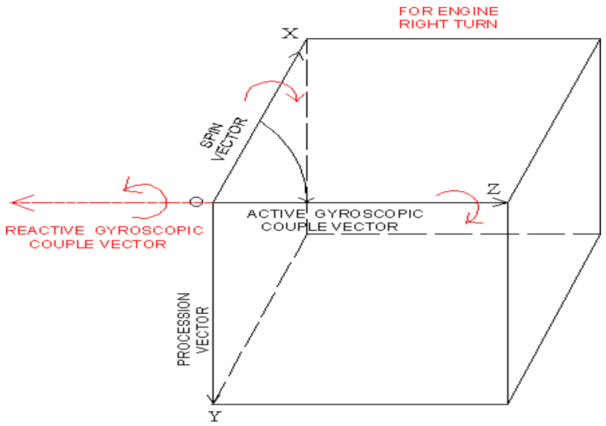


Fig. 93

The couple tends to press Rear wheels and Lift front wheels

Fig. 94

The road reaction is vertically upward for REAR and downward for FRONT wheels.

Fig.95

Road reaction on each Front/Rear wheels

$$\frac{Q}{2} = \frac{C_E}{2b} = 6.94 \text{ N}$$

Effect of Centrifugal Couple

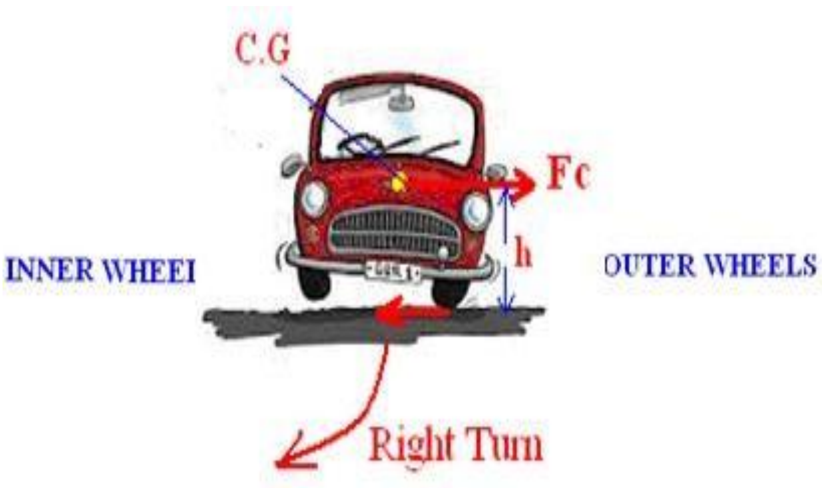


Fig.96

Centrifugal force, $F_c = \frac{MV^2}{R} = \mathbf{9263}$ N

Centrifugal Couple $C_C = \frac{mV^2}{R} \times h = \mathbf{4631.5}$ N

The gyroscopic couple tends to press the outer and lift the inner wheels.

Fig.97

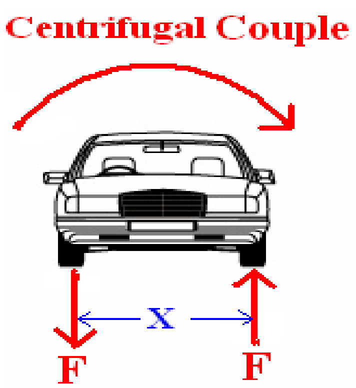


Fig. 98

The road reaction is vertically upward for outer wheels and downward for inner wheels.

Road reaction on each outer/Inner wheel

$$\frac{F}{2} = \frac{C_C}{2X} = \mathbf{1543.8}\text{N}$$

Engine crank shaft rotates clockwise direction seen from front, and Vehicle takes RIGHT turn

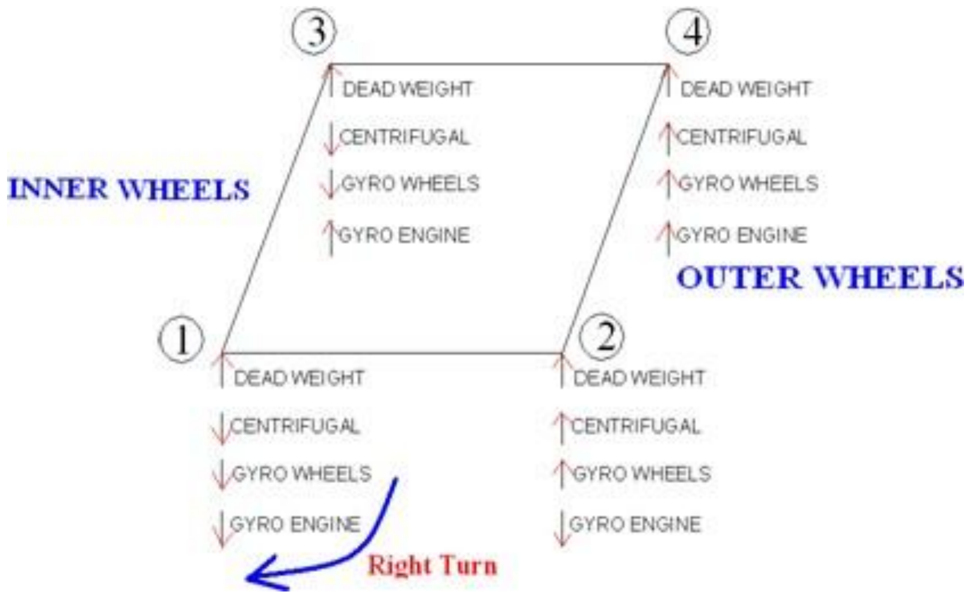


Fig.99

Load on front wheel 1 = 4322.86 N
Load on front wheel 2 = 7435.26 N
Load on rear wheel 3 = 2374.74 N
Load on rear wheel 4 = 5487.14 N

Problem 9

A section of an electric rail track of gauge 1.5 m has a left hand curve of radius 300 m, the superelevation of the outer rail being 260 mm. The approach to the curve is along a straight length of track, over the last 50 m there is a uniform increase in elevation of the outer rail from level track to the super elevation of 260 mm. Each motor used for traction has a rotor of mass 550 kg and radius of gyration 300 mm. The motor shaft is parallel to the axes of the running wheels. It is supported in bearings 780 mm apart and runs at four times the wheel speed but in opposite direction. The diameter of running wheel is 1.2 m. Determine the forces on the bearings due to gyroscopic action when the train is travelling at 90 kmph (a) on the last 50 m of approach track (b) on the curve track.

Solution Angular velocity:

$$\omega = \frac{\text{Gear ratio} \times v}{r}$$

$$= \frac{4 \times 90 \times 1000}{3600 \times 0.6} = 166.67 \text{ rad/s}$$

Let ω_p = angular velocity of precession.

Moment of inertia: $I = mk^2 = 550 \times 0.3^2 = 49.5$ kg m^2

Gyroscopic couple:

$$C = I\omega\omega_p$$
$$= 49.5 \times 166.67 \times \omega_p$$
$$= 8250.16\ \omega_p \text{ Nm}$$

$$P = \frac{8250.16\,\omega_p}{0.78}$$
$$= 10577.1\ \omega_p \text{ N}$$

Forces on bearings:

(a) Angle turned by engine shaft in the last 50 m track

$$= \frac{0.26}{1.5} = 0.1734 \text{ rad}$$

Time taken to cover this distanc$= \dfrac{50}{90/3.6} = 2$ sec

Velocity of precession: $\omega_p = \dfrac{0.1734}{2} = 0.0867$

Forces on bearings: P = 10577.1 x 0.0867 = 917.03 N

The change in momentum is represented by vector *oa* and *ob* as shown in Figure 15.18.

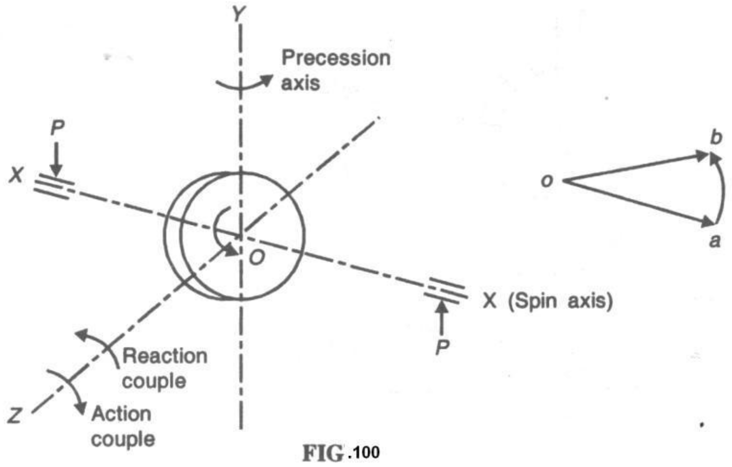


FIG.100

The couple required for precession is, therefore, acting in clockwise looking upward direction. The reaction couple acts in anticlockwise direction looking downward as the forces on the bearings are in the directions shown in Figure 100.

b) When electric rail moves on curved path, the effective angular velocity of precession about the axis perpendicular to the axis of rotation is:

$$\omega_p = \frac{v}{R}\cos\theta$$

where θ is angle due to superelevation of outer rail. Referring to Figure 15.19.

$$\cos\theta = \frac{AB}{AC} = \frac{1.4773}{1.5} = 0.9848$$

or $$\omega_p = \frac{90 \times 1000}{3600 \times 300} \times 0.9848 = 0.08206 \text{ rad/s}$$

Effective angular velocity of spin = ω - $\omega p \sin\theta$ = ω

Therefore,

Forces on bearings: P = 10577.1 ωp
= 10577.1 x 0.08206
= 867.95 N **Ans.**

The change in angular momentum vector and reaction couple shown in Figure 15.19 shows direction of forces on the bearings.

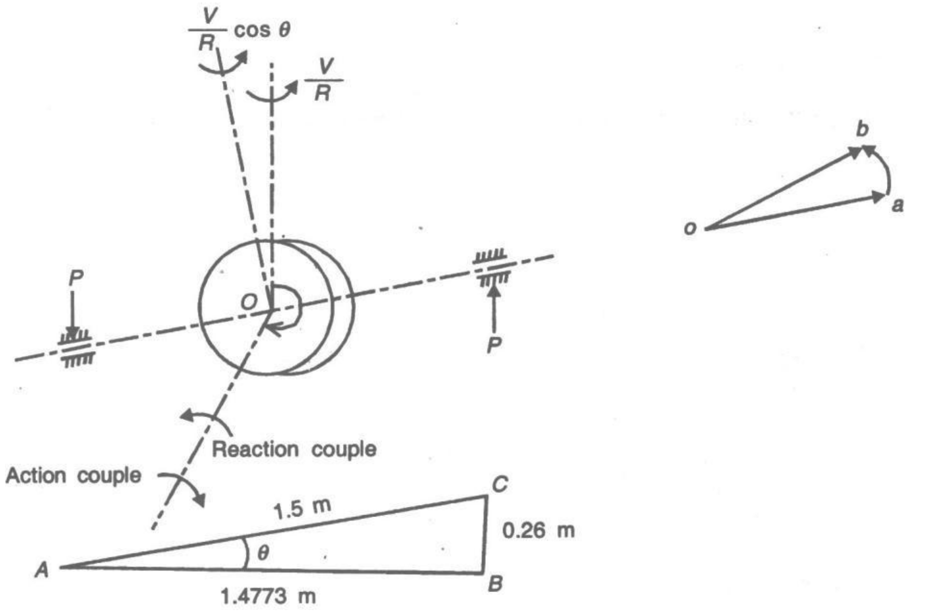


Fig.101

Problem 10.

A four wheeled trolley of total weight 20 kN running on rails of 1 m gauge rounds a curve of 30 m at 40 kmph on a track of embankment slope of 10^0. The wheels have external diameter of 0.6 m and each pair of axle weighs 2000 N and has a radius of gyration of 0.25 m. The height of the C.G of trolley above the wheel is 1 m. Calculate the reaction on the each rail due to gyroscopic and centrifugal couple.

Solution,

Weight of trolley = N = 20000 N
Wheel track = 2x
= 1 m
Radius of curve = R = 30 m
Trolley velocity = 40 kmph= 11.1 m/s
Track of embankment slope of = θ = 10^0
Diameter of wheel = d = 0.6 m
Weight of each pair of wheels = W_1 = 2000 N= mg
Radius of gyration k_g = 0.25 m
Height of C.G from wheel base = 1 m

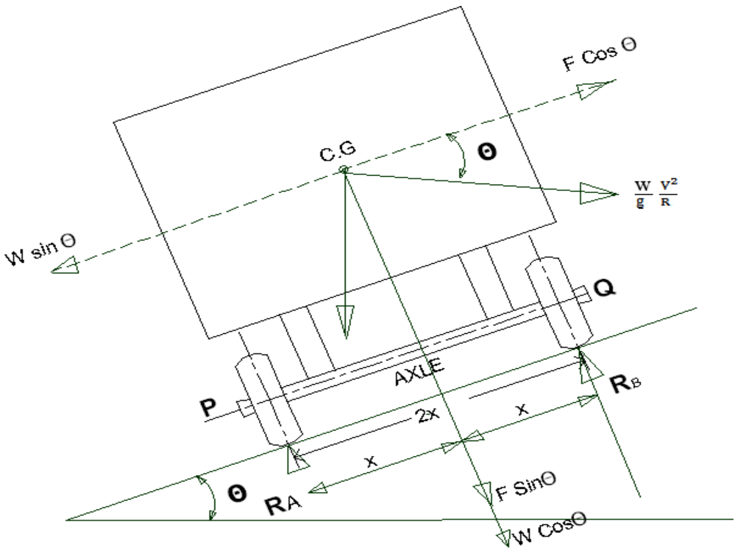


Fig.102

Referring to above Fig. 102,

Consider, the total effect of weight of trolley and that of centrifugal force F,

∴ The reaction RA and RB at the wheels X and Y,

Resolving forces perpendicular to the track,

R_A+R_B = mg Cos θ + F Sin θ

$$= mg \cos\theta + m\frac{V^2}{R}\sin\theta$$

$$= mg\left(\cos\theta + \frac{V^2}{gR}\sin\theta\right)$$

$$= 20000\left[0.9848 + \frac{11.1^2}{9.81*30}*0.1736\right]$$

R_A+R_B = 21.158 N

Taking moments about Q,

$R_A * x$ = (F sin θ + mg cos θ x - (F cos θ + mg sin θ) h

$$R_A = \frac{(\frac{mv^2}{R}\sin\theta + mg\cos\theta)}{2} - \frac{h}{2x}\left(\frac{mv^2}{R}\cos\theta - mg\sin\theta\right)$$

$$= \frac{mg(\frac{v^2}{gR}\sin\theta + \cos\theta)}{2} - \frac{hmg}{2x}\left(\frac{v^2}{gR}\cos\theta - \sin\theta\right)$$

$$= \frac{20000}{2}\left[\frac{11.1^2}{9.81*30}x\ 0.1736 + 0.9848\right] - \frac{1*20000}{1}\left[\frac{11.1^2}{9.81*40}*0.9848 - 0.1736\right]$$

R_A = 5751 N

R_B = 15407 N

Let the force at each pair of wheels or each rail due to gyroscopic couple = F_g

∴ Gyroscopic couple applied = Iω cosθ ω_p

$\therefore F_g * 2x = I\omega \cos\theta\, \omega_p$

$= \frac{I\omega \cos\theta\, \omega p}{2x}$

But, $I = mk^2_g$

$= \frac{2000}{9.81} * 0.25^2$

$= 12.74$ kg m^2

$\omega_p = \frac{V}{R} = \frac{11.1}{30}$

$= 0.37$ rad/s

$\therefore$ Reaction on inner rail = $R_A - F_g$

$\therefore$ Reaction on outer rail = $R_A + F_g$

= 15579 N

Dynamics of Machinery

Preamble

Relation between motion and forces causing is a fascinating subject. This study is a generally referred as dynamic. Modern Engineering aims at analysing and predicting dynamics behavior of physical systems

Theory of Mechanisms & Machines is used to understand the relationships between the geometry and motions of the parts of a machine or mechanism and forces which produce motion.

TOM (M&M theory) is divided into two parts:-

1) ***Kinematics of Machinery***: Study of motion of the components and basic geometry of the mechanism and is not concerned with the forces which cause or affect motion. Study includes the determination of velocity and acceleration of the machine members

2) ***Dynamics of Machinery***: Analyses the forces and couples on the members of the machine due to external forces (static force analysis) also analyses the forces and couples due to accelerations of machine members (Dynamic force analysis)

Deflections of the machine members are neglected in general by treating machine members as rigidbodies (also called rigid body dynamics). In other words the link must be properly designed to withstand the forces without undue deformation to facilitate proper functioning of the system.

In order to design the parts of a machine or mechanism for strength, it is necessary to determine the forces and torques acting on individual links. Each component however small, should be carefully analysed for its role in transmitting force.

The forces associated with the principal function of the machine are usually known or assumed.

Ex:

a) Piston type of engine: gas force on the piston is known or assumed
b) QRM – Resistance of the cutting tool is assumed.

a & b are called static forces.

Example of other static forces are:

i. Energy transmitted
ii. Forces due to assembly
iii. Forces due to applied loads
iv. Forces due to changes in temperature
v. Impact forces
vi. Spring forces
vii. Belt and pulley
viii. Weights of different parts

Apart from static forces, mechanism also experiences inertia forces when subjected to acceleration, called dynamic forces.

Static forces are predominant at lower speeds and dynamic forces are predominant at higher speeds.

Force analysis:

The analysis is aimed at determining the forces transmitted from one point to another, essentially from input point to out put point. This would be the starting point for strength design of a component/ system, basically to decide the dimensions of the components

Force analysis is essential to avoid either overestimation or under estimation of forces on machine member.

Under estimation: leads to design of insufficient strength and to early failure.
Overestimation: machine component would have more strength than required.
Over design leads to heavier machines, costlier and becomes not competitive

Graphical analysis of machine forces will be used here because of the simplification it offers to a problem, especially in cases of complex machines. Moreover, the graphical analysis of forces is a direct application of the equations of equilibrium.

General Principle of force analysis:

A machine / mechanism is a three dimensional object, with forces acting in three dimensions. For a complete force analysis, all the forces are projected on to three mutually perpendicular planes. Then, for each reference plane, it is necessary that, the vector sum of the applied forces in zero and that, the moment of the forces about any axis perpendicular to the reference plane or about any point in the plane is zero for equilibrium.

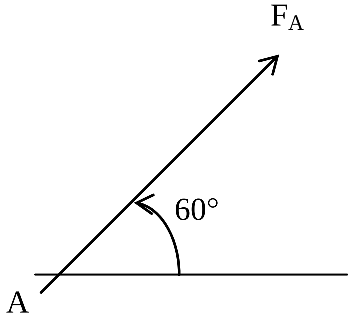
That is $\sum F = 0$ & $\sum M = 0$ *or*

$$\sum F_x = 0 \ \& \sum F_y = 0 \ and \sum M = 0$$

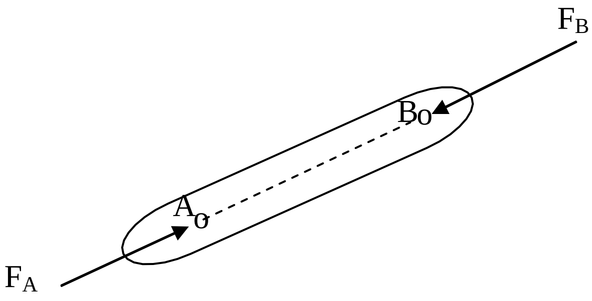
A force is a vector quantity and three in properties define a force completely;

i. Magnitude
ii. Direction
iii. Point of application

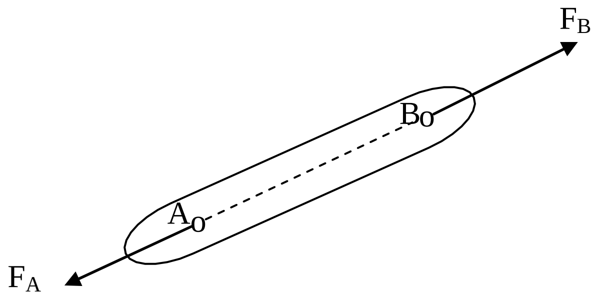
Some basic aspects and notations



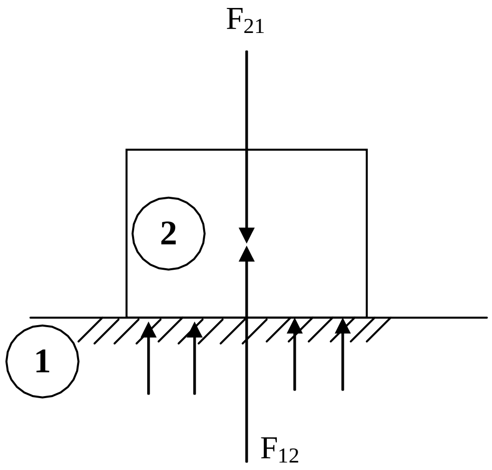
i. Force applied at A
ii. Force vector is inclined at 60° to the reference plane.
iii. Length of the vector represents the magnitude of the force to some scale.



Compressive forces



Tensile forces



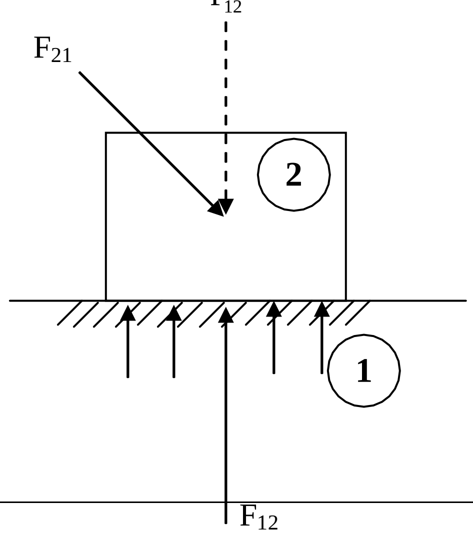
Forces are perpendicular to the line of contact

F_{21} : Force exerted by link 2 on link 1

F_{12} : Force exerted by link 1 on link 2

Here, $F_{21} = F_{12}$ Action and reaction are equal and opposite.

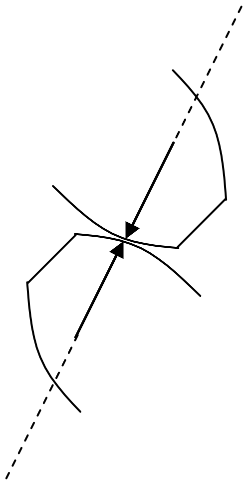
2 is sliding on 1



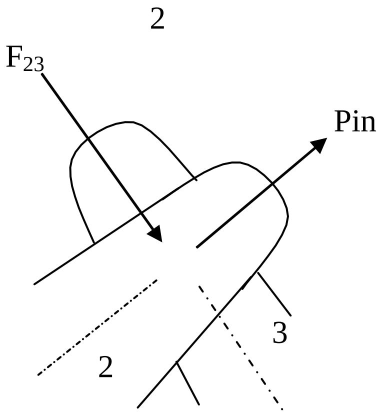
F_{12}^{n}: Vertical component of F_{21}

$F_{12} = F_{12}^{n}$

Gear meshing



Forces act along the common normal or line of action of gears.



Forces pass through the centre of the pin

Equilibrium

For a rigid body to be in Equilibrium

i) Sum of all the forces must be zero

ii) Sum of all the moments of all the forces about any axis must be zero

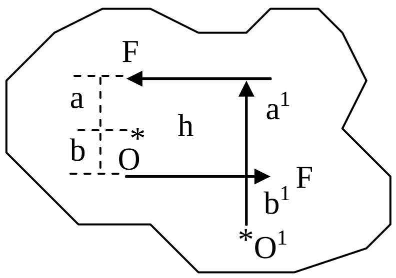
i.e, (i) $\sum F = 0$ (ii) $\sum M = 0$

or $\boxed{\begin{array}{l}\sum Fx = 0 \\ \sum Fy = 0\end{array}}$ $\sum TM = 0$ $\sum My = 0$

$\sum Fz = 0$ $\boxed{\sum Tz = 0}$ (For a planar system represented by 2D vectors)

Fx, Fy, Fz force Components along X, Y & Z axis

Similarly moments



O & O^1 are axis points

i) $Fa + Fb = F(a + b)$

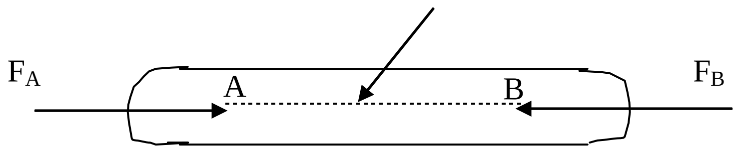
ii) $Fa^1 - Fb^1 = F(a^1 - b^1)$

(Clock wise)

"Axis point does not affect the couple"

<u>Very useful & important principles.</u>

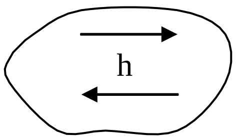
(i) *Equilibrium of a body under the action of two forces only (no torque)*



For body to the in Equilibrium under the action of 2 forces (only), the two forces must the equal opposite and collinear. The forces must be acting along the line joining A&B.

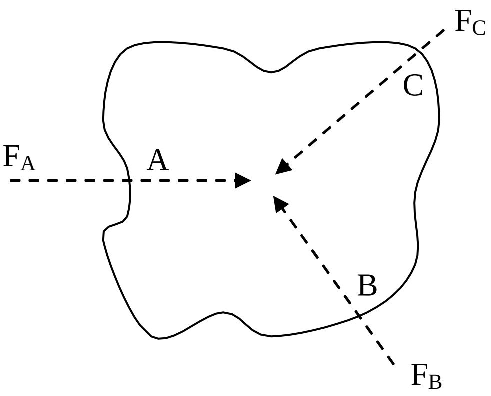
That is,

$F_A = - F_B$ (for equilibrium)



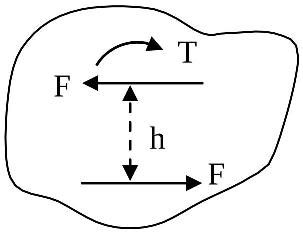
If this body is to be under equilibrium „h" should tend to zero

(ii) *Equilibrium of a body under the action of three forces only (no torque / couple)*



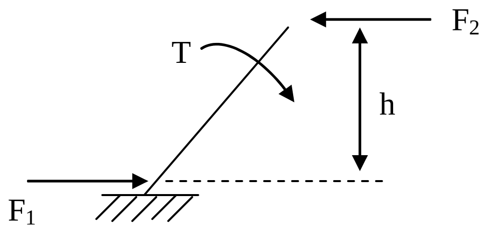
For equilibrium, the 3 forces must be concurrent and the force polygon will be a triangle.

(iii) *Equilibrium of a body acted upon by 2 forces and a torque.*



For equilibrium, the two forces must form a counter couple. Therefore the forces must be equal, opposite and parallel and their senses must be so as to oppose the couple acting on the body

<u>Example:</u>



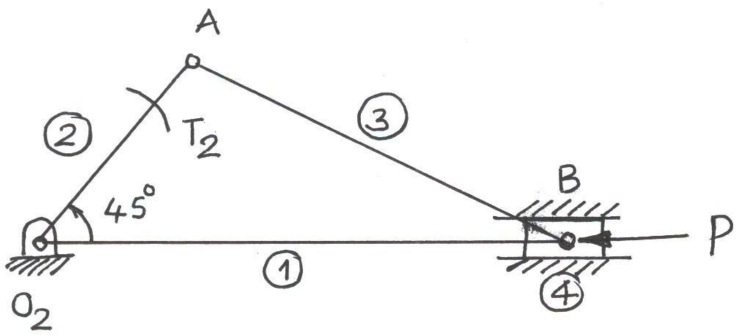
h = Perpendicular distance between F_1 & F_2

Free body diagram

The mass is separated from the system and all the forces acting on the mass are represented.

Problem No.1: Slider crank mechanism

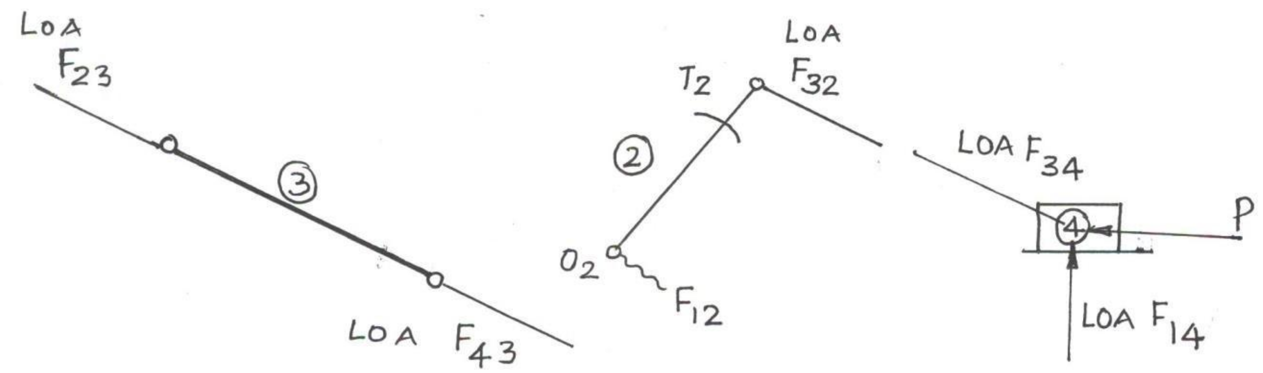
Figure shows a slider crank mechanism in which the resultant gas pressure 8 x 10^4 Nm^{-2} acts on the piston of cross sectional area 0.1 m^2. The system is kept in equilibrium as a result of the couple applied to the crank 2, through the shaft at O_2. Determine forces acting on all the links (including the pins) and the couple on 2.



$$P = (8 \times 10^4) \times (0.1)$$

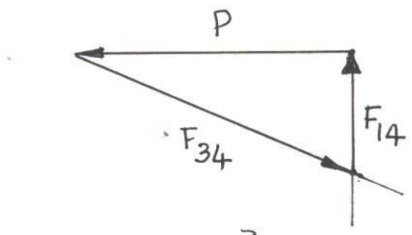
$$= 8 \times 10^3 N$$

Free body diagram



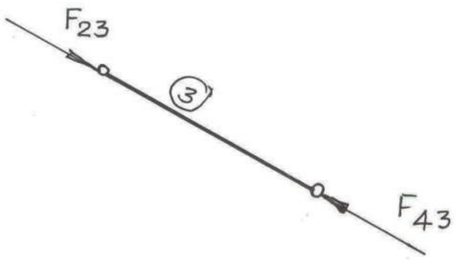
Force triangle for the forces acting on (4) is drawn to some suitable scale.

Magnitude and direction of P known and lines of action of F_{34} & F_{14} known.



Measure the lengths of vectors and multiply by the scale factor to get the magnitudes of F_{14} & F_{34}. Directions are also fixed.

$F_{34} = 8.8 \times 10^3\ N$

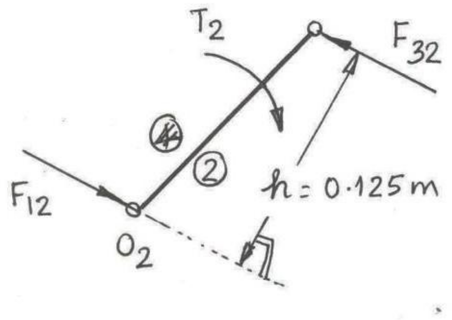


$i.e,\ F_{23} = -F_{32}$

Since link 3 is acted upon by only two forces, F_{43} and F_{23} are collinear, equal in magnitude and opposite in direction

$$i.e., \quad F_{43} = -F_{23} \quad = 8.8 \times 10^3\, N$$

Also, F_{23} = - F_{32} (equal in magnitude and opposite in direction).



Link 2 is acted upon by 2 forces and a torque (stated in the problem), for equilibrium the two forces must be equal, parallel and opposite and their sense must oppose T_2.

There fore,

$$F_{32} = -F_{12} = 8.8\times10^3\,N$$

F_{32} & F_{12} form a counter clock wise couple of magnitude,

$$\left({}_{23} \times h\right) = \left(F_{12} \times h\right) = \left(8.8\times10^3\right)\times 0.125 = 1100\,Nm.$$

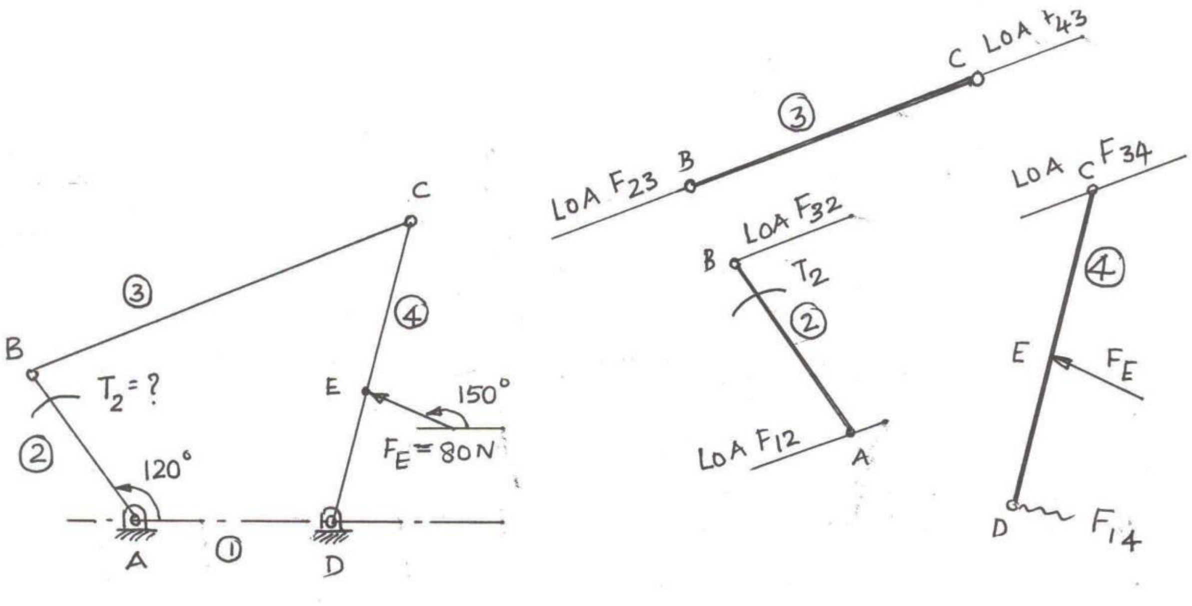
To keep 2 in equilibrium, T_2 should act clockwise and magnitude is 1100 Nm.
Important to note;

i) h is measured perpendicular to F_{32} & F_{12};
ii) always multiply back by scale factors.

Problem No 2. Four link mechanism.

A four link mechanism is acted upon by forces as shown in the figure. Determine the torque T_2 to be applied on link 2 to keep the mechanism in equilibrium.

AD=50mm, AB=40mm, BC=100mm, Dc=75mm, DE= 35mm,



Link 3 is acted upon by only two forces F_{23} & F_{43} and they must be collinear & along BC.

Link 4 is acted upon by three forces F_{14}, F_{34} & F_4 and they must be concurrent. LOA F_{34} is known and F_E completely given.

F_{32} & F_{12} from a CCW couple which is equaled by a clockwise couple T_2

Problem No 3.

Determine T_2 to keep the mechanism in equilibrium

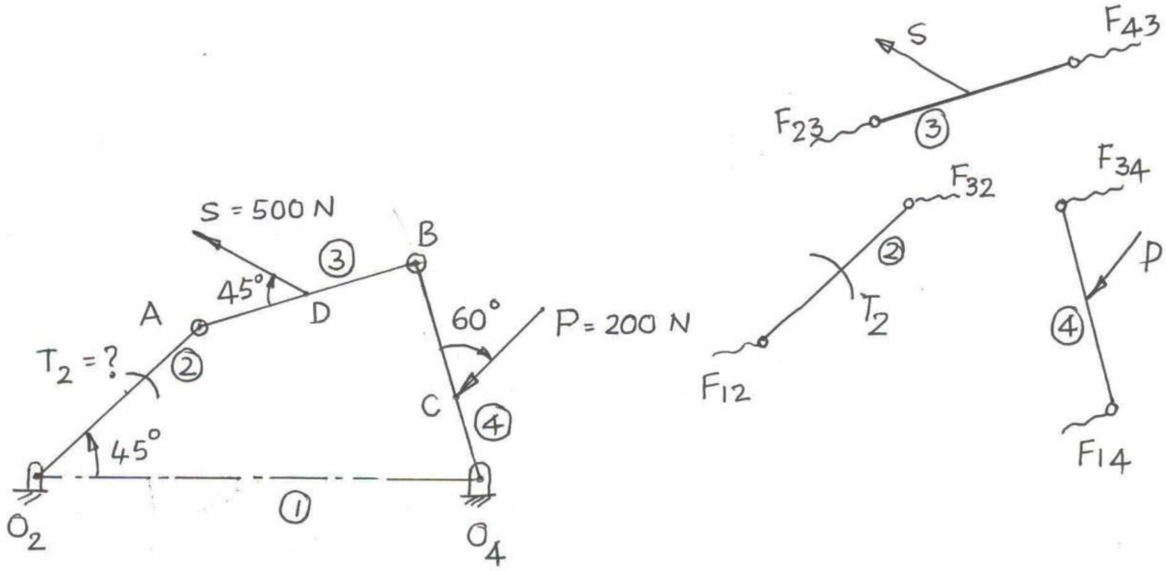
AC=70mm,
AB=150mm,
O_2A= 40mm

F_{32} and F_{12} form a CCW couple and hence T_2 acts clock wise.

Problem No 4.

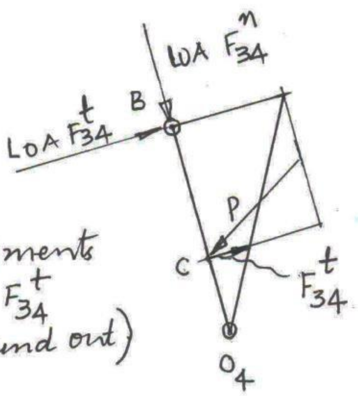
Determine the torque T_2 required to keep the given mechanism in equilibrium.

O_2A= 30mm,   = AB =O_4B, O_2O_4 =60mm, $\hat{AO_2O_4}$ = 60º, BC = 19mm, AD=15mm.

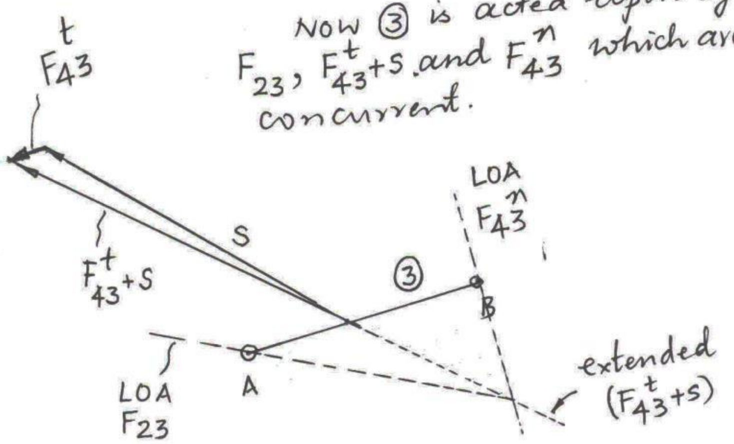


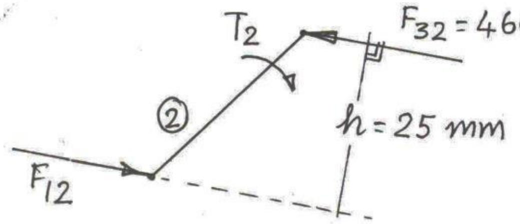
None of the links are acted upon by only 2 forces. Therefore links can‟t be analyzed individually.

(Also, taking moments about O_4, F_{34}^t can be found out)

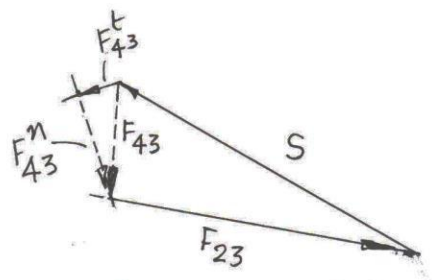


Now ③ is acted upon by F_{23}, $F_{43}^t + S$ and F_{43}^n which are concurrent.





$T_2 = 460 \times 25 = 11500$ N-mm.
(CW)



$F_{23} = 460$ N

$F_{43} = 160$ N

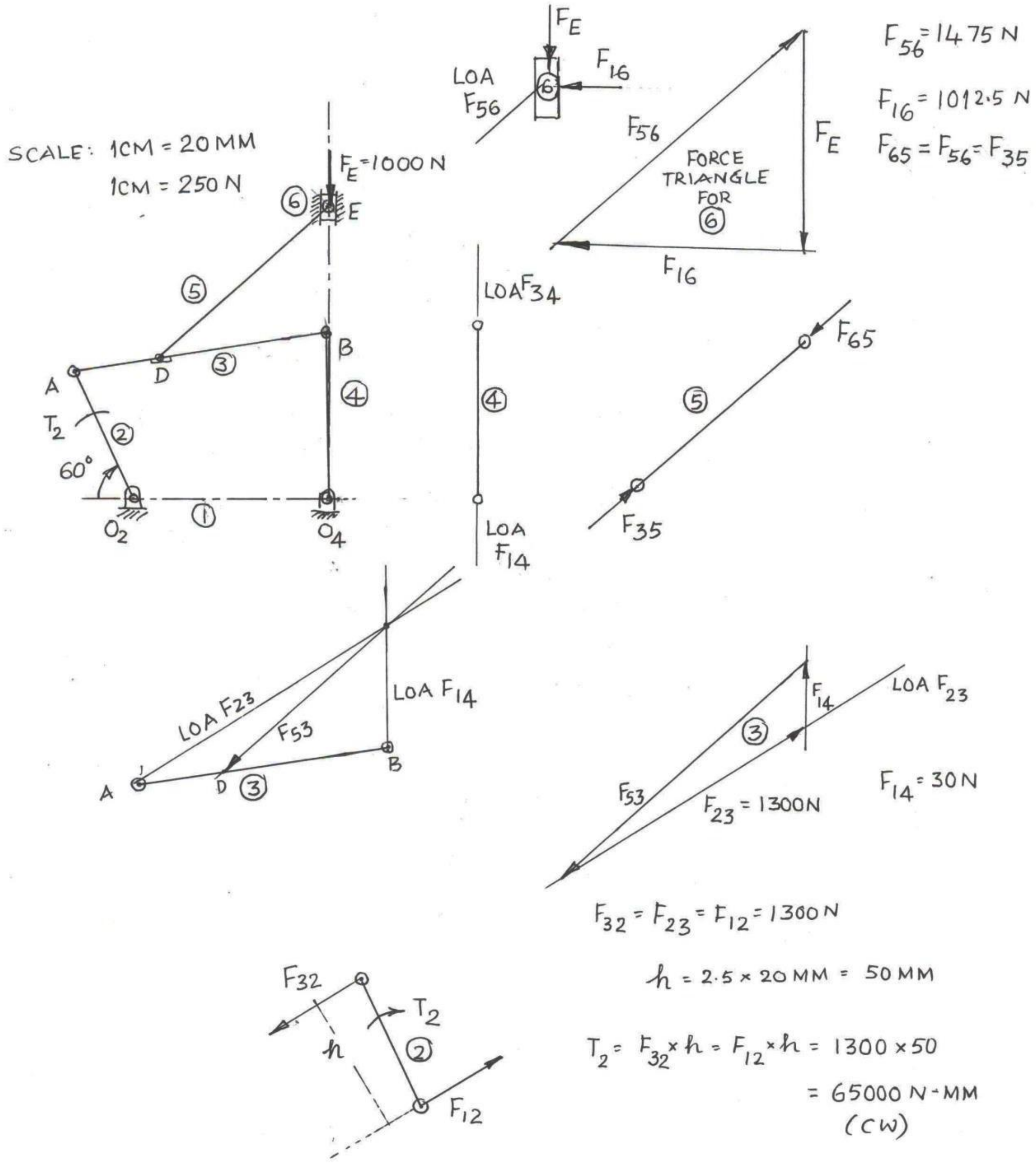
Force triangle for ③

Vector scale : 1cm = 100N

Problem No 5.

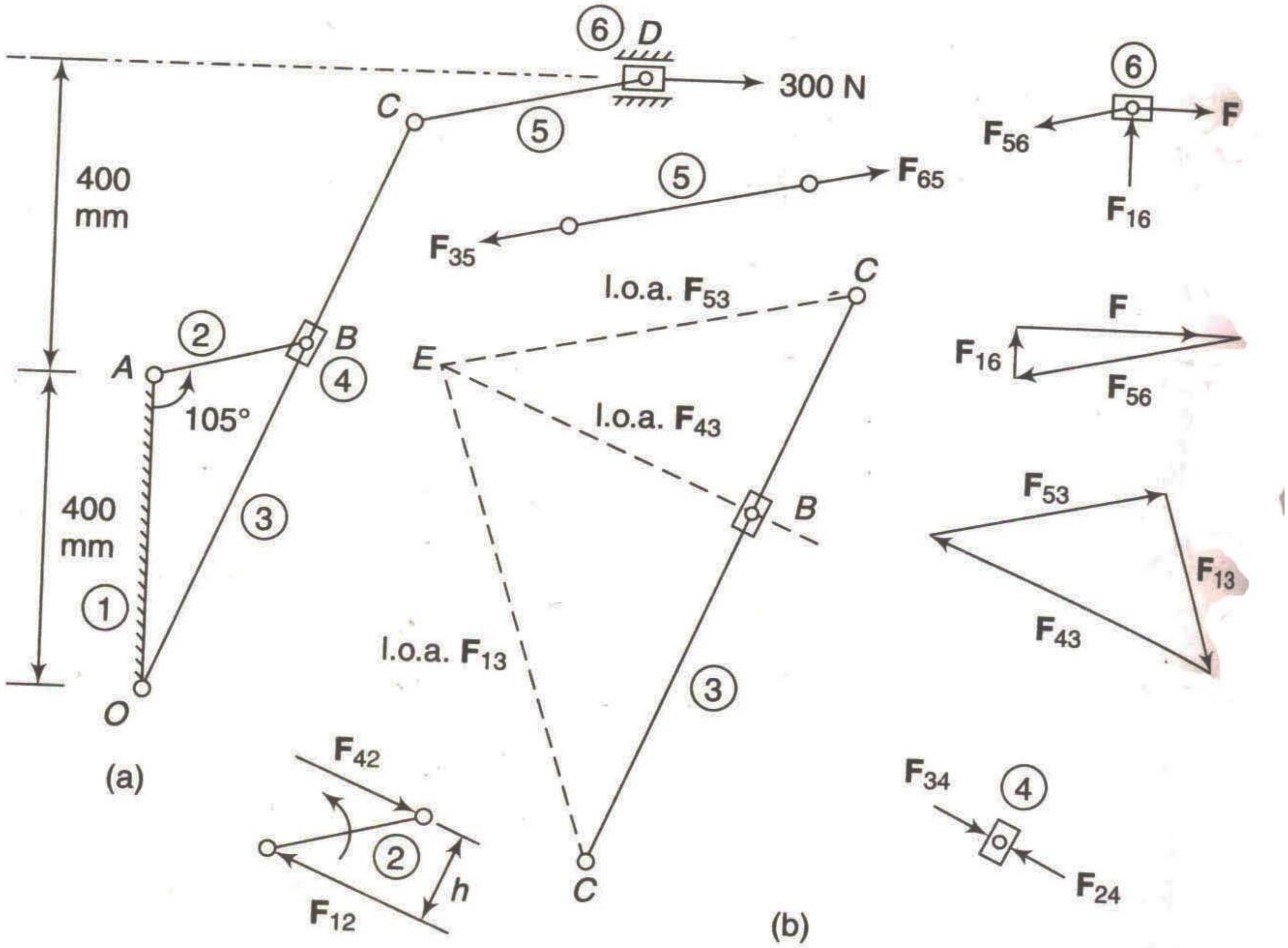
Determine the torque T_2 required to overcome the force F_E along the link 6.

AD=30mm, AB=90mm, O_4 B=60mm, DE=80mm, O_2 A=50mm, O_2 O_4 =70mm



Problem No 6

For the static equilibrium of the quick return mechanism shown in fig. 12.11 (a), determine the input torque T_2 to be applied on link AB for a force of 300N on the slider D. The dimensions of the various links are OA=400mm, AB=200mm, OC=800mm, CD=300mm



Than, torque on link 2,

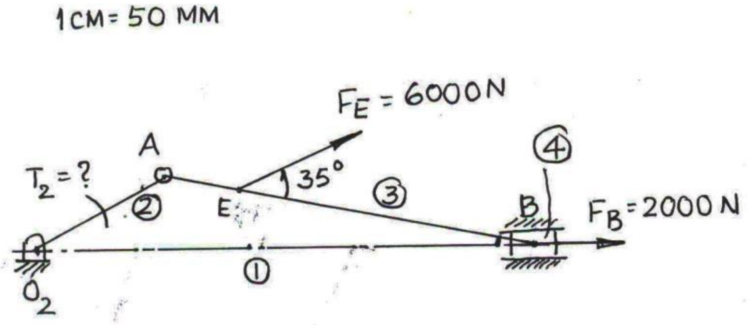
$T_2 = F_{42} x\ h = 403x120 = 48\ 360$ N counter - clockwise

Superposition method:

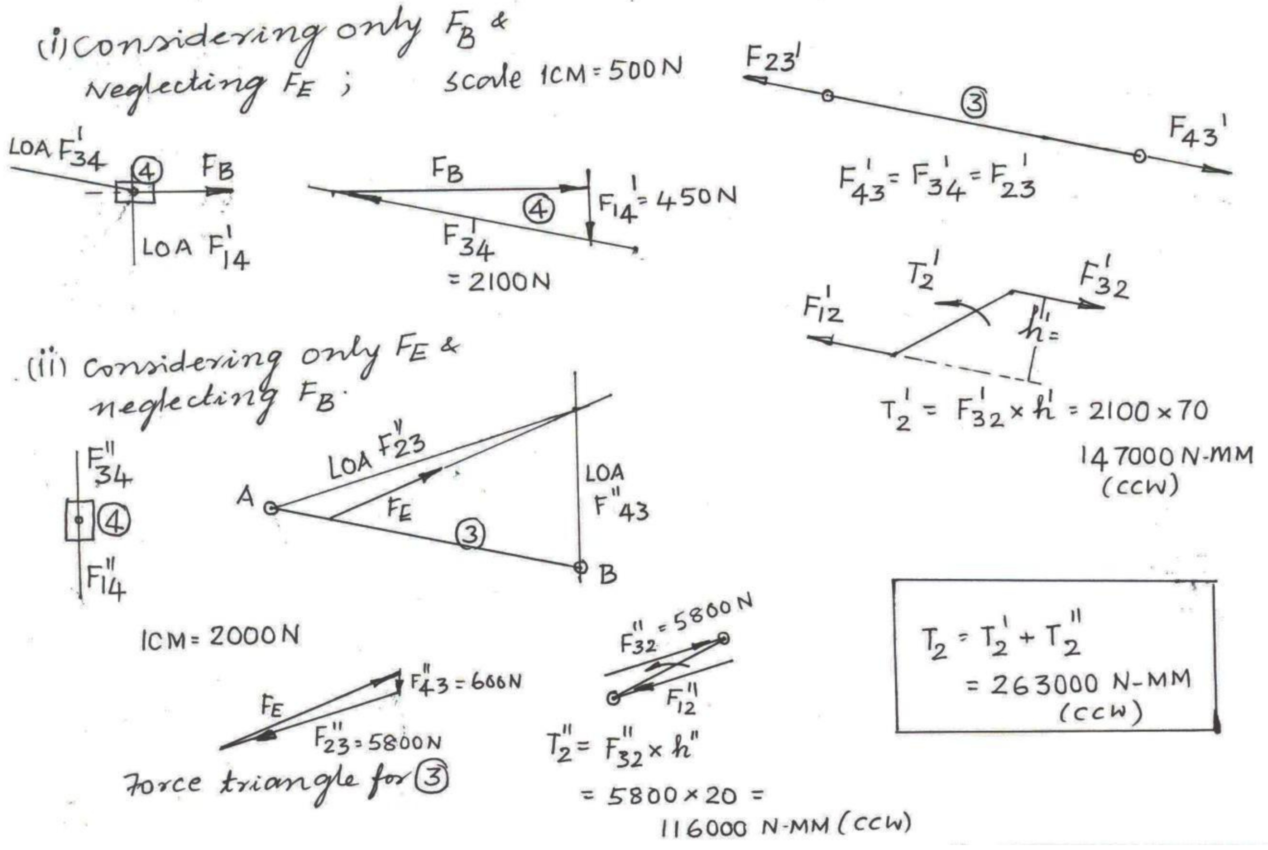
"When a number of forces (loads) act on a system (linear), the net effect is equal to the superposition of the effects of the individual forces (loads) taken one at a time" (Linear system: out put force is directly proportional to the input force)

Problem No 7. Determine T_2 to keep the body in equilibrium.

O_2A =100MM, AB=250MM, AE=50MM, $\hat{AO_2B} = 30^0$



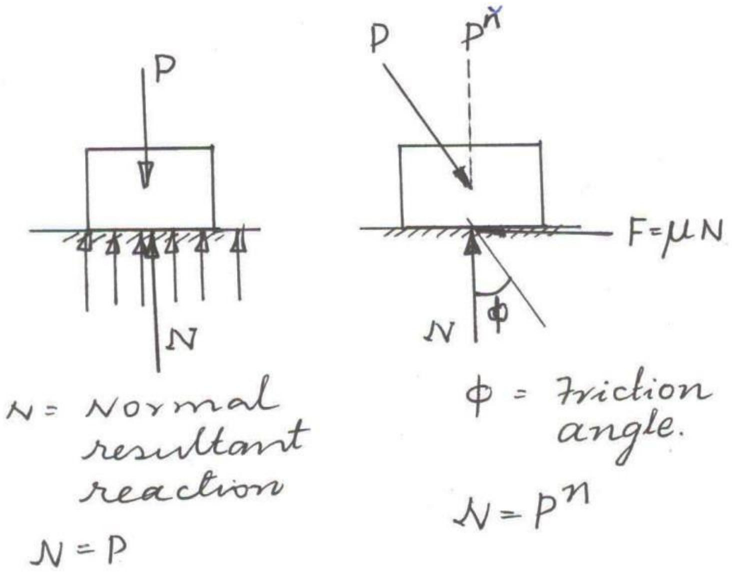
The problem is solved as two sub problems:
i) Considering only F_B
ii) Considering only F_E



Force Analysis considering friction.

If friction is considered in the analysis, the resultant force on a pin doesn"t pass through the centre of the pin. Coefficient of friction μ is assumed to the known and is independent of load and speed.

Friction in sliding member.

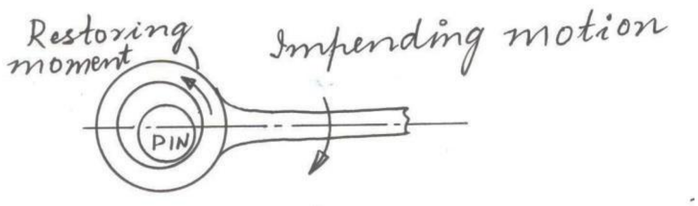


F = Frictional force

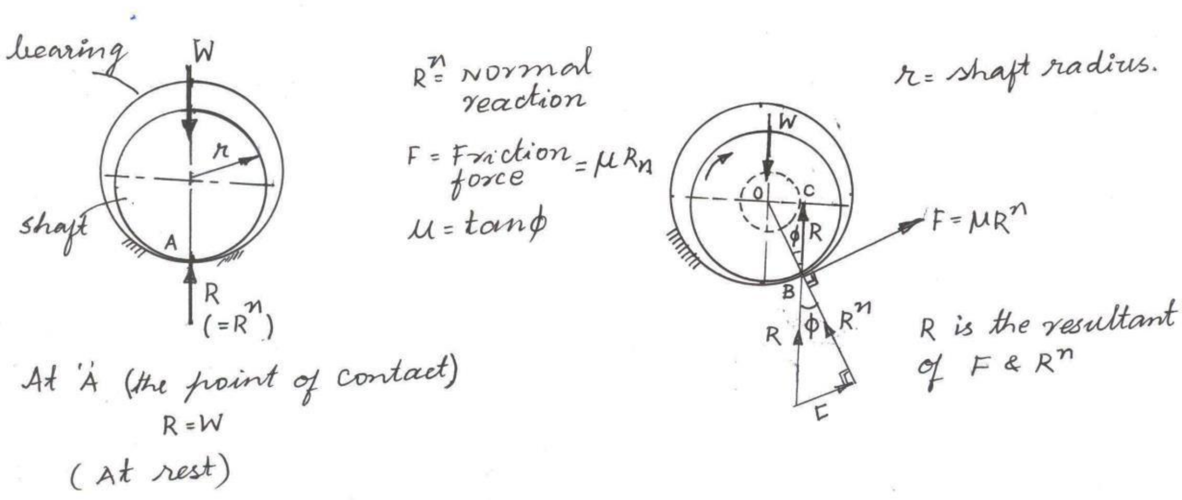
μ = coefficient of friction

$$\tan\phi = \mu = \frac{\mu N}{N}$$

Friction at pin points (bearings) & friction circle.



When a shaft revolves in a bearing, some power is lost due to friction between surfaces.



While rotating, the point of contact shifts to B; R^n passes

through B. The resultant „R‟ is in a direction opposite to ω.

The circle drawn at O, with OC as radius is called „FRICTION CIRCLE‟

For the shaft to be in equilibrium; W = R

Frictional moment M = R x OC

= W x OC

= W x r sin ϕ

= W x r tan ϕ

(sin ϕ ≈ tan ϕ, for small ϕ) i.e, M = w x r x μ

∴ Radius of the friction circle (OC) = μr.

The friction circle is used to locate the line of action of the force between the shaft (pin) and the bearing or a pin joint. The direction of the force is always be tangent to it (friction axis)

Friction axis: the new axis along which the thrust acts.

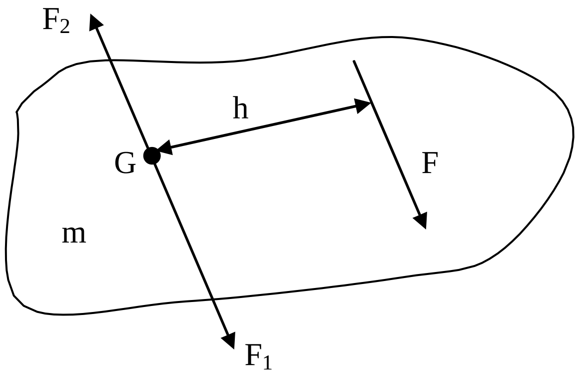
DYNAMIC FORCE ANALYSIS:

It is defined as the study of the force at the pin and guiding surfaces and the forces causing stresses in machine parts, such forces being the result of forces due to the motion of each part in the machine. The forces include both external and inertia forces. Inertia forces in high speed machines become very large and cannot be neglected, Ex: Inertia force of the piston of an automobile travelling at high speed might be thousand times the weight of the piston. The dynamic forces are associated with accelerating masses.

If each link, with its inertia force and force applied to the link can be considered to be in equilibrium, the entire system can also be considered to be in equilibrium.

Determination of force & couple of a link

(Resultant effect of a system of forces acting on a rigid body)



G = c .g point
F_1& F_2: equal and opposite forces acting through G (Parallel to F)

F: Resultant of all the forces acting on the rigid body.
h: perpendicular distance between F & G.
m = mass of the rigid body

Note: F_1=F_2 & opposite in direction; they can be cancelled with out affecting the equilibrium of the link. Thus, a single force „F" whose line of action is not through G, is capable of producing both linear & angular acceleration of CG of link.

F and F_2 form a couple.

T= F x h = I α = mk^2 α (Causes angular acceleration) ....(1)

Also, F_1 produces linear acceleration, f.

F_1= mf

Using 1 & 2, the values of „f" and „α" can be found out if F_1, m, k & h are known.

D’Alembert’s principle:

Final design takes into consideration the combined effect of both static and dynamic force systems. D‟Alembert‟s principle provides a method of converting dynamics problem into a static problem.

Statement:

The vector sum of all external forces and inertia forces acting upon a rigid body is zero. The vector sum of all external moments and the inertia torque, acting upon the rigid body is also separately zero.

In short, sum of forces in any direction and sum of their moments about any point must be zero.

Inertia force and couple:

Inertia: Tendency to resist change either from state of rest or of uniform motion

Let „R‟ be the resultant of all the external forces acting on the body, then this „R‟ will be equal to the product of mass of the body and the linear acceleration of c.g of body. The force opposing this „R‟ is the inertia force (equal in magnitude and opposite in direction).

(Inertia force is an Imaginary force equal and opposite force causing acceleration)

If the body opposes angular acceleration (α) in addition to inertia force R, at its cg, there exists an inertia couple Ig x α, Where Ig= M I about cg. The sense of this couple opposes α. i.e., inertia force and inertia couple are equal in magnitude to accelerating force and couple respectively but, they act in opposite direction.

Inertia force (F_i) = M x f,
(mass of the rigid body x linear acceleration of cg of body)

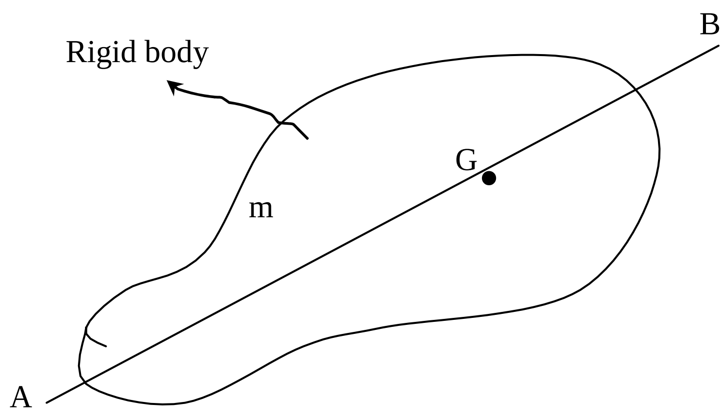
Inertia couple (C_i)=I x α,	MMI of the rigid body about an axis perpendicular to the plane of motion	Angular acceleration

Dynamic Equivalence:

The line of action of the accelerating force can also be determined by replacing the given link by a dynamically equivalent system. Two systems are said to be dynamically equivalent to one another, if by application of equal forces, equal linear and angular accelerations are produced in the two systems.

, the following conditions must be satisfied;

i) The masses of the two systems must be same.
ii) The cg‟s of the two systems must coinside.
iii) The moments of inertia of the two systems about same point must be equal, Ex: about an axis through cg.

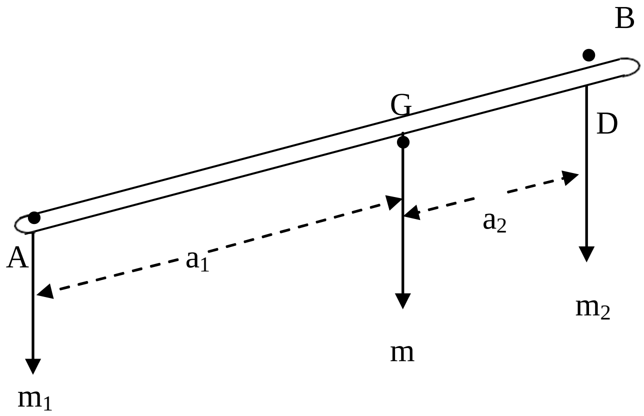


G = c.g.

m = mass of the rigid body

k_g = radius of gyration about an axis through G and perpendicular to the plane

Now, it is to be replaced by dynamically equivalent system.



m_1, m_2 – masses of dynamically equivalent system at a_1 & a_2 from G (respectively)

As per the conditions of dynamic equivalence,

$$m = m_1 + m_2 \quad .. (a)$$
$$m_1 a_1 = m_2 a_2 \quad .. (b)$$
$$m k_g^2 = m_1 a_1^2 + m_2 a_2^2 \quad .. (c)$$

Substituting (b) in (c),

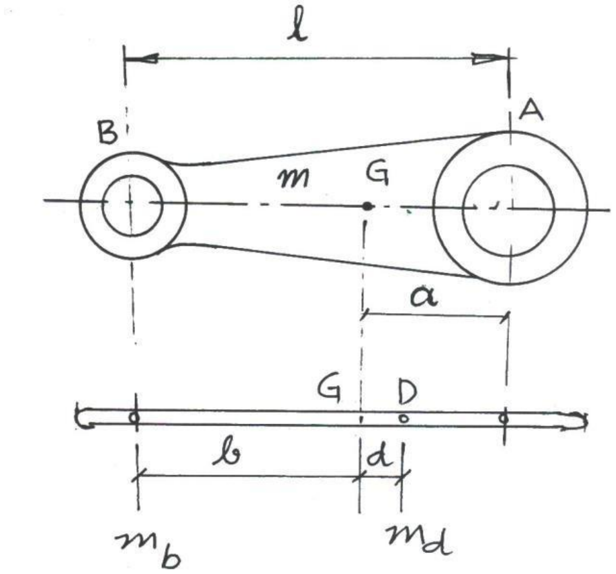
$$m k_g^2 = (m_2 a_2) a_1 + (m_1 a_1) a_2$$

$$= a_1 a_2 (m_2 + m_1) = a_1 a_2 (m)$$

i.e., $k_g^2 = a_1 a_2$ $\quad [I_g = m k_g^2 \text{ or } k_g^2 = \frac{I_g}{m}]$

or $\frac{I_g}{m} = a_1 a_2$

Inertia of the connecting rod:



Connecting rod to be replaced by a massless link with two point masses m_b & m_d.

m = Total mass of the CR m_b& m_d point masses at B& D.

$$m_b + m_d = m \qquad -- \quad (i)$$

$$m_b \times b = m_d \times d \qquad -- (ii)$$

Substituting (ii) in (i);

$$m_b + \left(m_b \times \frac{b}{d} \right) = m$$

$$m_b \left(1 + \frac{b}{d} \right) = m \qquad \text{or } m_b \left(\frac{b+d}{d} \right) = m$$

$$\text{or } m_b = m \left(\frac{d}{b+d} \right) \qquad -- (1)$$

Similarly;

$$m_d = m \left(\frac{b}{b+d} \right) \qquad -- (2)$$

Also; $I = m_b b^2 + m_d d^2$

$$= m \left(\frac{d}{b+d} \right) b^2 + m \left(\frac{b}{b+d} \right) d^2 \qquad [\,from\ (1)\ \&\ (2)]$$

$$I = mbd \left(\frac{b+d}{b+d} \right) = mbd$$

Then, $mk_g^2 = mbd,$ (since $I = mk_g^2$)

$$k_g^2 = bd$$

The result will be more useful if the 2 masses are located at the centers of bearings A & B.

Let m_a = mass at A and dist. AG = a

Then,

$$m_a + m_b = m$$

$$m_a = m\left(\frac{b}{a+b}\right) = m\frac{b}{l}\ ; \qquad Since\ (a+b=l)$$

Similarly, $$m_b = m\left(\frac{a}{a+b}\right) = m\frac{a}{l}\ ; \qquad (Since,\ a+b=l)$$

$$I^1 = m_a a^2 + m_b b^2 = \ldots \ldots = mbd$$ (Proceeding on similar lines it can be proved)

Assuming; $a > d,\ I^1 > I$

, by considering the 2 masses A & B instead of D and B, the inertia couple (torque) is increased from the actual value. i.e., there exists an error, which is corrected by applying a correction couple (opposite to the direction of applied inertia torque).

The correction couple,

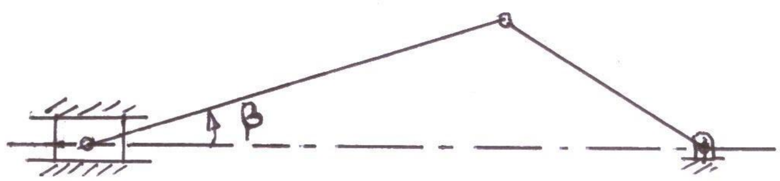
$$\Delta T = \alpha_c (mab - mbd)$$

$$= mb\,\alpha_c\ (a-d)$$

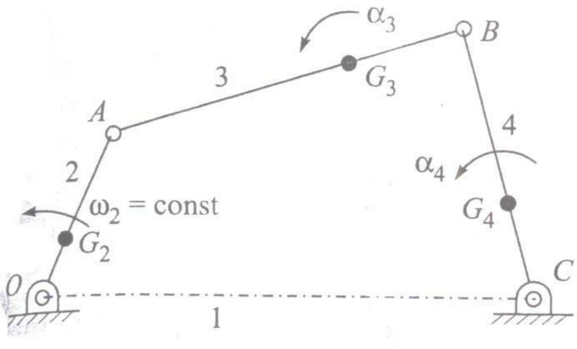
$$= mb\,\alpha_c \left[\ (a+b) - (b+d)\right]$$

$$= mb\,\alpha_c\ (l-L) \qquad because\ (b+d=L)$$

As the direction of applied inertia torque is always opposite to the direction of angular acceleration, the direction of the correction couple will be same as that of angular acceleration i.e., in the direction of decreasing angle β.



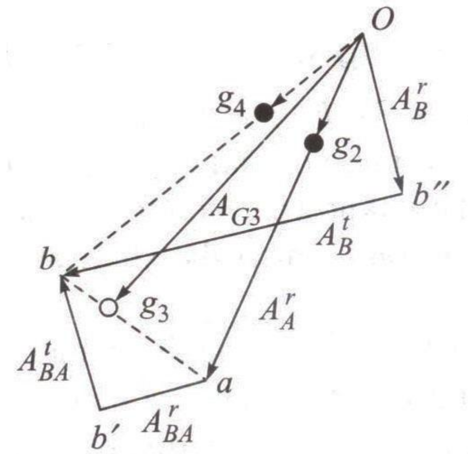
Dynamic force Analysis of a 4 – link mechanism.



OABC is a 4–bar mechanism. Link 2 rotates with constant ω_2. G_2, G_3 & G_4 are the cgs and M_1, M_2 & M_3 the masses of links 1, 2 & 3 respectively.

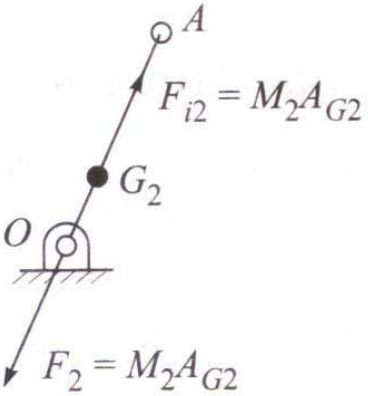
What is the torque required, which, the shaft at o must exert on link 2 to give the desired motion?

1. Draw the velocity & acceleration polygons for determing the linear acceleration of G_2, G_3 & G_4.
2. Magnitude and sense of α_3 & α_4 (angular acceleration) are determined using the results of step 1.



To determine inertia forces and couples

Link 2

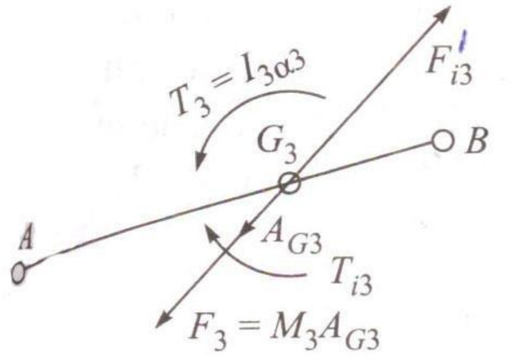


F_2 = accelerating force (towards O)

$F_{i\,2}$ = inertia force (away from O)

Since ω_2 is constant, $\alpha_2 = 0$ and no inertia torque involved.

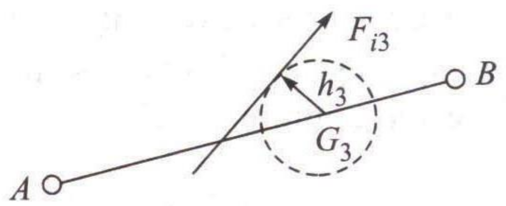
Link 3



Linear acceleration of G_3 (i.e., AG_3) is in the direction of Og_3 of acceleration polygon.

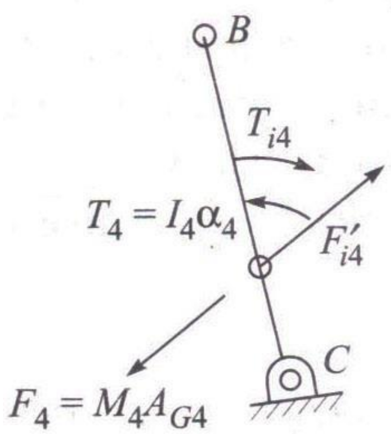
F_3 = accelerating force

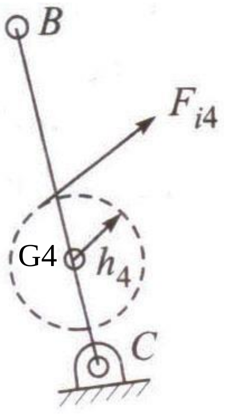
Inertia force F_{i3}' acts in opposite direction. Due to α_3, there must be a resultant torque $T_3 = I_3\ \alpha_3$ acting in the sense of α_3 (I_3 is MMI of the link about an axis through G_3, perpendicular to the plane of paper). The inertia torque T_{i3} is equal and opposite to T_3.



F_{i3} can replace the inertia force F_{i3}' and inertia torque T_{i3}. F_{i3} is tangent to circle of radius h_3 from G_3, on the top side of it so as to oppose the angular acceleration α_3. $h3 = \dfrac{I_3\alpha_3}{M_3 AG_3}$

<u>Link 4</u>





$$h4 = \frac{I_4\alpha_4}{M_4 AG_4}$$

<u>Problem 1 :</u>

It is required to carryout dynamic force analysis of the four bar mechanism shown in the figure.

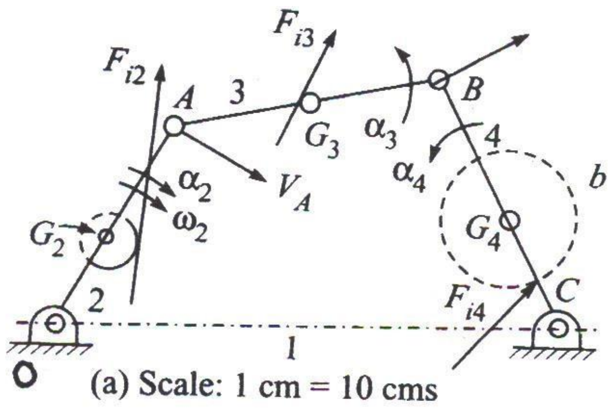
ω_2 = 20rad /s (cw), α_2 = 160 rad/s^2 (cw)

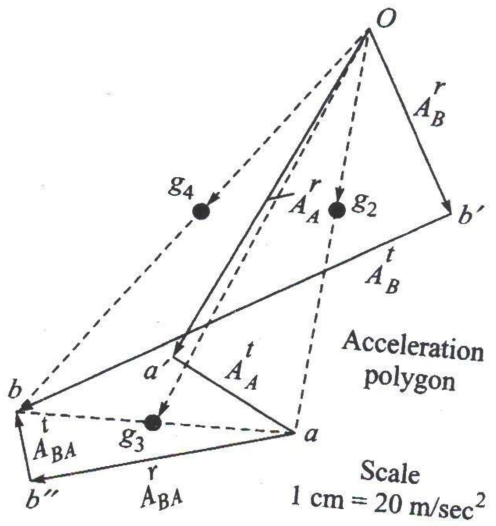
OA= 250mm, OG_2= 110mm, AB=300mm, AG_3=150mm, BC=300mm, CG_4=140mm, OC=550mm, $\angle AOC = 60°$

The masses & MMI of the various members are

Link	Mass, m	MMI (I_G, Kgm^2)
2	20.7kg	0.01872
3	9.66kg	0.01105
4	23.47kg	0.0277

Determine i) the inertia forces of the moving members
ii) Torque which must be applied to 2





A) Inertia forces:

(i) (from velocity & acceleration analysis)

$V_A = 250 \times 20;\ 5m/s, \quad V_B = 4\ m/s, \quad V_{BA} = 4.75\ m/s$

$a_A^r = 250 \times 20^2;\ 100 m/s^2, \quad a_A^t = 250 \times 160;\ 40 m/s^2$

Therefore;

$$A_B^r = \frac{V_B^2}{CB} = \frac{(4)^2}{0.3} = 53.33\ m/s^2$$

$$A_{BA}^r = \frac{V_{BA}^2}{B_A} = \frac{(4.75)^2}{0.3} = 75.21\ m/s^2$$

$$Og_2 = A_{G2} = 48\ m/s^2;\ Og_3 = AG_3 = 120 m/s^2$$

$$Og_4 = A_{G4} = 65.4\ m/s^2$$

$$\alpha_3 = \frac{A_{BA}^t}{AB} = \frac{19}{0.3} = 63.3\ rad/s^2$$

$$\alpha_4 = \frac{A_B^t}{CB} = \frac{129}{0.3} = 430\ rad/s^2$$

Inertia forces (accelerating forces)

$$F_{G2} = m_2\ A_{G2} = \frac{20.7}{9.81} \times 48 = 993.6\ N\ (\ in\ the direction\ of\ Og_2\ )$$

$$F_{G3} = m_3\ A_{G3} = 9.66 \times 120 = 1159.2\ N\ (\ in\ the\ direction\ of\ Og_3\ )$$

$$= F_{G4} = m_4 A_{G4} = 23.47 \times 65.4 = 1534.94\ N\ (in\ the direction\ of\ Og_4)$$

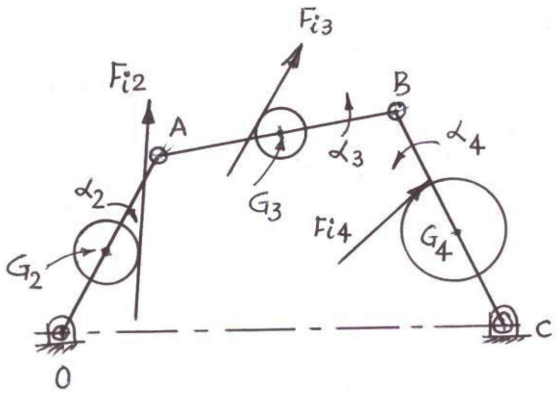
$$h_2 = \frac{I_{G2}(\alpha_2)}{F_2} = \frac{(0.01872 \times 160)}{993.6} = 3.01 \times 10^{-3}\, m$$

$$h_3 = \frac{I_{G3}(\alpha_3)}{F_3} = \frac{(0.01105 \times 63.3)}{1159.2} = 6.03 \times 10^{-4}\, m$$

$$h_4 = \frac{I_{G4}(\alpha_4)}{F_4} = \frac{(0.0277 \times 430)}{1534.94} = 7.76 \times 10^{-3}\ m$$

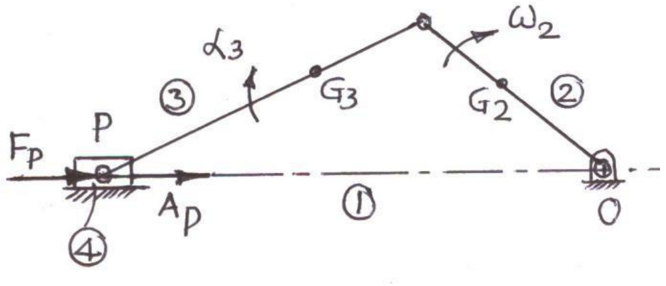
The inertia force F_{i2}, F_{i3} & F_{i4} have magnitudes equal and direction opposite to the respective accelerating forces and will be tangents to the circles of radius h_2, h_3 & h_4 from G_2, G_3 & G_4 so as to oppose α_2, α_3 & α_4.

$F_{i2} = 993.6\ N$, $F_{i3} = 1159.2\ N$ $F_{i4} = 1534.94N$



Further, each of the links is analysed for static equilibrium under the action of all external force on that link plus the inertia force.

<u>Dynamic force analysis of a slider crank mechanism.</u>



F_p = load on the piston

Link	mass	MMI
2	m_2	I_2
3	m_3	I_3
4	m_4	-

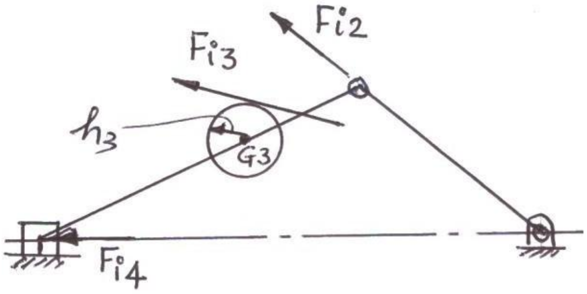
ω_2 assumed to be constant

Steps involved:

1. Draw velocity & acceleration diagrams
2. Consider links 3 & 4 together and single FBD written (elimination F_{34} & F_{43})
3. Since, weights of links are smaller compared to inertia forces, they are neglected unless specified.
4. Accelerating forces F_2 , F_3 & F_4 act in the directions of respective acceleration vectors Og_2, Og_3 & Og_p

Magnitudes: $F_2 = m_2 AG_2$ $\quad F_3 = m_3 AG_3$ $\quad F_4 = m_4 A_p$

$\boldsymbol{F_{i2} = F_2 ,\ F_{i3} = F_3 ,\ F_{i4} = F_4}$ (Opposite in direction)



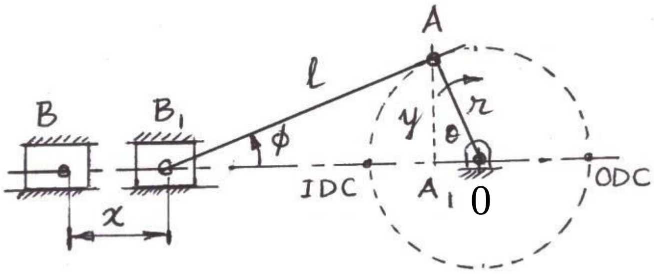
$$\boldsymbol{h_3 = \frac{I_3 \alpha_3}{M_3 \alpha_{g_3}}}$$

F_{i3} is tangent to the circle with h_3 radius on the RHS to oppose α_3

Solve for T_2 by solving the configuration for both static & inertia forces.

Dynamic Analysis of slider crank mechanism (Analytical approach)

Displacement of piston



x = displacement from IDC

$$x = BB_1 = BO - B_1O$$
$$= BO - (B_1A_1 + A_1O)$$
$$= (l + r) - (l\cos\phi + r\cos\theta)$$

$$\left(\text{since}, \frac{l}{r} = n\right)$$

$$= (nr + r) - (rn\cos\phi + r\cos\theta)$$

$$= r\big[(n+1) - (n\cos\phi + \cos\theta)\big]$$

$$\cos\phi = \sqrt{1 - \sin^2\phi}$$

$$= r\left[(n+1) - (\sqrt{n^2 - \sin^2\theta} + \cos\theta)\right]$$

$$= \sqrt{1 - \frac{y^2}{l^2}}$$

$$= r\left[(1-\cos\theta) + (n - \sqrt{n^2 - \sin^2\theta})\right]$$

$$= \sqrt{1 - \frac{(r\sin\theta)^2}{l^2}}$$

(*similary* $l >> r, \frac{l}{r} = n >> 1$ & max *valueof* $\sin\theta = 1$)

$$= \sqrt{1 - \frac{\sin^2\theta}{n^2}}$$

$\therefore \sqrt{n^2 - \sin^2\theta} \rightarrow \sqrt{n^2}$ *or* n),

$$\boxed{x = r\,(1-\cos\theta)}$$

$$= \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

This represents SHM and therefore Piston executes SHM.

Velocity of Piston:

$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\ \frac{d\theta}{dt}$$

$$\frac{d}{d\theta}\left[r\,(1-\cos\theta) + n - (n^2 - \sin 2\theta)^{-\frac{1}{2}}\right]\frac{d\theta}{dt}$$

$$= r\left[0 + \sin\theta + 0 - \frac{1}{2}(n^2 - \sin 2\theta)^{-1/2}(-2\sin\theta\cos\theta)\right]\omega$$

$$= r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right]$$

Since, $n^2 >> \sin^2\theta$,

$$\therefore\ v = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]$$

Since n is quite large, $\frac{\sin 2\theta}{2n}$ can be neglected.

$$\boxed{\therefore\ v = r\omega\sin\theta}$$

Acceleration of piston:

$$a = \frac{dv}{dt} = \frac{dv}{d\theta}\,\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left[r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)\right]\omega$$

$$= r\omega\left[\cos\theta + \frac{2\cos 2\theta}{2n}\right]$$

$$= r\omega\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

If n is very large;

$$\boxed{a = r\omega^2\cos\theta}$$ **(as in SHM)**

When θ = 0, at IDC,

$$a = r\omega^2\left(1 + \frac{1}{n}\right)$$

When θ = 180, at 0DC,

$$a = r\omega^2\left(-1 + \frac{1}{n}\right)$$

At θ = 180, when the direction is reversed,

$$a = r\omega^2\left(1 - \frac{1}{n}\right)$$

Angular velocity & angular acceleration of CR (α_c)

$$y = l\sin\phi = r\sin\theta$$

$$\sin\phi = \frac{\sin\theta}{n}$$

Differentiating w.r.t time,

$$\cos\phi\,\frac{d\phi}{dt} = \frac{1}{n}\cos\theta\,\frac{d\theta}{dt} \qquad \frac{d\phi}{dt} = \omega_c$$

$$\omega_c = \omega\frac{\cos\theta}{n\frac{1}{n}\sqrt{n^2 - \sin^2\theta}} \qquad \frac{d\theta}{dt} = \omega$$

$$\cos\phi = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$$\omega_c = \omega \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

$$\alpha_c = \frac{d\omega_c}{dt} = \frac{d\omega_c}{d\theta}\frac{d\theta}{dt}$$

$$= \omega \frac{d}{d\theta}\left[\cos\theta\,(n^2 - \sin^2\theta)^{-\frac{1}{2}}\right]\omega$$

$$= \omega^2\left[\cos\theta\,\frac{1}{2}(n^2 - \sin^2\theta)^{-\frac{3}{2}}(-2\sin\theta\cos\theta) + (n^2 - \sin^2\theta)^{-\frac{1}{2}}(-\sin\theta)\right]$$

$$= \omega^2 \sin^2\theta\left[\frac{\cos^2\theta - (n^2 - \sin^2\theta)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}}\right]$$

$$= -\omega^2 \sin\theta\left[\frac{(n^2 - 1)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}}\right]$$

Negative sign indicates that, ϕ reduces (in the case, the angular acceleration of CR is CW)

Engine force Analysis:

Forces acting on the engine are weight of reciprocating masses & CR, gas forces, Friction & inertia forces (due to acceleration & retardation of engine elements)

i) Piston effort (effective driving force)

- Net or effective force applied on the piston.

In reciprocating engine:

The reciprocating parts (masses) accelerate during the first half of the stroke and the inertia forces tend to resist the same. Thus, the net force on the piston is reduced. During the later half of the stroke, the reciprocating masses decelerate and the inertia forces oppose this deceleration or acts in the direction of applied gas pressure and thus effective force on piston is increased.

In vertical engine, the weights of the reciprocating masses assist the piston during out stroke (down) this in creasing the piston effort by an amount equal to the weight of the piston. During the in stroke (up) piston effect is decreased by the same amount.

Force on the piston due to gas pressure; $F_P = P_1 A_1 - P_2$

P_1 = Pressure on the cover end, P_2 = Pressure on the rod
A_1 = area of cover end, A_2 = area of rod end, m = mass of the reciprocating parts.

Inertia force (F_i) = m a

$$= m.r\omega^2\left(Cos\theta + \frac{Cos2\theta}{n}\right)$$ (Opposite to acceleration of piston)

Force on the piston F = F_P - F_i

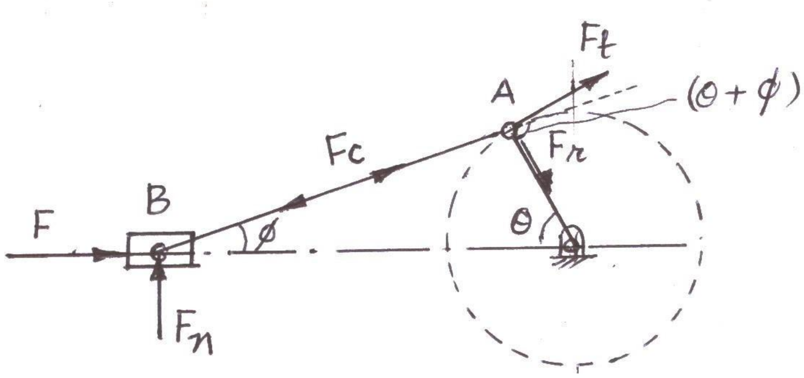
(if F_f frictional resistance is also considered)

F = F_P – F_i – F_i

In case of vertical engine, weight of the piston or reciprocating parts also acts as force.

$\therefore$ F = F_P + mg – F_i – F_i

ii) Force (Thrust on the CR)



F_c = force on the CR

Equating the horizontal components;

$$F_c\ Cos\phi = F\ \ or\ \ F_c\ \frac{F}{Cos^2\phi}$$

iii) Thrust on the sides of the cylinder

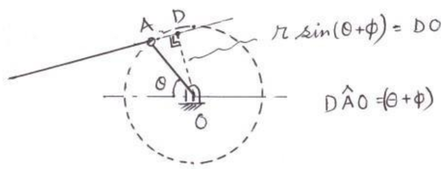
It is the normal reaction on the cylinder walls

$$F_n = F_c \sin\phi = F\tan\phi$$

iv) Crank effort (T)

It is the net force applied at the crank pin perpendicular to the crank which gives the required TM on the crank shaft.

$$F_t \times r = F_c\, r\sin(\theta+\phi)$$



$$F_t = F_c\sin(\theta+\phi)$$

$$= \frac{F}{\cos\phi}\sin(\theta+\phi)$$

v) Thrust on bearings (F_r)

The component of F_C along the crank (radial) produces thrust on bearings

$$F_r = F_c \quad Cos(\theta + \phi) = \frac{F}{Cos\ \phi}\ Cos(\theta + \phi)$$

vi) Turning moment of Crank shaft

$$T = F_t \times r$$

$$= \frac{F}{\cos\phi}\sin(\theta+\phi) \times r \;=\; \frac{F_r}{\cos\phi}\ (\sin\ \theta + \cos\phi + \cos\theta \sin\ \phi)$$

$$= F \times r \left(\sin\theta + \cos\theta\ \frac{\sin\ \phi}{\cos\phi} \right)$$

$$= F \times r \left(\sin\ \theta + \cos\theta\ \frac{\sin\theta}{n}\ \frac{1}{\frac{1}{n}\sqrt{n^2 - \sin^2\theta}} \right)$$

Proved earlier

$$\cos\phi = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$$\sin\ \phi = \frac{\sin\ \theta}{n}$$

$$= F \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$

Also,

$$r\ \sin(\theta + \phi) = OD\ \cos\phi$$

$$T = F_t \times r$$

$$= \frac{F}{\cos\phi} .\ r\ \sin\ (\theta + \phi)$$

$$= \frac{F}{\cos\phi} .\ OD\ \ \cos\phi$$

$$T = F \times OD.$$

TURNING MOMENT DIAGRAMS AND FLYWHEEL

Introduction:

A flywheel is nothing but a rotating mass which is used as an energy reservoir in a machine which absorbs the energy when the speed in more and releases the energy when the speed is less, thus maintaining the fluctuation of speed within prescribed limits. The kinetic energy of a rotating body is given as ½ I_0 ω^2, where I_0 is the mass moment of inertia of the body about the axis of rotation and ω is the angular speed of rotation. If the speed should decrease; energy will be given up by the flywheel, and, conversely, if the speed should increase energy will be stored up in the flywheel.

There are two types of machines which benefit from the action of a flywheel. The first type is a punch press, where the punching operation is intermittent Energy is required in spurts and then only during the actual punching operation. This energy can be provided in the following two ways: (i) with a large motor which is capable of providing the energy when required; or (ii) with a small motor and a flywheel, where the small motor may provide the energy to a flywheel gradually during the time when the punching operation is not being carried out. The latter method would definitely be the cheaper and would provide for less sudden drain of power from the power lines to the motor, which is very desirable.

The second type is a steam engine or an internal combustion engine, where energy is supplied to the machine at a non-uniform rate and withdrawn from the engine at nearly a constant rate. Under such a condition, the output shaft varies in speed. The speed increases where there is an excess of supplied energy; and the speed decreases where there is a deficiency of energy. The use of a flywheel would allow the engine to operate with a minimum speed variation because it would act as a reservoir for absorbing the excess energy; during the period when an excess of energy was being supplied, to be redistributed when the energy supplied was not sufficient for the load on the engine. It is evident that, it is not possible to obtain an absolutely uniform speed of rotation of the output shaft if the power is supplied at a variable rate even with a flywheel because a change of speed of the flywheel is necessary to permit redistribution of the energy. However, for a given change of energy in the flywheel, the speed variation may be made very small by using a large mass. Practically, there is no need of using masses any larger than necessary for the proper operation of a given machine. The analysis is aimed to determine the size of flywheel necessary.

Difference between Governor and Flywheel:

A governor controls the speed of the output shaft within close limits, but its action depends upon controlling the amount of working fluid to the engine as required by the load on the engine. The flywheel, on the other hand, serves only to smooth out the energy transfer in each energy cycle. For example, if an engine is operating at quarter load, with the governor in a particular position controlling the amount of working media to the engine; the flywheel would take care of redistributing the energy throughout a cycle. If the load was increased to full load the governor action would permit more working fluid to the engine maintaining the speed of the engine, but when balance of working fluid to the engine and load on the engine was reached, the flywheel would continue its action of redistributing the energy throughout a cycle. Changes of seed in an engine will cause the governor to respond and attempt to do the flywheels job. Usually, the effect of the governor is disregarded in the design of the flywheel. The flywheel analysis is limited to engines receiving power at a variable rate and delivering it to a shaft at an approximately constant rate.

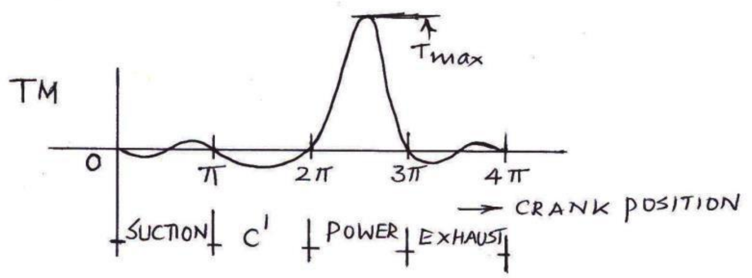
Crank effort diagrams or Turing moment diagrams:

It is the graphical representation of turning moment or crank effort for the various positions of the crank. The TM is plotted on „y‟ axis and crank angle on „x‟ axis.

Uses of turning moment Diagram

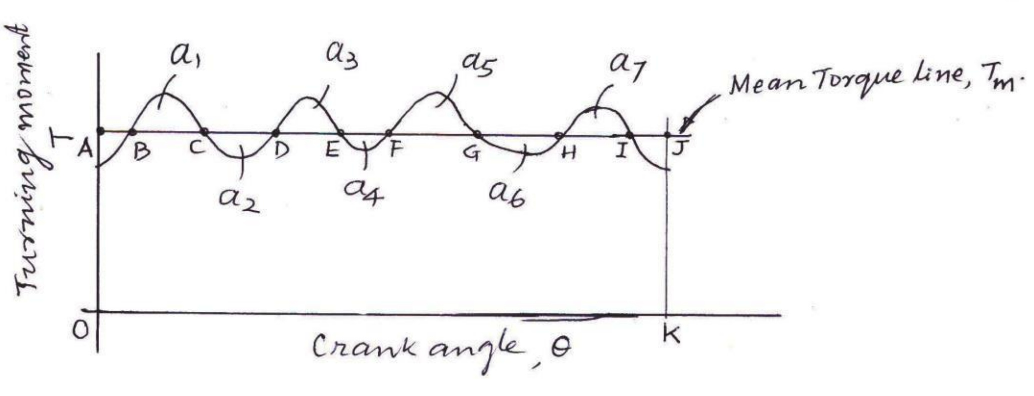
1) The area under the turning moment diagram represents work done per cycle. The area multiplied by number of cycles per second gives the power developed by the engine.
2) By dividing the area of the turning moment diagram with the length of the base we get the mean turning moment. This enables us the find the fluctuation of energy.
3) The max. ordinate of the turning moment diagram gives the maximum torque to which the crank shaft is subjected. This enables us the find diameter of the crank shaft.

TMD for a four stroke I.C. Engine



We know that four stroke cycle internal combustion engine there is one working stroke after the crank has turned through two revolutions (4π or 720°). Since the pressure inside the engine cylinder is less than the atmospheric pressure during suction stroke therefore a negative loop is formed as shown in figure. During compression stroke the work done on engine the gases therefore a higher negative loop is obtained. During expansion or working stroke the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases; therefore a negative loop is firmed.

Fluctuation of energy



The fluctuation of the energy is the excess energy developed by the engine between two crank position or difference between maximum and minimum energies is known as fluctuation of energy. TMD for a multi cylinder engine is as shown in figure. The horizontal line AG represents mean torque line. Let a_1, a_3, a_5 be the areas above the mean torque line a_2, a_4 & a_6 be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving part of the engine.

Let the energy in the fly wheel at $A = E$
Energy at $B = E + a_1$
Energy at $C = E + a_1 - a_2$
Energy at $D = E + a_1 - a_2 + a_3$
Energy at $E = E + a_1 - a_2 + a_3 - a_4$
Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$
Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

Suppose greatest of these energies is at B and least at E,

Maximum energy in the fly wheel $= E + a_1$
Minimum energy in the fly wheel $= E + a_1 - a_2 + a_3 - a_4$
$\therefore$ Maximum fluctuation of energy (ΔE) = max. energy – min. energy

$$\Delta E = (E + a_1) - (E + a_1 - a_2 + a_3 - a_4)$$
$$\Delta E = a_2 - a_3 + a_4$$

Co-efficient of fluctuation of energy

It may be defined as the ratio of maximum fluctuation of energy to the work done per cycle:

Co-efficient of fluctuation of energy $= \dfrac{\Delta E}{W.D/Cycle}$

$$W.D / Cycle = \frac{P \times 60}{n}$$

Where P= power transmitted
n = number of working strokes/ minute

Fluctuation of energy and speed in Terms of Torques:

The driving torque T produced by an engine (crank effort) fluctuates during any one cycle, the manner in which it varies depending on the type of engine, number of cylinders, etc. It can usually be assumed that, the resisting torque due to the load T_m is constant, and when $T > T_m$ the engine will be accelerating, and vice versa. If there are N complete cycles per minute and n rpm, then the engine power is given by

$$\text{Power} = N \int T \, d\theta = 2\pi n \, T_m$$

T_m = mean height of turning moment diagram. For any period during which $T > T_m$, the area cut off on the turning moment diagram represents "excess energy", which goes to increase the speed of the rotating parts, i.e., excess energy,

$$\Delta E = \int (T - T_m)\, d\theta = \frac{1}{2} I_0 \, (\omega_{\max}^2 - \omega_{\min}^2) \text{ same as before.}$$

In simple cases, ΔE is given by the area of one "loop" intercepted between T and T_m but for a multi cylinder engine a further analysis is necessary.

Coefficient of Fluctuation of speed:

The coefficient of fluctuation of speed is defined as

$$\delta = \frac{\omega_{\max} - \omega_{\min}}{\omega_{mean}}$$

Where $\omega_{\max}$ = max. angular speed of the flywheel

$\omega_{\min}$ = min. angular speed of the flywheel

ω_{mean} = average angular speed of the flywheel

or

$$\delta = \frac{V_{\max} - V_{\min}}{V_{mean}}$$

The maximum permissible coefficients for different applications are as follows:

δ = 0.2 for pumps, crushing machines

= 0.003 for alternating current generators.

In general, δ varies between the above values for all machines.

Weight of a flywheel for given value of δ:

The kinetic energy (K.E.) of a body rotating about a fixed centre is,

$$\text{K.E} = \frac{1}{2} I_0 \, \omega_{mean}^2$$

The maximum fluctuation of K.E ΔE is given by

$$\Delta E = \frac{1}{2} I_0 \, (\omega_{\max}^2 - \omega_{\min}^2)$$

Multiply and divide by r^2 on the right hand side, we have

$$\Delta E = \frac{I_0}{2r^2}\,(r\omega_{\max})^2 - (r\omega_{\min})^2$$

Where r is the mean radius of the flywheel rim.

$$\therefore\ \Delta E = \frac{I_0}{2r^2}\ (V_{\max}^2 - V_{\min}^2) = \frac{I_0}{2r^2}\ (V_{\max} + V_{\min})\ (V_{\max} - V_{\min})$$

but, $V_{mean} = \frac{V_{\max} + V_{\min}}{2}$ *and* $\delta = \frac{V_{\max} - V_{\min}}{V_{mean}}$

we have, $\Delta E = \frac{I_0}{2r^2}\ (2V_{mean})(V_{mean}\delta) = \frac{I_0 V_{mean}^2 \delta}{r^2} = \frac{(mk^2)V_{mean}^2 \delta}{r^2} = mk^2\omega^2\delta$

It is usual practice, in flywheel analysis, to consider the mass of the flywheel concentrated at the mean radius of the rim, and to make corrections later for the fact that the arms and hubs contributed to the flywheel effect. That is, k is assumed to be equal to *r*, the mean radius of the rum.

$$\therefore\ \Delta E = (m\ V_{mean}^2\ \delta) = mr^2\ \omega^2\ \delta$$

or the mass of the flywheel, $m = \frac{(\Delta E)}{V_{mean}^2 \delta}$

$$m = \frac{2\ (\Delta E)}{V_{mean}^2\ \delta} \quad \text{for flat circular plate.}$$

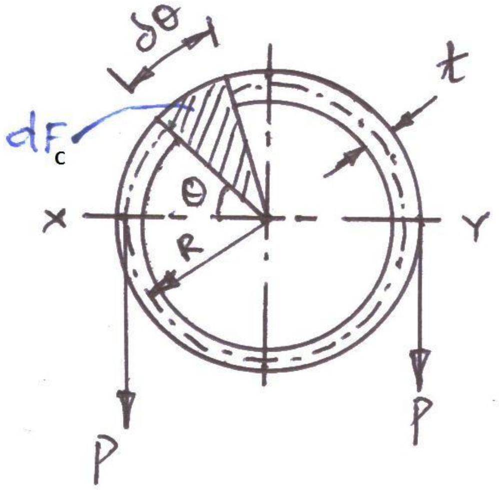
For a solid disc of diameter D, $k^2 = \frac{D^2}{8}$ and for ring or rim of diameters D and d, $k^2 = \left(\frac{D^2 + d^2}{8}\right)$

Notice that, as a result of the above assumption, the actual mass of the rim of the flywheel may be taken as approximately 10% less than that calculated by the above formula to allow for the effect of the arms and hub of the flywheel and other rotating parts, which is sufficient for the usual designs encountered.

For a given engine with a flywheel of a given material, the safe allowable mean rim velocity V_{mean} is determined by the material and the centrifugal stresses set in the rim. Consequently, with a velocity established for a given type of flywheel, the δ set by the type of application. The problem now is to find the maximum excess or deficiency of energy (ΔE), during an energy cycle which causes the speed of the flywheel to change from $V_{\max}$ to $V_{\min}$ or vice versa.

Size of fly wheel and hoop stress developed in a fly wheel.

Consider a rim of the fly wheel as shown in figure. Let D = mean diameter of rim, R = mean radius of rim, *t* = thickness of the fly wheel, A = cross sectional as area of rim in m^2 and ρ be the density of the rim material in Kg/m^3, N be the speed of the fly wheel in rpm, ω= angular velocity in rad/sec, V = linear velocity in m/σ, hoop stress in N/m^2 due to centrifugal force.



Consider small element of the rim. Let it subtend an angle $\delta\theta$ at the centre of flywheel.
Volume of the small element = $R\delta\theta.A$.
Mass of the small element = $dm = R\delta\theta.A$ ρ

The centrifugal force on the small element

$$dF_C = dm\omega^2 R$$
$$= R\delta\theta.A\omega^2 R \,\rho$$
$$= R^2 A.\omega^2 \delta\theta \,\rho$$

Resolving the centrifugal force vertically

$$dF_C = dF_C Sin\theta$$

$$= \rho\, R^2 A\omega^2 Sin\theta.\ \delta\theta \qquad ---(1)$$

Total Vertical upward force across diameter X & Y

$$= \int_0^{\pi} \rho R^2 A\omega^2 Sin\theta.\ \delta\theta$$
$$= \rho R^2 A\omega^2 \int_0^{\pi} Sin\theta.\ \delta\theta$$
$$2\rho = 2\rho A R^2 \omega^2$$

This vertical upward force will produce tensile stress on loop stress developed & it is resisted by 2P.

We know that, $\sigma = P / A$

$$P = \sigma A$$

$$\therefore\ 2P = 2\sigma A$$

$$PAR^2\omega^2 = 2\sigma A$$

$\boxed{o = \rho R^2\omega^2}$ % up to this deviation

Also,

Linear velocity V=Rx ω

$$o = \delta V^2$$

$$\boxed{V = \sqrt{\sigma/\delta}}$$

Mass of the rim = volume x density

$$\boxed{m = \pi\, dA \times \rho}$$

Problem 1:

A shaft fitted with a flywheel rotates at 250 r.p.m. and drives a machine. The torque of the machine varies in a cyclic manner over a period of 3 revolutions. The torque rises from 750 N-m to 3000 N-m uniformly during ½ revolution and remains constant fore the following revolution. It then falls uniformly to 750 N-m during the next ½ revolution and remains constant for one revolution, the cycle being repeated thereafter.

Determine the power required to drive the machine and percentage fluctuation in speed, if the driving torque applied to the shaft is constant and the mass of the flywheel is 500 kg with radius of gyration of 600 mm.

Solution.

Given: $N = 250\ r.p.m\ or\ \omega = 2\pi \times 250 / 60 = 26.2 rad/s;\ m = 500 kg;\ k = 600\ m = 0.6$

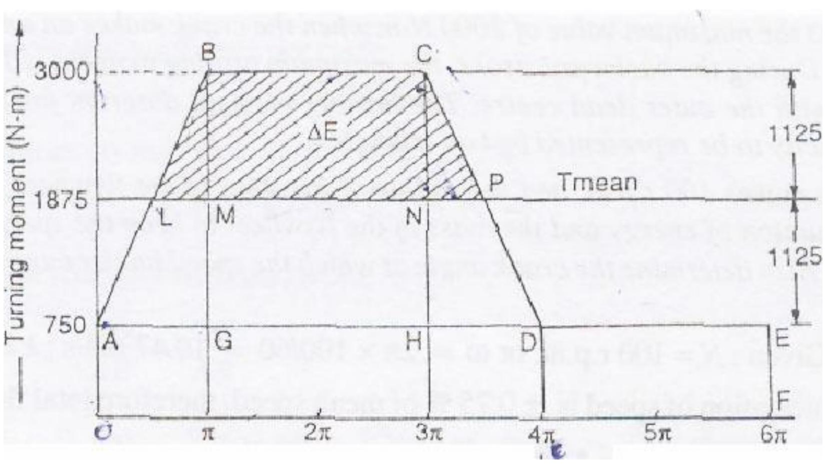
The turning moment diagram for the complete cycle is drawn.

The torque required for one complete cycle

$$
\begin{aligned}
&= Area\ of\ \ figure\ OABCDEF \\
&= Area\ OAEF + Area\ ABG + Area BCHG + Area\ CDH \\
&= OF \times OA + \frac{1}{2} \times AG \times BG + GH \times CH + \frac{1}{2} \times HD \times CH \\
&= 6\pi \times 750 + \frac{1}{2} \times \pi (3000 - 750) + 2\pi (3000 - 750) + \frac{1}{2} \times \pi (3000 - 750) \\
&= 11250\ \pi N - m
\end{aligned}
$$

Torque required for one complete cycle = $T_{mean} \times \pi N - m$

$$\therefore T_{mean} = 11250\pi / 6\pi = 1875\ N - m$$



Power required to drive the machine, $P = T_{mean} \times \omega = 11875 \times 26.2 = 49125 W = 49.125 kW$.

To find Coefficient of fluctuation of speed, δ.

Find the values of *LM* and *NP*.

From similar triangles *ABG* and *BLM*,

$$\frac{LM}{AG} = \frac{BM}{BG} \quad or \ \frac{LM}{\pi} = \frac{3000-1875}{3000-750} = 0.5 \quad or\ LM = 0.5\pi$$

From similar triangles *CHD* and *CNP*,

$$\frac{NP}{HD} = \frac{CN}{CH} \quad or\ \frac{NP}{\pi} = \frac{3000-1875}{3000-750} = 0.5 \quad or\ NP = 0.5\pi$$

From the figure, we find that,

$BM = CN = 3000 - 1875 = 1125$ N-m

The area above the mean torque line represents the maximum fluctuation of energy. Therefore the maximum fluctuation of energy, ΔE

$= \text{Area } LBCP = \text{Area } LBM + \text{Area } MBCN + \text{Area } PNC$

$$= \frac{1}{2} \times LM \times BM + MN \times BM + \frac{1}{2} \times NP \times CN$$

$$= \frac{1}{2} \times 0.5\,\pi \times 1125 + 2\pi \times 1125 + \frac{1}{2} \times 0.5\,\pi \times 1125 = 8837\ N-m$$

We know that maximum fluctuation of energy (ΔE),

$$8837 = m.k^2.\omega^2.\delta = 500\,(0.6)^2\ (26.2)^2\,\delta = 123\,559\ \delta$$

$$\boxed{\delta = 0.071}$$

Problem 2

The torque delivered by two stroke engine is represented by T = 1000+300 sin 2θ-500 cos 2θ where θ is angle turned by the crack from inner dead under the engine speed. Determine work done per cycle and the power developed.

Solution

θ,deg.	$T, N-m$
0	500
90	1500
180	500
270	1500
360	500

Work done / cycle = Area under the turning moment diagram.

$$= \int_0^{2\pi} T\, d\theta$$

$$= \int_0^{2\pi} \left(1000 + 300\sin 2\theta - 500\cos 2\theta\right) d\theta$$

$$= 2000\pi\, N - m$$

$$T_{mean} = \frac{W.D/cycle}{2\pi}$$

$$= \frac{2000\pi}{2\pi} = 1000\ N - m$$

Power developed = $T_{mean} \times \omega_{mean}$

$$= 1000 \times \frac{2\pi N}{60}$$

$$= 1000 \times \frac{2\pi \times 200}{60}$$

$$= 26179 W$$

Problem: 3

The turning moment curve for an engine is represented by the equation,

T = (20 000 + 9500 sin 2θ - 5700 cos 2θ) N-m, where θ is the angle moved by the crank from inner dead centre. If the resisting torque is constant, find:

1. Power developed by the engine;
2. Moment of inertia of flywheel in kg-m^2, if the total fluctuation of speed is not the exceed 1% of mean speed which is 180 r.p.m. and
3. Angular acceleration of the flywheel when the crank has turned through 45° from inner dead centre.

Solution:

Given, *T* = (20 000 + 9500 sin 2θ - 5700 cos 2θ) N-m ;

N = 180 r.p.m. or ω = 2π x 180/60 = 18.85 rad/s

Since the total fluctuation of speed (ω_1-ω_2) is 1% of mean speed (ω), coefficient of fluctuation of speed,

$$\delta = \frac{\omega_1 - \omega_2}{\omega} = 1\% = 0.01$$

1. Power developed by the engine.

Work done per revolution

$$= \int_0^{2\pi} T d\theta = \int_0^{2\pi} (20000 + 9500 \sin 2\theta - 5700 \cos 2\theta)\, d\theta$$

$$= \left[20000\,\theta - \frac{9500 \cos 2\theta}{2} - \frac{5700 \sin 2\theta}{2} \right]_0^{2\pi}$$

$$= 20000 \times 2\pi = 40\,000\,\pi \;\; N - m$$

Mean resisting torque of the engine,

$$T_{mean} = \frac{Work\,done\,per\,revolution}{2\pi} = \frac{40\,000\,\pi}{2\pi} = 20000\; N - m$$

Power developed by the engine

$$= T_{mean}\,.\,\omega = 20\,000 \times 18.85 = 377\,000 W \;\; = 377\; kW.$$

2. Moment of inertia of the flywheel

The turning moment diagram for one stroke (i. e. half revolution of the crankshaft) is shown in the Fig. Since at points *B* and *D*, the torque exerted on the crankshaft is equal to the mean resisting torque on the flywheel, therefore,

$$T = T_{mean}$$

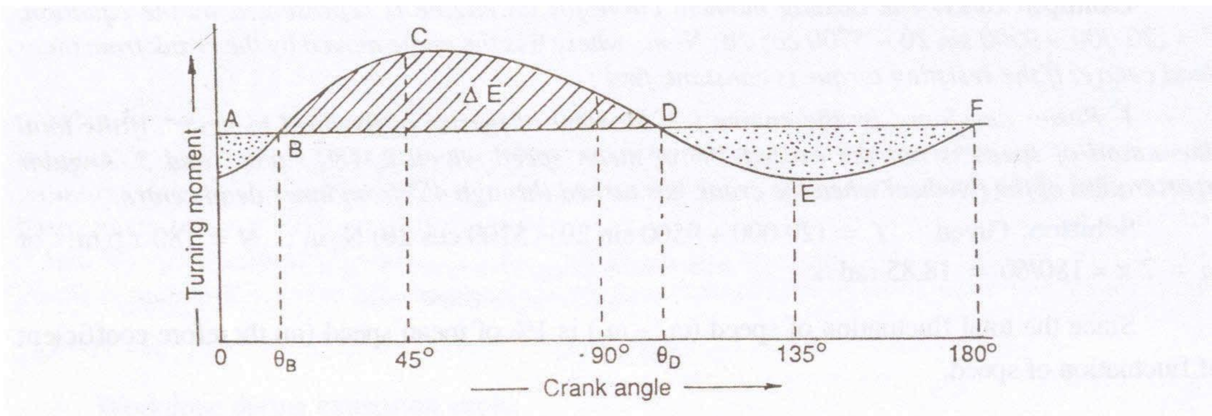
$$20\,000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20\,000$$

or $$9500 \sin 2\theta = 5700 \cos 2\theta$$

$$\tan\, 2\theta = \sin 2\theta \;/ \cos 2\theta = 5700 / 9500 = 0.6$$

$\therefore$ $$2\theta = 31° \; or\, \theta = 15.5°$$

$\therefore$ $$\text{i.e.,}\, \theta_B = 15.5° \; and \;\, \theta_D = 90° + 15.5° = 105.5°$$



Maximum fluctuation of energy,

$$\Delta E = \int_{\theta_B}^{\theta_D} (T - T_{mean}) d\theta$$

$$= \int_{15.5°}^{105.5°} (20000 + 9500 \sin 2\theta - 5700 \cos 2\theta - 20\ 000)\, d\theta$$

$$\Delta E = \int_{\theta_B}^{\theta_D} (T - T_{mean}) d\theta = \left[-\frac{9500 \sin 2\theta}{2} - \frac{5700 \cos 2\theta}{2} \right]_{15.5°}^{105.5°} = 11078\ N - m$$

Maximum fluctuation of energy (ΔE),

$$11\ 078 = I.\omega.\ \delta = I(18.85)^2\ 0.01 = 3.55\ I$$

$$I = 11078/3.55 = 3121 \text{ kg-m}^2.$$

3. Angular acceleration of the flywheel

Let α = Angular acceleration of the flywheel, and

θ = Angle turned by the crank from inner dead centre = 45°... (Given)

The angular acceleration in the flywheel is produced by the excess torque over the mean torque. Excess torque at any instant,

$$T_{excess} = T - T_{mean}$$

$$20\ 000 + 9500 \sin 2\theta - 5700 \cos 2\theta = 20\ 000$$

$$9500 \sin 2\theta - 5700 \cos 2\theta$$

$\therefore$ Excess torque at 45°= 9500 sin 90° - 5700 cos 90° =9500Nm

We also know that excess torque= $I.\alpha$ = 3121 xα

From equations (i) and (ii),

$$\alpha = 9500 / 3121 = 3.044 \text{ rad/s}^2.$$

Problem 4

The torque exerted on the crankshaft is given by the equation

$$T_m = 1500 + 240 \sin 2\theta - 200 \cos 2\theta \quad Nm.$$

Where θ is the crank angle displacement from the inner dead centre. Assuming the resisting torque to be constant, determine (a) the power of the engine when the speed is 150 rpm (b) the moment of inertia of the flywheel if the speed variation is not to exceed ±0.5% of the mean speed and (c) the angular acceleration of the flywheel when the crank has turned through 30° from the inner dead center.

SOLUTION: (a) Since the fluctuating terms sin2θ and cos2θ have zero mean, we have

$T_m = 1500\ Nm$

$\therefore$ Power of the engine $= \dfrac{2\pi}{60} n\, T_m$

$= \dfrac{2\pi \times 150 \times 1500}{60}$

= 23.5 kW

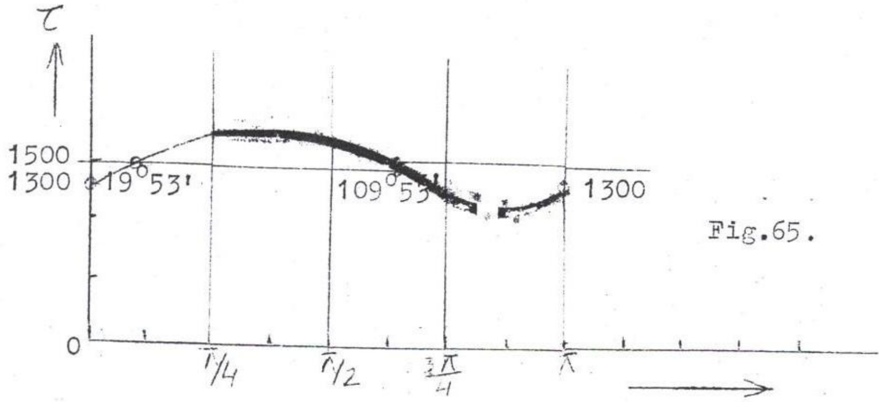


Fig.65.

(b) $T = T_m$ when tan 2θ $= \dfrac{200}{240} = 0.8350$

$$\therefore 2\theta = 39°46' \;\; or \;\; 180 + 39°46'$$

$$\theta = 19°53' \; and\, 109°53'$$

and, excess energy, $$\Delta E = \int_{109°53'}^{109°53'} (240\sin 2\theta - 200\cos 2\theta)\,d\theta$$

= 314Nm.

Speed variation $\pm 0.5\% = \pm \dfrac{\Delta E}{2I\omega_{mean}} \times 100$

$$\therefore\ I = \frac{314 \times 100}{\left(\dfrac{150\pi}{30}\right)} = 126.5 Nm^2$$

(c) At θ = 30°, accelerating torque

$T - T_m = 240 \times 0.866 - 20 \times 0.5$

= 108 Nm.

$\therefore$ Angular acceleration, $\alpha = \dfrac{108}{I} = 0.855\ rad / \sec^2$

Problem 5: The equation of the turning moment diagram of a three crank engine is 21000+7000 sin3θ Nm. Where θ in radians is the crank angle. The moment of inertia of the flywheel is $4.5 \times 10^3 \ Nm^2$ and the mean engine speed is 300 rpm. Calculate the power of the engine and the total percentage fluctuation of speed of the flywheel (i) if the resisting torque is constant (ii) if the resisting torque is 21000 + 3000 sinθ Nm.

a) $T_m = 21000 \ Nm.$

$$\text{Power} = \frac{2\pi \times 21000 \times 300}{60} = 660 \ kW.$$

b) (i) $$\Delta E = \int_0^{\frac{\pi}{3}} 7000 \sin 3\theta d\theta = 4666.7 \ Nm.$$

$$\therefore \text{ Total percent fluctuation of speed } = \frac{100 \Delta E}{I\omega^2{}_{mean}}$$

$$= \frac{100 \times 4666.7 \times 9.8}{45 \times 10^3 \times \left(\frac{300\pi}{30} \right)^2}$$

$$= 1.04\%$$

(ii) Engine torque = load torque, at crack angles given by

7000 sin3θ = 3000 sinθ

i.e., 2.33 (3sinθ - 4 sin^{3}θ) = sinθ

One solution is sinθ =0, i.e., θ = 0 and 180º, and the other is sinθ = ±0.803, i.e., θ = 53°24‘‘or 126°36‘‘ between 0° and 180º. The intersections are shown in figure and the areas between the curves represent increase or decrease of total energy. The numerically longest is between θ = 53° 24‘‘ and 126º36‘‘.

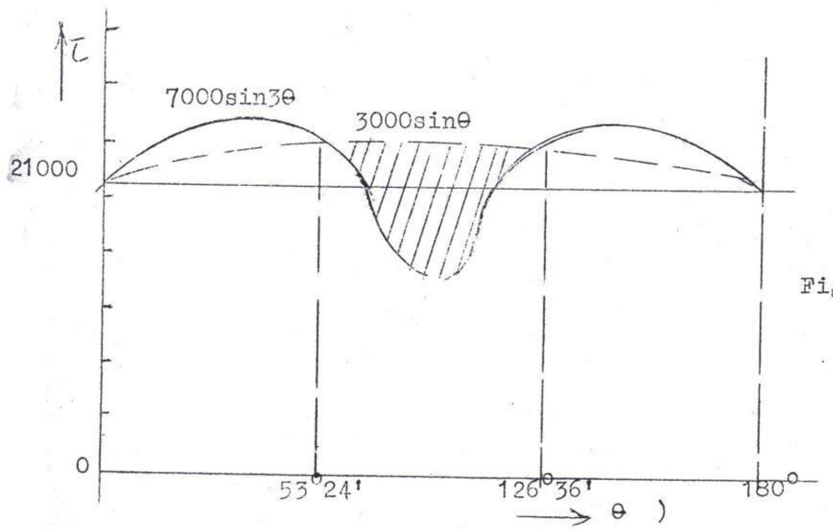


Fig.66.

$$i.e., \Delta E = \int_{53°24'}^{126°36'} (7000 \sin 3\theta - 3000 \sin\theta)\, d\theta$$

=7960 Nm.

Therefore, the total (percentage) fluctuation of speed $\dfrac{100\Delta E}{I\omega^2_{mean}}$

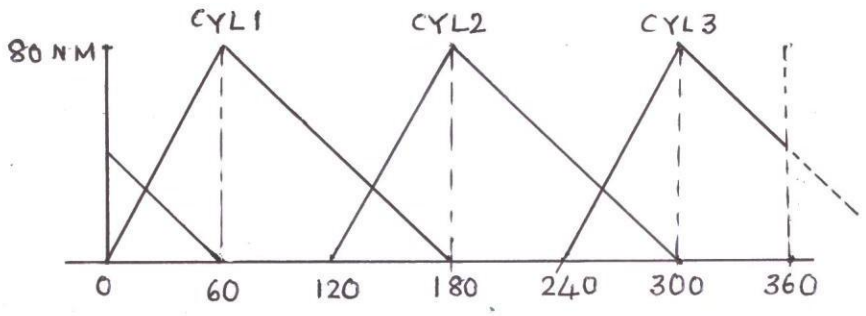
$$= \frac{100 \times 7960 \times 9.8}{4.5 \times 10^3 \times \left(\dfrac{300\pi}{30}\right)^2}$$

$$= 1.65\%$$

Problem 6:

A 3 cylinder single acting engine has its cranks set equally at 120° and it runs at 600 rpm. The Torque crank angle diagram for each cylinder is a triangle for the power with maximum torque 80 N-m at 60° after dead centre of the corresponding crank. The torque on the return stroke is sensibly zero.

Determine the (a) Power developed
(b) K if the flywheel used has a mass of 10 Kg. and radius of gyration is 8 cm
(c) Coefficient of fluctuation of energy
(d) Maximum angle of the flywheel



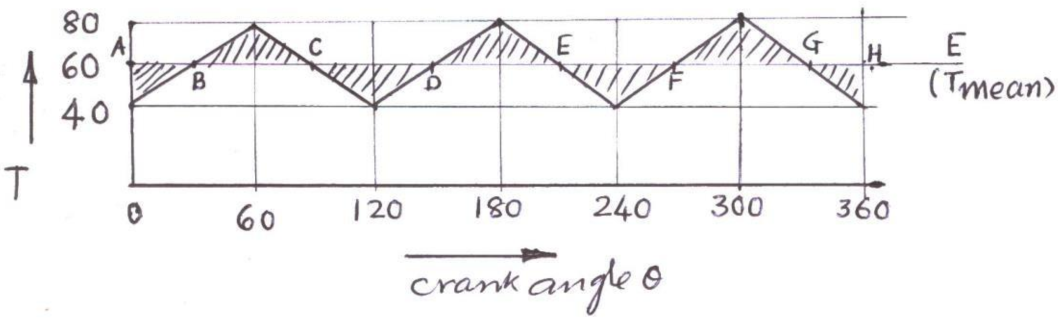
Work done / cycle = Area of 3 triangles

$$3 \times \frac{1}{2}\pi \times 80 = 120\pi \quad N-m$$

(a) Power developed $= \dfrac{work\ done / cycle \times cycle / \min}{60 \times 1000} kW$

$$= \frac{120 \times \pi \times 600}{60 \times 1000} = 3.75\ kW$$

(b) $T_{mean} = \dfrac{work\ done / cycle}{crank\ angle / cycle} = \dfrac{120\ \pi}{2\ \pi} = 60\ N-m$



Energy at $A = E$

Energy at $B = (E - \frac{1}{2}.\frac{\pi}{6}.20) = \frac{10\pi}{6}$

Energy at $C = (E - \frac{10\pi}{6} + \frac{1}{2}.\frac{\pi}{3}.20) = E + \frac{10\pi}{6}$

Energies at D, E, F, G & H will be,

$$E - \frac{10\pi}{6}, E + \frac{10\pi}{6}, E - \frac{10\pi}{6}, E + \frac{10\pi}{6}\ E \textit{ respectively}$$

$$\underbrace{\Delta E}_{\text{Maximum Fluctuation of energy}} = \underbrace{\left(E + \frac{10\pi}{6}\right)}_{\text{Maximum energy}} - \underbrace{\left(E - \frac{10\pi}{6}\right)}_{\text{Minimum energy}} = \frac{10\pi}{3} \quad N-m$$

$$\omega = \frac{2\pi 600}{60} = 20\ \pi \quad rad/s$$

$$\Delta E = I\ \omega^2 \delta = I\ \omega\,(\omega_1 - \omega_2)$$

$$= mk^2\ \omega\,(\omega_1 - \omega_2)$$

$$(\omega_1 - \omega_2) = \frac{\Delta E}{mk^2\omega} = \frac{10\pi}{3} \times \frac{1}{10 \times (0.08)^2\, 20\pi} = 2.604\ rad/s$$

$$\delta = \frac{\omega_1 - \omega_2}{\omega} = \frac{2.604}{20\ \pi} \times 100 = 4.44\%$$

(c) Coefficient of fluctuation of energy $= \dfrac{\textit{Maximum fluctuation of energy}}{\textit{work done / cycle}}$

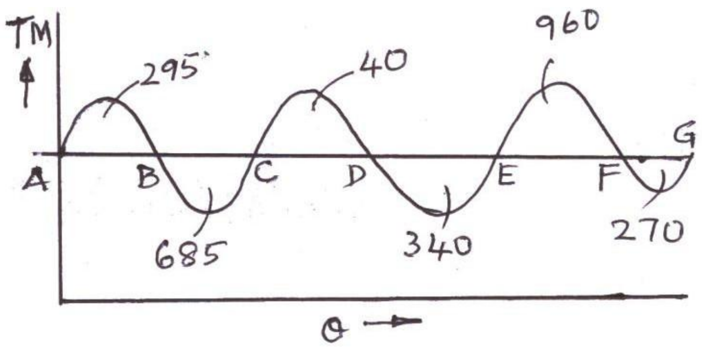
$$= \frac{10\pi}{3}.\frac{1}{120\ \pi} = 0.0278 \textit{ or } 2.78\%$$

(d) $T_{max} - T_m = I\,\alpha$

$$\therefore\ \alpha = \frac{T_{max} - Tm}{I} = \frac{80-60}{10\times(0.08)^2} = 312.5\ rad/s^2$$

Problem 7:

The TMD for a petrol engine is drawn to the following scale, turning moment, 1mm = 5Nm, crank 1mm = 1°. The TMD repeats itself at every half revolution of the engine & areas above & below the mean turning moment line taken in order are 295, 685, 40, 340, 960, 270 mm^2. The rotating parts are equivalent to a mass of 36 kg at a radius of gyration of 150mm. Calculate the maximum fluctuation of energy & co-efficient of fluctuation of speed when engine runs at 1800rpm



Energy at $A = E$
Energy at $B = E + a_1$
$= E + 295$
Energy at $C = E + 295 - 685 = E - 390$
Energy at $D = E + 295 - 685 + 40 = E - 350$
Energy at $E = E - 350 - 340 = E - 690$
Energy at $F = E - 690 + 960 = E + 270$
Energy at $G = E + 270 - 270 = E$
$\therefore\ A = G$

Max Energy $= E + 295$
Min Energy $= E - 690$

Maximum Fluctuation of Energy $\Delta E = E + 295 - (E - 690)$

$= 985 mm^2$

Scale: $1mm = 5Nm$ & $1mm = 1°$

$$Torque \times \theta = \frac{5}{180}\pi \times 1 = \frac{\pi}{36} Nm$$

$$\Delta E = 985 \times \frac{\pi}{36} = 85.95 Nm$$

$m = 36kg,\ k = 150mm, \quad N = 1800rpm$

$$\Delta E = mk^2\omega^2\delta$$

$$86 = 36 \times 0.15^2 \times \left(\frac{2\Pi(1800)}{60}\right)^2 \delta$$

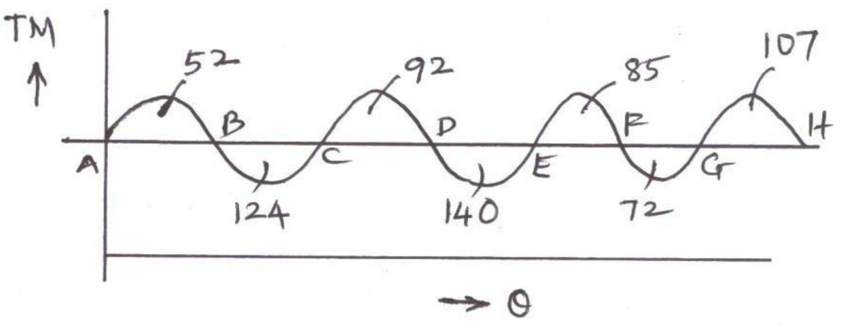
$$\delta = 0.003\ or \quad 0.3\%$$

Problem 8:

The turning moment diagram for a multi cylinder engine has been drawn to a scale 1mm = 600 Nm vertically and 1mm = 3° horizontally. The intercepted areas between the output torque curve and mean resistance line taken in order from one end are as follows + 52, -124, +92, -140; 85, -72 and 107 mm^2 when the engine is running at a speed of 600 rpm. If the total fluctuation of speed is not exceed 1.5% of the mean, find the necessary mass of the fly wheel of radius 0.5 m.

Solution:

$N = 600 \quad rpm$



Co-efficient of fluctuation of speed, $\delta = \dfrac{\omega_1 - \omega_2}{\omega} = 1.5 + 1.5 = 3\%$

$\Delta E = mR^2\omega^2\delta$

Energy at $A = E$
Energy at $B = E + 52$
Energy at $C = E + 52 - 124 = E - 702$
Energy at $D = E - 72 + 92 = E + 20$
Energy at $E = E + 20 - 140 = E - 120$
Energy at $F = E - 120 + 85 = E - 35$
Energy at $G = E - 35 - 72 = E - 107$
Energy at $H = E - 107 + 107 = E$

$\Delta E = E + 52 - (E - 120) = 172\ mm^2$

Scale: $T\theta = 1\ mm^2 = 600 \times 3 \times \dfrac{\pi}{180} = 31.41\ Nm$

$$\Delta E = 172 \times 31.41 = 5402.52\ Nm$$

$$\Delta E = mR^2 \omega^2 \delta$$

$$5402.5^2 = m(0.5)^2 \left(\frac{2\pi \times 600}{60} \right)^2 \times \frac{3}{100}$$

$$m = 182.47\ kg$$

Problem 9:

The TMD for a multi cylinder engine has been drawn to a scale 1mm to 500Nm torque & 1mm to 6º of crank displacement. The intercepted area in order from one end is mm^2 are -30, 410, -280, 320, -330, 250, -360, +280, -260 mm^2 when engine is running at 800rpm. The engine has a stroke of 300mm & fluctuation of speed is not to exceed ±2% of the mean speed, determine

1. a suitable diameter & cross section of the fly wheel rim for a limiting value of the safe centrifugal stress of 7MPa. The material density may be assumed as 7200 kg/m³. The width of the rim is to be 5times the thickness.

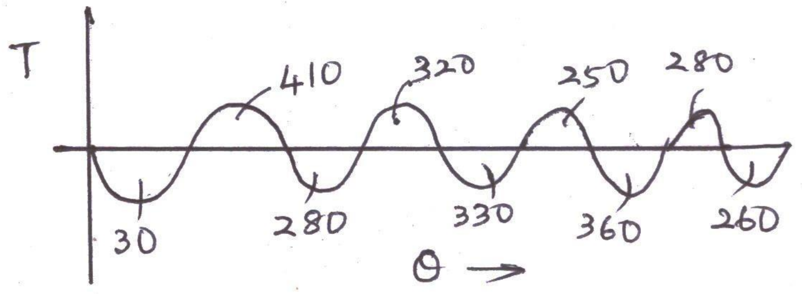
Solution:

$N = 800\ \ rpm$

$\pm 2\ \% means,\ \delta = 4\% = 0.04$

$o = 7\ Mpa = 7\ N / m2$

$\rho = 7\ 200\ kg / m^3$



Energy at $A = E$
Energy at $B = E - 30$
Energy at $C = E - 30 + 410 = E + 380$
Energy at $D = E + 380 - 280 = E + 100$
Energy at $E = E + 100 + 320 = E + 420$
Energy at $F = E + 420 - 330 = E + 90$
Energy at $G = E + 90 + 250 = E + 340$
Energy at $H = E + 340 - 360 = E - 20$
Energy at $I = E - 20 + 280 = E + 260$
Energy at $J = E + 260 - 260 = E$

$$\Delta E = E + 420 - (E - 30)$$
$$= 450mm^2$$

$1mm = 500Nm\ ,\quad 1mm = 6°\ (0.1047\ radians),\quad 1mm^2 = 52.35Nm$

$\Delta E = 450 \times 52.35 =$23557.5 Nm

$o = \rho\ V^2$ $\qquad\qquad$ $\Delta E = mr^2\ \omega^2 \delta$

$7 \times 10^6 = 7200\ V^2$ $\qquad$ $= mV^2 \delta$

$V = r\omega$

$V = 31.18\ m/s$

$$V = \frac{\pi DN}{60} \quad , D = 0.745m$$

Cross sectional area $A = bt$

$A = (5t)t = 5t^2$

Fluctuation of energy $\Delta E = mV^2\delta$

$$23.56 \times 10^3 = m(31.18)^2 \ (0.04)$$

$$m = 605 \ kg$$

$m = Volume \times Density$

$\pi DA \times \rho$

$605 = \pi(0.745)(5t^2)7200$

$t = 0.084m$

Area $= 5t^2 = 0.035m^2$

Problem 10:

The T M diagram for a multi cylinder engine has been drawn to a scale of 1 cm = 5000 N-m and 1 cm = 60° respectively. The intercepted areas between output torque curve and mean resistance line is taken in order from one end are – 0.3, +4.1, -2.8, +3.2, -3.3, + 2.5, - 3.6, +2.8, - 2.6 cm^2 , when the engine is running at 800 rpm. The engine has a stroke of 300 mm and fluctuation of speed is not to exceed 2% of the mean speed. Determine a suitable diameter & cross section of the flywheel rim for a limiting value of the shaft centrifugal stress of $280 \times 10^5 \ N/m^2$. The material density can be assumed as 7.2 gm / cm^3. Assume the thickness of the rim to be ¼ of the width.

$$E_{max} = E + 4.2$$

$$E_{min} \ E - 0.3$$

$$(\Delta E) = 4.5cm^2 \, or$$

$$= 4.5 \times 5000 \times \frac{\pi}{3} = 23,562 \quad N - m$$

$$\Delta E = I\omega^2\delta$$

$$I = \frac{\Delta E}{\omega^2\delta} = \frac{23.562}{\left(\frac{2\pi 800}{60}\right)^2 0.02} = 168 \ kgm^2$$

Safe peripheral velocity is given by;

$f = \rho v^2 \; N/m^2$ f = safe stress N/m^2

or $V = \sqrt{\dfrac{f}{\rho}} \; m/s$ V= velocity m/s (peripheral)

ρ = density Kg/m^3

$= \sqrt{\dfrac{28 \times 10^5}{7.2 \times \dfrac{10^6}{1000}}} = 62.36 \; m/s$

$\Delta E = I\omega^2\delta$

$\delta = \dfrac{\Delta E}{I\omega^2} = \dfrac{\Delta E}{2 \times \frac{1}{2} I\omega^2} = \dfrac{\Delta E}{2\,(KE)}$

$(KE) = \dfrac{\Delta E}{2\delta}$

Also, $V = \dfrac{\pi DN}{60}$

$\therefore = \dfrac{\pi DN}{60} = 62.36$;

$D = 1.4887 \; m$

Energy of the flywheel $(KE) = \dfrac{\Delta E}{2\delta} = \dfrac{23562}{2 \times 0.02} = 589050 \; N-m$

But $KE = \dfrac{1}{2} mV^2$

$\therefore \; 589050 = \dfrac{1}{2} m\,(62.36)^2$

$\therefore \; m = 303 Kg.$

Also $m = \pi DA\rho$

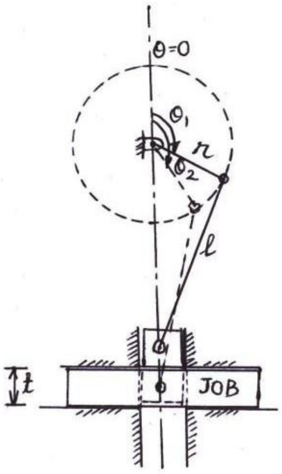
or $A = \dfrac{m}{\pi D\rho} = \dfrac{303}{\pi \times 1.4887 \times \dfrac{7.2 \times 10^6}{1000}} = 89.98 \; cm^2$

Area of cross section $A = t \times b = t \times 4t = 4t^2 = 89.89 \; cm^2$

$t = \sqrt{\dfrac{89.98}{4}} = 4.75 \; cm$

$b = 4 \times 4.75 = 19 cm$

Flywheel in punching press / Riveting machine



(a) Crank is driven by motor for which supplies a uniform torque.

(b) Load acts from $\theta = \theta_1 \; to \; \theta = \theta_2$ (during Punching). Load is zero for the remaining period.

(c) If flywheel is not there speed increases from $\theta = \theta_2 \; to \; \theta = 2\pi$ (= 0) and again from $\theta = 0 \; to \; \theta = \theta_1$

(d) From $\theta_1 \; to \; \theta_2$ big drop in speed.

(e) Use flywheel of suitable I for uniform speed

Let, E = Energy required for one punch

E is determined by Size of the hole, thickness of the blank to be punched and Material property

For stable operation (constant speed), energy supplied to the crank / rev = E (assuming 1 punch / revolution)

Energy supplied to the crank shaft from motor during punching = $E\left[\frac{(\theta_1 - \theta_2)}{(2\pi)}\right]$, if crank rotation is constant (when flywheel is there it is possible)

i.e., $E\left[1 - \frac{(\theta_1 - \theta_2)}{(2\pi)}\right]$ is supplied by flywheel by the decrease in its E_k (Kinetic energy) when the speed falls from $\omega_{max} \; to \; \omega_{min}$

$$\therefore (\Delta E)_{k\,max} = E\left[1 - \frac{(\theta_1 - \theta_2)}{(2\pi)}\right] = \frac{1}{2} I(\omega_{max}^2 - \omega_{min}^2) = I\omega^2\delta \quad \text{(same as before)}$$

θ_1 and θ_2 can be computed only if l,t, r and relative position of job w.r.t. crank shaft are given.

In the absence of data assuming (taking velocity of tool to be constant),

$$\frac{(\theta_2 - \theta_1)}{(2\pi)} \approx \frac{t}{2S} = \frac{t}{4r}$$

S = stroke of the punch = 2r

Problem.1

A machine punching 3.8 cm dia hole in a 3.2 cm thick plate does 600 J of work / sq. cm of sheared area. The punch has a stroke of 10.2 cm and punches 6 holed / min. The maximum speed of the flywheel at its radius of gyration is 27.5 m/s. Find the mass of the flywheel so that its speed at the same radius does not fall below 24.5 m/s. Also determine the power of the motor, driving this machine.

d = 3.8cm, t = 3.2 cm, A= 38.2 cm^2

Energy required / punch = 600 x 38.2 = 22.920 J

Assuming, $\dfrac{(\theta_2 - \theta_1)}{(2\pi)} = \dfrac{t}{2S} = \dfrac{3.2}{20.4}$

$$\therefore (\Delta K)_{E\max} = E\left[1 - \frac{t}{2S}\right] = \frac{1}{2} I\left(\omega^2_{\max} - \omega^2_{\min}\right)$$

$$= 22.920\left[1 - \frac{3.2}{20.4}\right] = \frac{1}{2} mk^2\left(\omega^2_{\max} - \omega^2_{\min}\right)$$

$$V_{\max} = k\ \omega_{\max} = 27.5 m/s$$

$$V_{\min} = k\ \omega_{\min} = 24.5 m/s$$

We get,

$$22920\left[1 - \frac{3.2}{20.4}\right] = \frac{1}{2} m\left(27.5^2 - 24.5^2\right) = \frac{1}{2} m\ 158$$

$$\therefore\ m = 244 kg.$$

The energy required / minute is $6 \times 22920 J$

$$\therefore\ Motor\ \ power\ = \frac{6 \times 22920}{1000 \times 60} k\omega = 2.292 kW$$

Problem.2

A riveting machine is driven by a constant torque 3 kW motor. The moving parts including the flywheel are equivalent to 150 kg at 0.6 m radius. One riveting operation takes 1 second and absorbs 10 000 N-m of energy. The speed of the flywheel is 300 r.p.m. before riveting. Find the speed immediately after riveting. How many rivets can be closed per minute.

Solution.

Given: P = 3 *kW; m*= 150 *kg; k* = 0.6m; N_1 = 300 r.p.m. or $\omega_1 = 2\pi \times 300/60 = 31.42$rad/s

Speed of the flywheel immediately after riveting

Let ω_2 = Angular speed of the flywheel immediately after riveting.

We know that, energy supplied by the motor,

$$E_2 = 3 \ kW = 3000 \ W \ = 3000 \ N - m/s \qquad (\because 1W = 1 \ N - m/s)$$

But, energy absorbed during one riveting operation which takes 1 second,

$$E_1 = 10\ 000 \ N - m$$

$\therefore$ Energy to be supplied by the flywheel for each riveting operation per second or the maximum fluctuation of energy,

$$\Delta E = E_1 - E_2 = 10\ 000 - 3000 = 7000 \text{ N-m}$$

We know that maximum fluctuation of energy (ΔE),

$$7000 = \frac{1}{2} \times m.k^2 \left[(\omega_1)^2 - (\omega_2)^2\right] = \frac{1}{2} \times 150\ (0.6)^2 \left[(31.42)^2 - (\omega_2)^2\right]$$

$$= 27\left[987.2 - (\omega_2)^2\right]$$

$$\therefore \qquad (\omega_2)^2 = 987.2 - 7000\ /27 = 728 \ or \ \omega_2 = 26.98 \ rad/s$$

Corresponding speed in r.p.m.,

$$N_2 = 26.98 \times 60 / 2\pi \ = 257.6 \ r.p.m.$$

Number of rivets that can be closed per minute.

Since, the energy absorbed by each riveting operation which takes 1 second is 10 000 N-m, therefore number of rivets that can be closed per minute,

$$\frac{E2}{E1} X60 = \frac{3000}{1000} X60 = 18 \ Rivets$$

Introduction

Flywheel which minimizes fluctuations of speed within the cycle but it cannot minimize fluctuations due to load variation. This means flywheel does not exercise any control over mean speed of the engine. To minimize fluctuations in the mean speed which may occur due to load variation, governor is used. The governor has no influence over cyclic speed fluctuations but it controls the mean speed over a long period during which load on the engine may vary.
The function of governor is to increase the supply of working fluid going to the prime- mover when the load on the prime-mover increases and to decrease the supply when the load decreases so as to keep the speed of the prime-mover almost constant at different loads.
Example: when the load on an engine increases, its speed decreases, therefore it becomes necessary to increase the supply of working fluid. On the other hand, when the load on the engine decreases, its speed increases and hence less working fluid is required.
When there is change in load, variation in speed also takes place then governor operates a regulatory control and adjusts the fuel supply to maintain the mean speed nearly constant. Therefore, the governor automatically regulates through linkages, the energy supply to the engine as demanded by variation of load so that the engine speed is maintained nearly constant.

6.1.1 Objectives

After studying this unit, you should be able to

- classify governors,
- analyse different type of governors,
- know characteristics of governors,
- know stability of spring controlled governors, and
- compare different type of governors.

Types of Governors

The governors may, broadly, be classified as
1. Centrifugal governors, and
2. Inertia governors.

The centrifugal governors, may further be classified (Fig. 1) as follows :

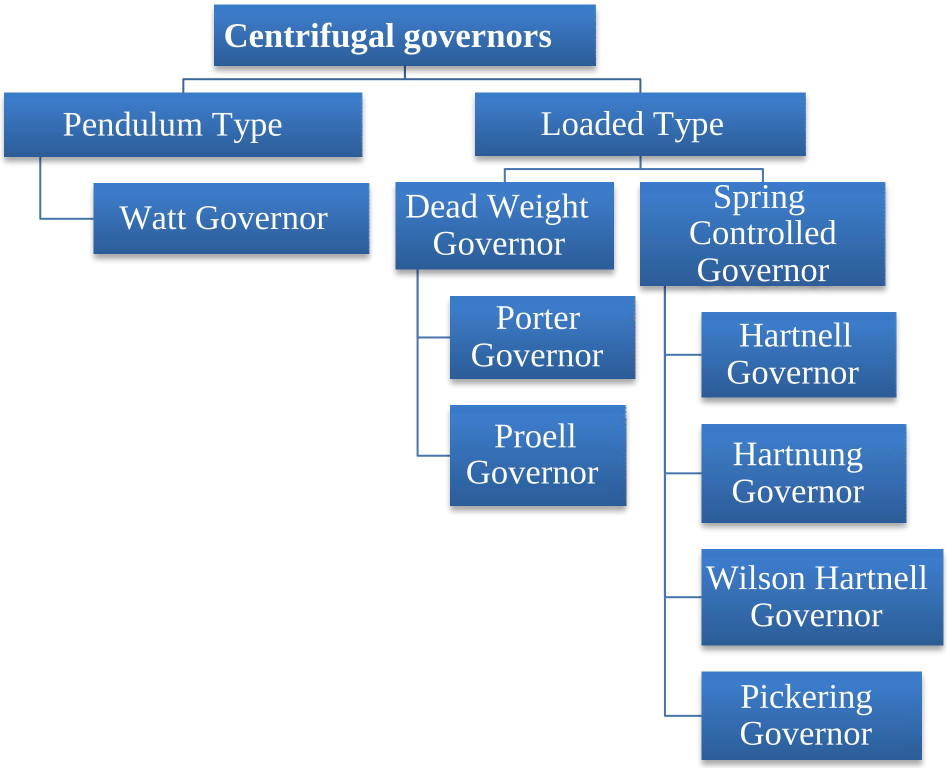


Fig.1 Classification of Centrifugal Governors

Centrifugal Governors

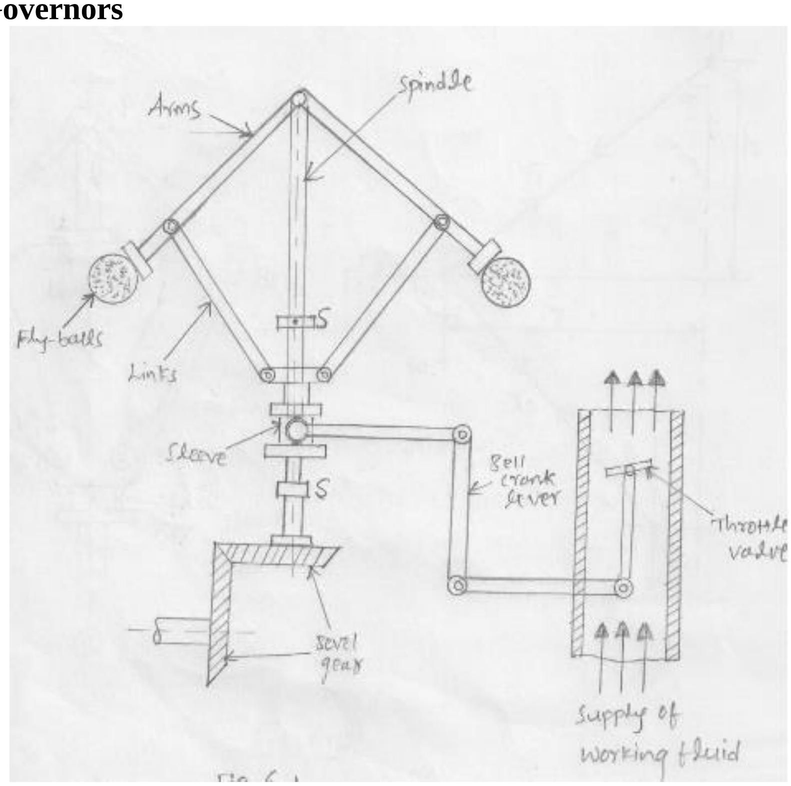


Fig.6.2 Schematic diagram of a centrifugal Governor

The centrifugal governors are based on the balancing of centrifugal force on the rotating balls by an equal and opposite radial force, known as the ***controlling force********.It consists of two balls of equal mass, which are attached to the arms as shown in Fig 6.2 These balls are known as ***governor balls*** **or** ***fly balls*****.** The balls revolve with a spindle, which is driven by the engine through bevel gears. The upper ends of the arms are pivoted to the spindle, so that the balls may rise up or fall down as they revolve about the vertical axis. The sleeve revolves with the spindle; but can slide up and down. The balls and the sleeve rises when the spindle speed increases, and falls when the speed decreases. In order to limit the travel of the sleeve in upward and downward directions, two stops *S*, *S* are provided on the spindle. The sleeve is connected by a bell crank lever to a throttle valve. The supply of the working fluid decreases when the sleeve rises and increases when it falls. When the load on the engine increases, the engine and the governor speed decreases. This results in the decrease of centrifugal force on the balls. Hence the balls move inwards and the sleeve moves downwards. The downward movement of the sleeve operates a throttle valve at the other end of the bell crank lever to increase the supply of working fluid and thus the engine speed is increased. Hence, the extra power output is provided to balance the increased load. When the load on the engine decreases, the engine and the governor speed increases, which results in the increase of centrifugal force on the balls. Thus the balls move outwards and the sleeve rises upwards. This upward movement of the sleeve reduces the supply of the working fluid and hence the speed is decreased. Hence, the power output is reduced.

Inertia Governors

This works on a different principle. The governor balls are arranged so that the inertia forces caused by angular acceleration or retardation of the governor shaft tend to alter their positions. The amount of the displacement of the balls is controlled by springs and the governor mechanism to alter the supply of energy to the engine.

The advantage of this type of governor is that the positions of the balls are affected by the rate of change of speed of the governor shaft. Consequently, a more rapid response to a change of load is obtained, since the action of the governor is due to acceleration and not to a finite change of speed. The advantage is offset, however, by the practical difficulty of arranging for a complete balance of the revolving parts of the governor. For this reason centrifugal governors are much more frequently used.

Porter Governor and its Force analysis

The Porter governor is a modification of a Watt’s governor, with central load attached to the sleeve as shown in Fig. 6.3(*a*). The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level.

Consider the forces acting on one-half of the governor as shown in Fig. 6.3 (*b*).

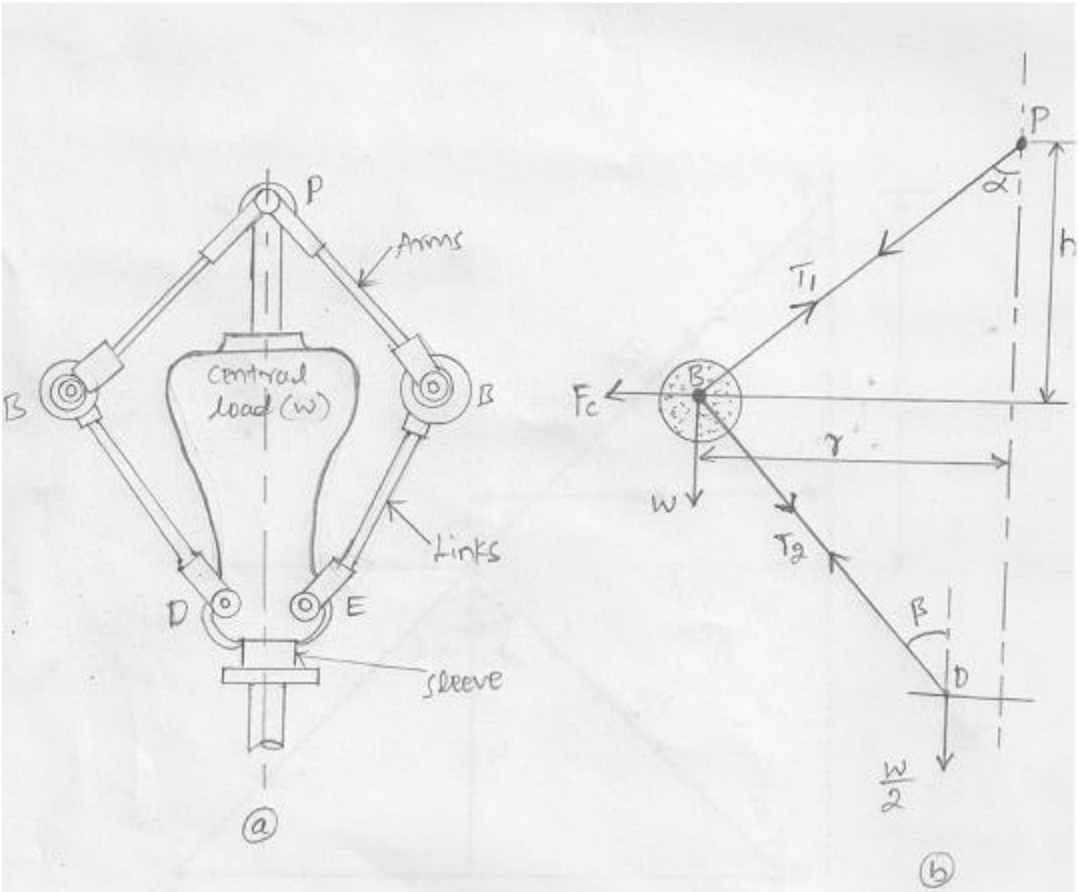


Fig 6.3 Porter governor.

m = Mass of each ball in kg,

w = Weight of each ball in N,

M = Mass of the central load in kg, *W*

= Weight of the central load in N, *r* =

Radius of rotation in m,

h = Height of governor in m,

N = Speed of the balls in rpm.

ω = Angular speed of the balls (2 Π*N*/60) rad/s,

*F*C = Centrifugal force acting on the ball N,

*T*1 = Force in the arm in N,

*T*2 = Force in the link in N,

α= Angle of inclination of the arm (or upper link) to the vertical, and

= Angle of inclination of the link (or lower link) to the vertical.

Relation between the height of the governor (h) and the angular speed of balls(ω).

1. Method of resolution of forces.
2. Instantaneous centre method.

6.3.1 Method of resolution of forces

Considering the equilibrium of the forces acting at *D*, we have the equilibrium of the forces acting on *B*. *T*he point *B* is in equilibrium under the action of the following forces, as shown in Fig. 6.3(*b*).
(i) The weight of ball (*w*),
(ii) The centrifugal force (FC),
(iii) The tension in the arm (T1), and
(iv) The tension in the link (T2).

Resolving the forces vertically

Resolving the forces horizontally

Dividing equation (3) by equation (2)

Substituting and we have

When the length of arms are equal to the length of links and the points *P* and *D* lie on the same vertical line, then

and q=1,

( ) ( )

When the loaded sleeve moves up and down the spindle; the frictional force acts on it in a direction opposite to that of the motion of sleeve.

)

~~q=1~~)

The + sign is used when the sleeve moves upwards or the governor speed increases and - sign is used when the sleeve moves downwards or the governor speed decreases.

6.4 Instantaneous centre method

In this method, equilibrium of the forces acting on the link *BD* were considered. The instantaneous centre *I* lies at the point of intersection of *PB* produced and a line through *D* perpendicular to the spindle axis, as shown in Fig. 6.4. Taking moments about the point *I*,

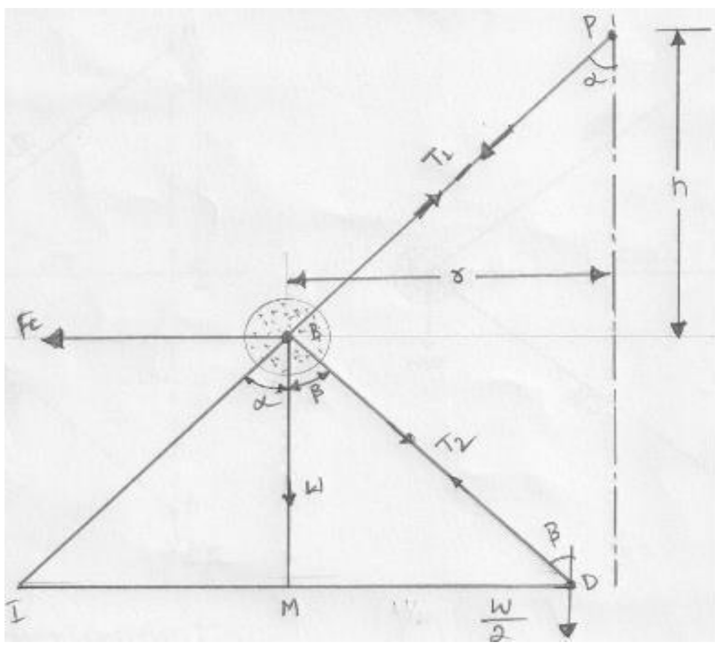


Fig 6.4 (Instantaneous centre method)

-

Porter Governor Problems

Problem 6.1

A Porter governor has equal arms each 250 mm long and pivoted on the axis of rotation. Each ball has a mass of 5 kg and the mass of the central load on the sleeve is 15 kg. The radius of rotation of the ball is 150 mm when the governor begins to lift and 200 mm when the governor is at maximum speed. Find the minimum and maximum speeds and range of speed of the governor.

Solution.

Given : $BP = BD = 250$ mm = 0.25 m ; $m = 5$ kg ; $M = 15$ kg ; $r1 = 150$ mm = 0.15m; $r2 = 200$ mm = 0.2 m

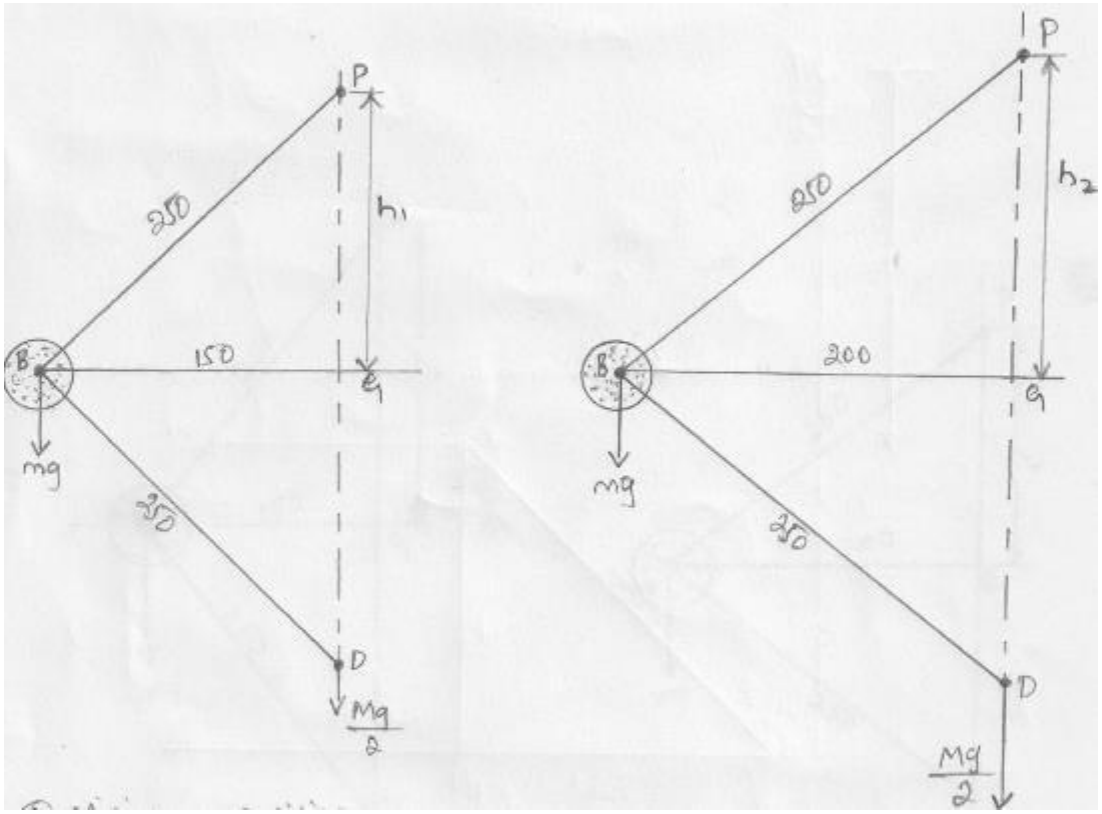


Fig 6p.1

The minimum and maximum positions of the governor are shown in Fig. 6p.1 (*a*) and (*b*) respectively.

Minimum speed

when r_1 = BG = 0.15 m

N1 = Minimum speed

Referring Fig. 6p.1(a), height of the governor,

,

Minimum speed of the governor is given by

( ) (   )   (   )   ( )

Maximum speed when r_2 = BG = 0.2 m

Let

Referring Fig. 6p.1(b), height of the governor, Maximum speed of the governor is given by

( ) ( ) ( ) ( )

Range of speed

Problem 6.2 The arms of a Porter governor are each 250 mm long and pivoted on the governor axis. The mass of each ball is 5 kg and the mass of the central sleeve is 30 kg. The radius of rotation of the balls is 150 mm when the sleeve begins to rise and reaches a value of 200 mm for maximum speed. Determine the speed range of the governor. If the friction at the sleeve is equivalent of 20 N of load at the sleeve, determine how the speed range is modified.

Solution;. Given : BP = BD = 250 mm ; m = 5 kg ; M = 30 kg ; r_1 = 150 mm ; r_2 = 200 mm

Minimum and maximum speed of the governor

The minimum and maximum position of the governor is shown in Fig. 6p.2 (*a*) and (*b*) respectively.

$N1$ = Minimum speed when $r1 = BG$ = 150 mm, and

$N2$ = Maximum speed when $r2 = BG$ = 200 mm.

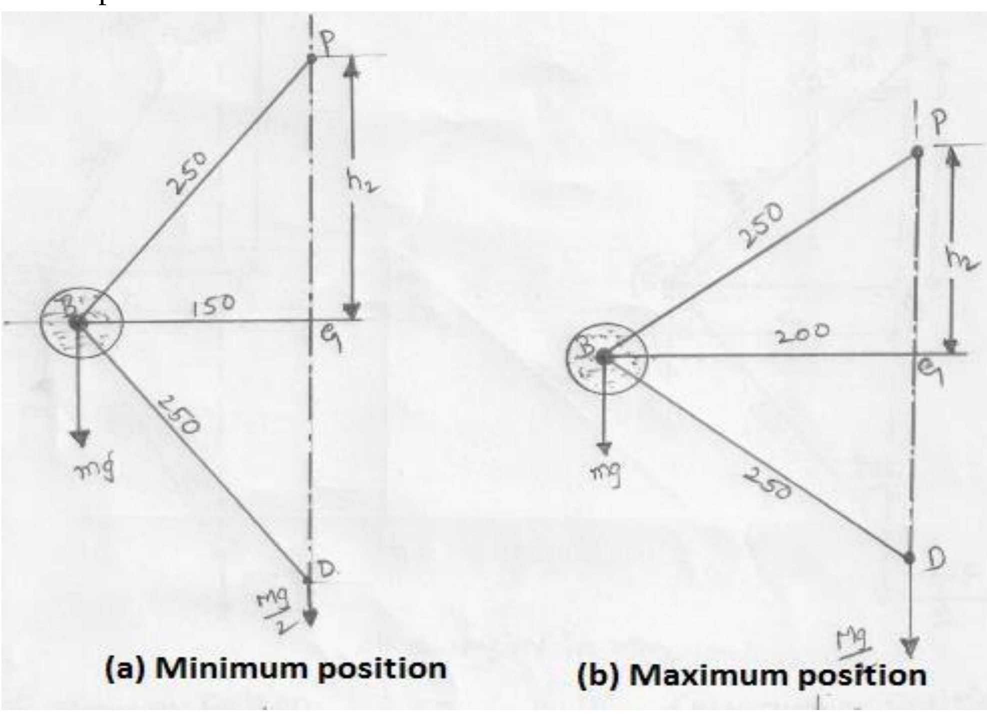


Fig 6p.2

To find Speed range of the governor Referring Fig. 6p.2 (a), height of the governor,

Minimum speed of the governor is given by

( ) ( ) ( ) ( )

Referring fig 6p.2(b) height of the governor,

Maximum speed of the governor is given by

( ) (( ) ( ) ( )

Speed range of the governor is given by

$= N_2 - N_1 = 204.4 - 177 = 27.4$ rpm.

Speed range when friction at the sleeve is equivalent of 20 N of load (i.e. when F = 20 N)

When the sleeve moves downwards, the friction force (F) acts upwards and the minimum speed is given by

( ) (

We also know that when the sleeve moves upwards, the frictional force (F) acts downwards and the maximum speed is given by

) ( ) (

( ) ( )

Speed range of the governor **= $N_2 – N_1$ = 210 – 172 = 38 rpm.**

Problem 6.3 In an engine governor of the Porter type, the upper and lower arms are 200 mm and 250 mm respectively and pivoted on the axis of rotation. The mass of the central load is 15 kg, the mass of each ball is 2 kg and friction of the sleeve together with the resistance of the operating gear is equal to a load of 25 N at the sleeve. If the limiting inclinations of the upper arms to the vertical are 30° and 40°, find, taking friction into account, range of speed of the governor.

Solution.

Given : *BP* = 200 mm = 0.2 m ; *BD* = 250 mm = 0.25 m ; *M* = 15 kg ; *m* = 2 kg ;
F = 25 N ; = 30°; = 40°

Minimum and maximum speed of the governor

The minimum and maximum position of the governor is shown Fig. 6p.3(*a*) and (*b*) respectively.

= Minimum speed,

= Maximum speed.

Referring Fig. 6p.3 (*a*), minimum speed, = *BG* = *BP* sin 30° = 0.2 × 0.5 = 0.1 m Height of the governor, = *PG* = *BP* cos 30° = 0.2 × 0.866 = 0.1732 m

DG = .23 m

When the sleeve moves downwards, the frictional force (F) acts upwards and the minimum speed is given by

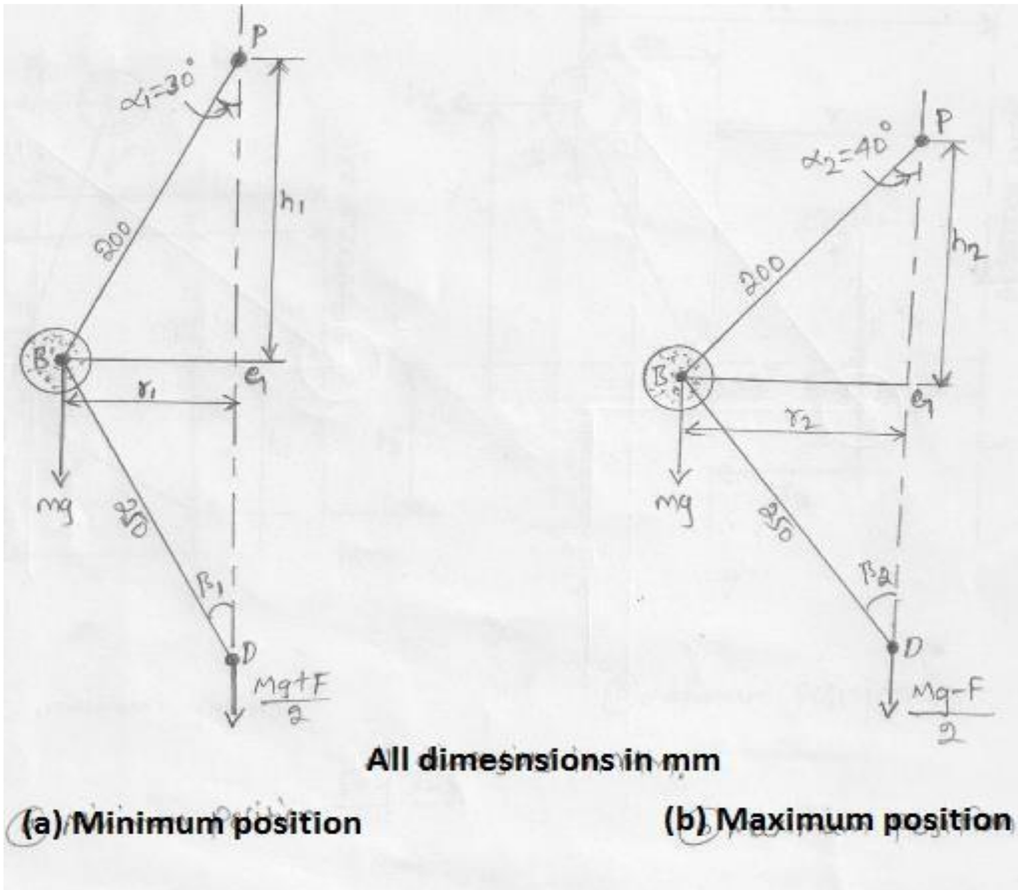


Fig 6p.3

Referring Fig. 6p.3(b),maximum speed,

DG = 0.2154 m

When the sleeve moves upwards, the frictional force (F) acts downwards and the maximum speed is given by

Range of speed

6.4 Hartnell Governor

A Hartnell governor is a spring loaded governor as shown in Fig. 6.4. It consists of two bell crank levers pivoted at the points O,O to the frame. The frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm OB and a roller at the end of the horizontal arm OR. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.

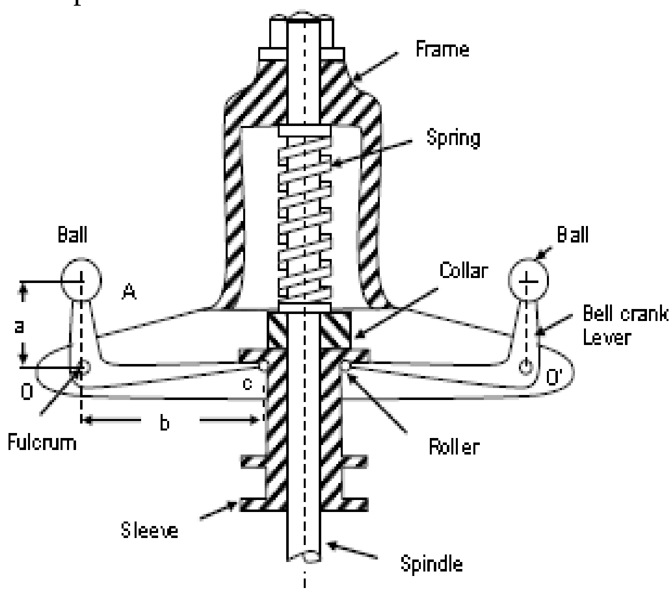


Fig. 6.5 Hartnell Governor

Consider the forces acting at one bell crank lever. The minimum and maximum position is shown in Fig.6.5. Let *h* be the compression of the spring when the radius of rotation changes from

For the minimum position *i.e.* when the radius of rotation changes from , as shown in Fig. 6.5 (*a*), the compression of the spring or the lift of sleeve is given by

Similarly, for the maximum position *i.e.* when the radius of rotation changes from , as shown in Fig. 6.5 (*b*), the compression of the spring or lift of sleeve is given by

Adding equation 1 and 2

–

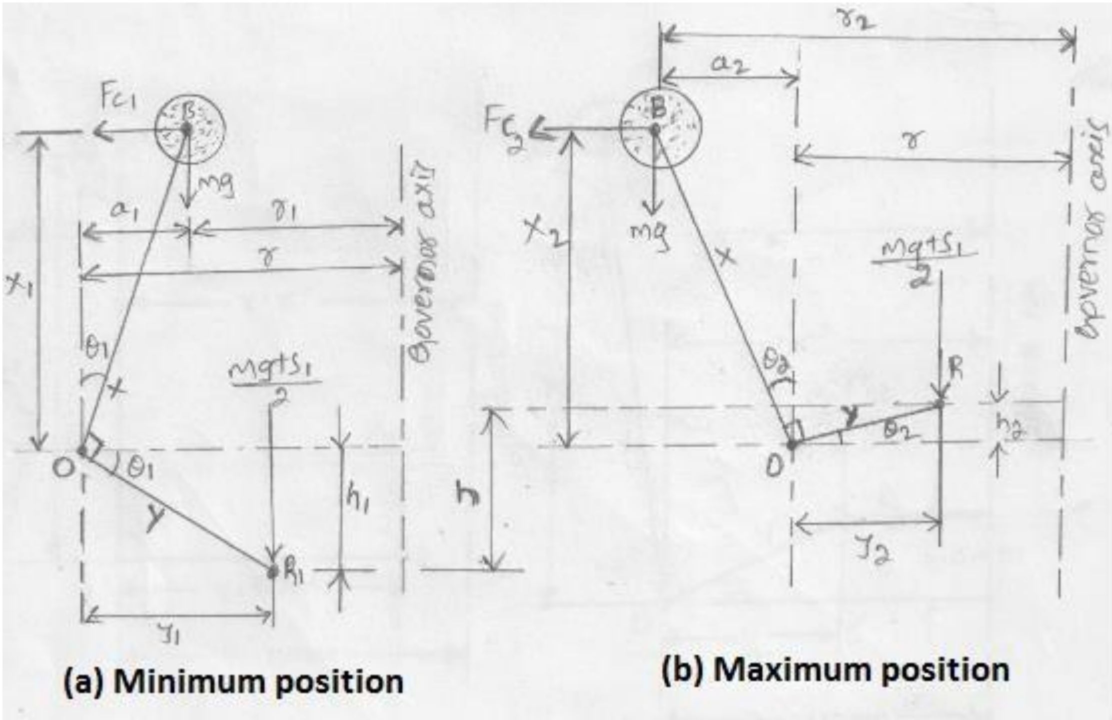


Fig 6.5 Hartnell Governor.

Now for minimum position, taking moments about point O, we get

Again for maximum position, taking moments about point O, we get

Subtracting equation (4) from equation (5),

We know that

Neglecting the obliquity effect of the arms and the moment due to weight of the balls ( ), we have for minimum position,

Similarly for maximum position,

Subtracting equation (6) from equation (7),

Notes :

1. Unless otherwise stated, the obliquity effect of the arms and the moment due to the weight of the balls is neglected, in actual practice.

2. When friction is taken into account, the weight of the sleeve may be replaced by

3. The centrifugal force for any intermediate position (*i*.*e*. between the minimum and maximum position) at a radius of rotation may be obtained as discussed below: Since the stiffness for a given spring is constant for all positions, therefore for minimum and intermediate position,

and for intermediate and maximum position,

–

From equation 9,10 and 11

Problems on Hartnell Governor

Problem 6.4

A Hartnell governor having a central sleeve spring and two right-angled bell crank levers moves between 290 rpm. and 310 rpm. for a sleeve lift of 15 mm. The sleeve arms and the ball arms are 80 mm and 120 mm respectively. The levers are pivoted at 120 mm from the governor axis and mass of each ball is 2.5 kg. The ball arms are parallel to the governor axis at the lowest equilibrium speed. Determine

1. Loads on the spring at the lowest and the highest equilibrium speeds, and

2. Stiffness of the spring.

1. Loads on the spring at the lowest and highest equilibrium speeds

Let S = spring load at lowest equilibrium speed, and S2 =
spring load at highest equilibrium speed.

Since the ball arms are parallel to governor axis at the lowest equilibrium speed (*i.e.* at *N*1 = 290 rpm), as shown in Fig. 6p.4(*a*),

Therefore

Centrifugal force at the minimum speed,

let us find the radius of rotation at the highest equilibrium speed, *i.e.* at *N*2 = 310 rpm. The position of ball arm and sleeve arm at the highest equilibrium speed is shown in Fig. 6p.4 (*b*).

Let

We know that

Centrifugal force at the maximum speed,

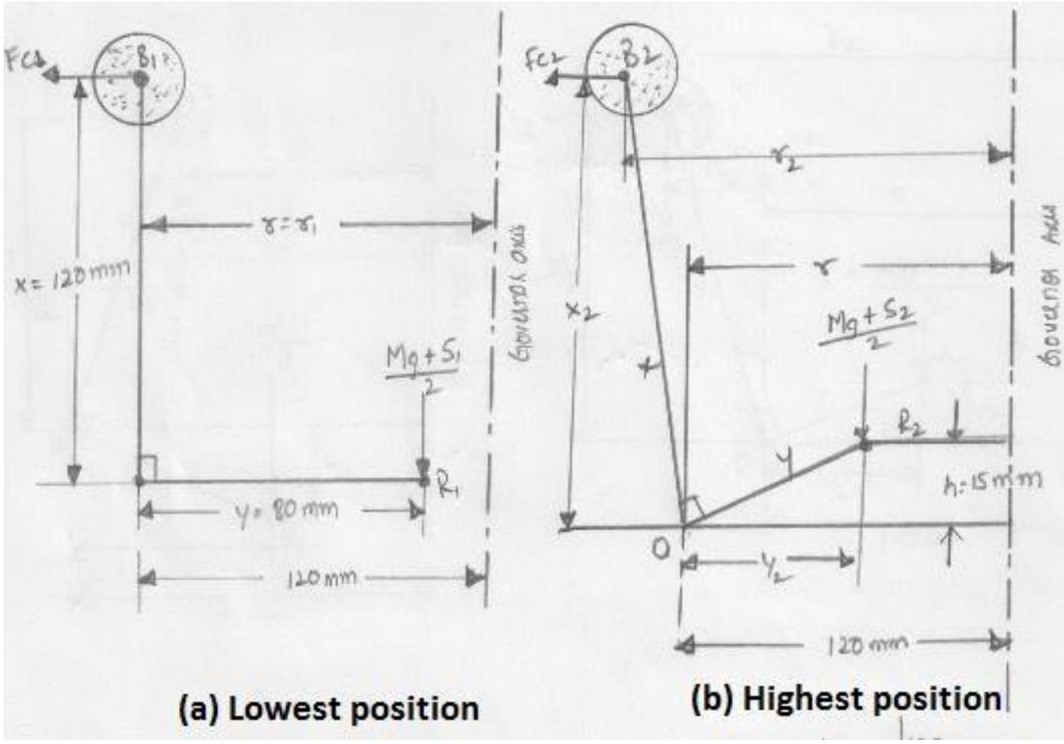


Fig 6p.4

Neglecting the obliquity effect of arms and the moment due to the weight of the balls, we have for lowest position,

–

2. Stiffness of the spring
We know that stiffness of the spring,

6.5

A spring loaded governor of the Hartnell type has arms of equal length. The masses rotate in a circle of 130 mm diameter when the sleeve is in the mid position and the ball arms are vertical. The equilibrium speed for this position is 450 r.p.m., neglecting friction. The maximum sleeve movement is to be 25 mm and the maximum variation of speed taking in account the friction to be 5 per cent of the mid position speed. The mass of the sleeve is 4 kg and the friction may be considered equivalent to 30 N at the sleeve. The power of the governor must be sufficient to overcome the friction by one per cent change of speed either way at mid-position. Determine, neglecting obliquity effect of arms;

1. The value of each rotating mass :
2. The spring stiffness in N/mm ; and
3. The initial compression of spring.

——   ——

1. Value of each rotating mass

Let m = Value of each rotating mass in kg, and

S = Spring force on the sleeve at mid position in N.

Since the change of speed at mid position to overcome friction is 1 per cent either way (*i.e.* ± 1%), therefore

Minimum speed at mid position,

and maximum speed at mid-position Centrifugal force at the minimum speed,

ω

and centrifugal force at the maximum speed,

We know that for minimum speed at mid position,

–

and for maximum speed at mid-position,

–

From equation (1) and (2)

2. Spring stiffness in N/mm

Let s = spring stiffness in N/mm.

Since the maximum variation of speed, considering friction is ± 5% of the mid-position speed, therefore,

Minimum speed considering friction,

and maximum speed considering friction,

Mnimum radius of rotation considering friction,

{

and maximum radius of rotation considering friction,

Centrifugal force at the minimum speed considering friction,

and centrifugal force at the maximum speed considering friction,

S_1 = Spring force at minimum speed considering friction, and
S_2 = Spring force at maximum speed considering friction.

Minimum speed considering friction,

Maximum speed considering friction,

Stiffness of the spring,

3. Initial compression of the spring

We know that initial compression of the spring

Problem6.6

In a spring loaded governor of the Hartnell type, the mass of each ball is 5 kg and the lift of the sleeve is 50 mm. The speed at which the governor begins to float is 240 rpm., and at this speed the radius of the ball path is 110 mm. The mean working speed of the governor is 20 times the range of speed when friction is neglected. If the lengths of ball and roller arm of the bell crank lever are 120 mm and 100 mm respectively. If the distance between the centre of pivot of bell crank lever and axis of governor spindle is 140 mm, determine the initial compression of the spring taking into account the obliquity of arms. If friction is equivalent to a force of 30 N at the sleeve, find the total alteration in speed before the sleeve begins to move from mid-position.

Initial compression of the spring taking into account the obliquity of arms

First of all, let us find out the maximum speed of rotation in rad/s. We know that mean working speed,

and range of speed, neglecting friction

Since the mean working speed is 20 times the range of speed, therefore

The minimum and maximum position of the governor balls is shown in Fig. 6p.5 (*a*) and (*b*) respectively.

r_2 = Maximum radius of rotation.

Lift of the sleeve,

–

–

We know that centrifugal force at the minimum speed,

and centrifugal force at the maximum speed,

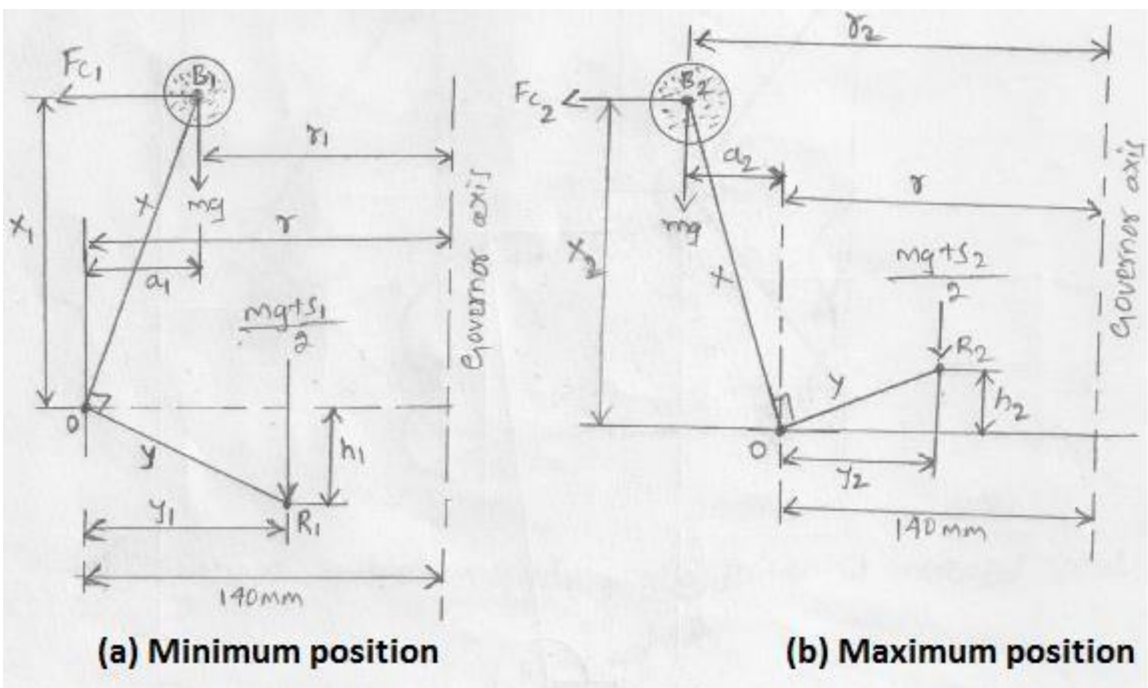


Fig 6p.5

Since the obliquity of arms is to be taken into account, therefore from the minimum position as shown in Fig. 6p.5 (*a*), Similarly, for the maximum position, as shown in Fig. 6p.5 (*b*),

Now taking moments about point O for the minimum position as shown in Fig. 6p.5 (a),

Similarly, taking moments about point O for the maximum position as shown in Fig. 6p.5(b),

We know that stiffness of the spring

3. Initial compression of the spring
We know that initial compression of the spring

Total alternation in speed when friction is taken into account

spring force for the mid-position, and mean angular speed

–

Speed when the sleeve begins to move downwards from the mid-position,

and speed when the sleeve begins to move upwards from the mid-position,

Alteration in speed

Controlling Force

It is the resultant of all the external forces which oppose the centrifugal force. It can be regarded as single radial inward force acting on the centre of ball. When the ball is in equilibrium the controlling force is equal, in magnitude, to the centrifugal force acting on the ball.

When a body rotates in a circular path, there is an inward radial force or centripetal force acting on it. In case of a governor running at a steady speed, the inward force acting on the rotating balls is known as ***controlling force.*** It is equal and opposite to the centrifugal reaction. Controlling force,
The controlling force is provided by the weight of the sleeve and balls as in Porter governor and by the spring and weight as in Hartnell governor (or spring controlled governor). When the graph between the controlling force (*F*C) as ordinate and radius of

rotation of the balls (r) as abscissa is drawn, then the graph obtained is known as ***controlling force diagram*****.** This diagram enables the stability and sensitiveness of the governor to be examined and also shows clearly the effect of friction.

Controlling Force Diagram for Porter Governor

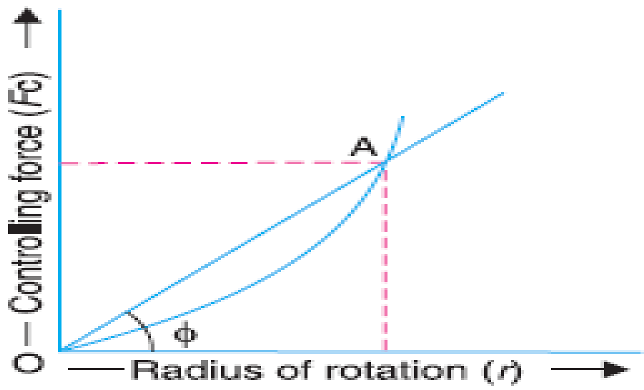


Fig 6.6

The controlling force diagram for a Porter governor is a curve as shown in Fig.

We know that controlling force,

—

Where Φ is the angle between the axis of radius of rotation and a line joining a given point (say *A*) on the curve to the origin *O*.

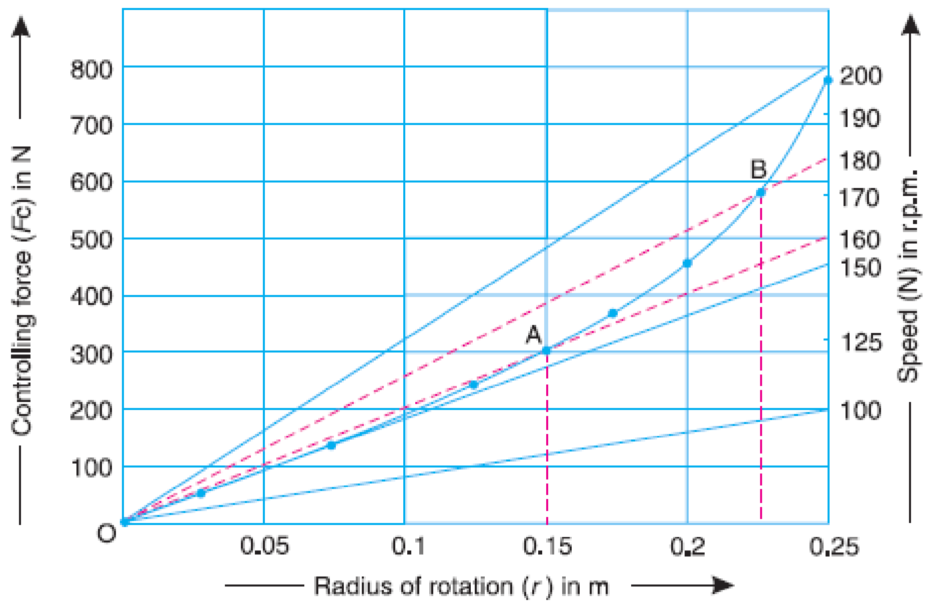
Notes : 1. In case the governor satisfies the condition for stability, the angle φ must increase with radius of rotation of the governor balls. In other words, the equilibrium speed must increase with the increase of radius of rotation of the governor balls.

2. For the governor to be more sensitive, the change in the value of φ over the change of radius of rotation should be as small as possible.

3. For the isochronous governor, the controlling force curve is a straight line passing through the origin. The angle φ will be constant for all values of the radius of rotation of the governor. From equation **(*1*)**

—

U~~sing~~ the above relation, the angle may be determined for different values of *N*



and the lines are drawn from the origin These lines enable the equilibrium speed corresponding to a given radius of rotation to be determined. Alternatively, the same results may be obtained more simply by setting-off a speed scale along any arbitrarily chosen ordinate. The controlling force is calculated for one constant radius of rotation and for different arbitrarily chosen values of speed. The values thus obtained are set-off along the ordinate that corresponds to the chosen radius and marked with the appropriate speeds.

Controlling Force Diagram for Spring-controlled Governors

The controlling force diagram for the spring controlled governors is a straight line, as shown in Fig. 6.6a. We know that controlling force,

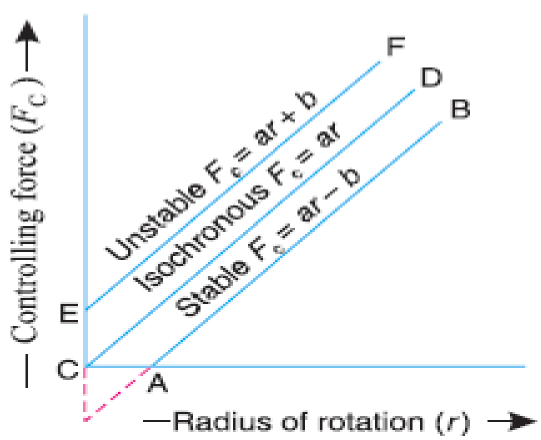


Fig 6.6a

Characteristics of Governors

Different governors can be compared on the basis of following characteristics.

Stability

A governor is said to be stable when for every speed within the working range there is a definite configuration i.e. there is only one radius of rotation of the governor balls at which the governor is in equilibrium. For a stable governor, if the equilibrium speed increases, the radius of governor balls must also increase.

Sensitiveness of Governors

If a governor operates between the speed limits N1 and N2, then sensitiveness is defined as the ratio of the mean speed to the difference between the maximum and minimum speeds. Thus,

N1 = Minimum equilibrium speed,
N2 = Maximum equilibrium speed, and

N = Mean equilibrium speed

Sensitiveness of the governor

6.6.3. Isochronous Governors

A governor is said to be ***isochronous*** when the equilibrium speed is constant (*i.e.* range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronism is the stage of infinite sensitivity.

Let us consider the case of a Porter governor running at speeds N_1 and N_2 rpm.

For isochronism, range of speed should be zero *i.e.* $N_2 - N_1 = 0$ or $N_2 = N_1$. Therefore from equations **(*1*)** and (***2***)**,** $h_1 = h_2$, which is impossible in case of a Porter governor. Hence a ***Porter governor cannot be isochronous.***

Now consider the case of a Hartnell governor running at speeds N_1 and N_2 r.p.m.

—

For isochronism, $N_2 = N_1$. Therefore from equations (3) and (4),

Note : The isochronous governor is not of practical use because the sleeve will move to one of its extreme positions immediately the speed deviates from the isochronous speed.

6.6.5 Hunting

Hunting is the name given to a condition in which the speed of the engine controlled by the governor fluctuates continuously above and below the mean speed. It is caused by a governor which is too sensitive and which, therefore, changes by large amount the supply of fuel to the engine.

The following points, for the stability of spring-controlled governors, may be noted

1. For the governor to be stable, the controlling force (F_C) must increase as the radius of rotation (r) increases,

i.e. F_C / r must increase as r increases. Hence the controlling force line *AB* when produced must intersect the controlling force axis below the origin, as shown in Fig. 6.6a. The relation between the controlling force (F_C) and the radius of rotation (r) for the ***stability*** of spring controlled governors is given by the following equation

where a and b are constants

2. The value of b in equation (i) may be made either zero or positive by increasing the initial tension of the spring. If b is zero, the controlling force line *CD* passes through the origin and the governor becomes ***isochronous*** because F_C /r will remain constant for all radii of rotation.

The relation between the controlling force and the radius of rotation, for an ***isochronous governor*** is, therefore,

3. If b is greater than zero or positive, then F_C /r decreases as r increases, so that the equilibrium speed of the governor decreases with an increase of the radius of rotation of balls, which is impracticable Such a governor is said to be *unstable* and the relation between the controlling force and the radius of rotation is, therefore

6.5.5 Effort and Power of a Governor

The effort of a governor is the mean force exerted at the sleeve for a given percentage change of speed* (or lift of the sleeve). It may be noted that when the governor is running steadily, there is no force at the sleeve. But, when the speed changes, there is a resistance at the sleeve which opposes its motion. It is assumed that this resistance which is equal to the effort, varies uniformly from a

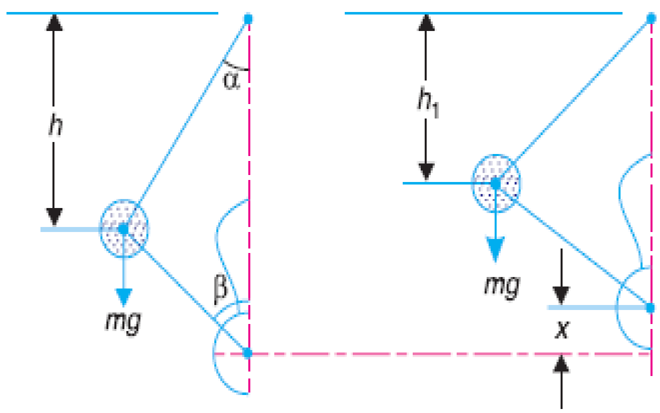
maximum value to zero while the governor moves into its new position of equilibrium. The power of a governor is the work done at the sleeve for a given percentage change of speed. It is the product of the mean value of the effort and the distance through which the sleeve moves.

i.e., Power = Mean effort × lift of sleeve

Effort and Power of a Porter Governor

The effort and power of a Porter governor may be determined as discussed below.
Let *N* = Equilibrium speed corresponding to the configuration as shown in Fig. 6.7 (*a*), and

The equilibrium position of the governor at the increased speed is shown in Fig. 6.7 (*b*).



(*a*) Position at equilibrium speed. (*a*) Position at increased speed.

Fig 6.7

We have discussed that when the speed is N rpm., the sleeve load is M.g. Assuming that the angles α and β are equal, so that q = 1, then the height of the governor, When the increase of speed takes place, a downward force P will have to be exerted on the sleeve in order to prevent the sleeve from rising. If the speed increases to (1 + c) N rpm. and the height of the governor remains the same, the load on the sleeve increases to $M_1.g$. Therefore

Equating equation (1) and (2) we have

A little consideration will show that $(M_1 - M)g$ is the downward force which must be applied in order to prevent the sleeve from rising as the speed increases. It is the same force which acts on the governor sleeve immediately after the increase of speed has taken place and before the sleeve begins to move. When the sleeve takes the new position as shown in Fig. 6.7 (*b*), this force gradually diminishes to zero.

Let
P = Mean force exerted on the sleeve during the increase in speed or the
effort of the governor.

If F is the frictional force (in newtons) at the sleeve, then

We have already discussed that the power of a governor is the product of the governor effort and the lift of the sleeve.

If the height of the governor at speed N is h and at an increased speed $(1 + c)\ N$ is h_1, then

As there is no resultant force at the sleeve in the two equilibrium positions, therefore

Or

We know that *

Substituting the values of P and x in equation (5), we have

When speed increases to (1 + c) N and height of the governor remains the same, then

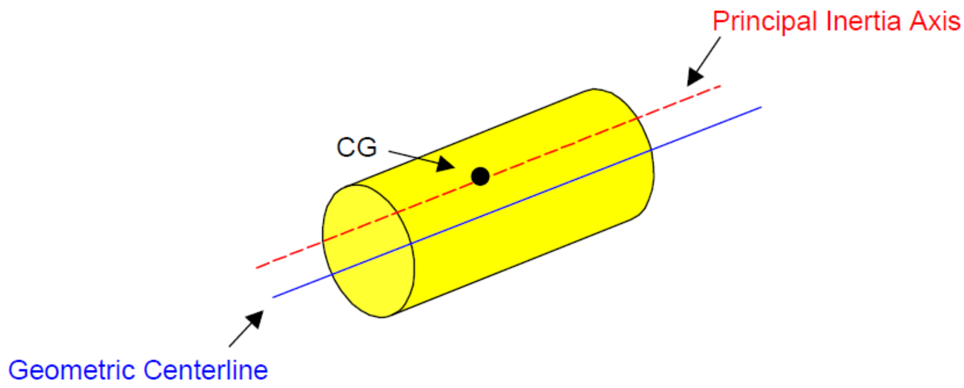
From equations (8) and (9), we have

The equation (6) for the lift of the sleeve becomes

INTRODUCTION:

When man invented the wheel, he very quickly learnt that if it wasn"t completely round and if it didn"t rotate evenly about it"s central axis, then he had a problem!
What the problem he had?
The wheel would vibrate causing damage to itself and it"s support mechanism and in severe cases, is unusable.
A method had to be found to minimize the problem. The mass had to be evenly distributed about the rotating centerline so that the resultant vibration was at a minimum.

The condition which exists in a rotor when vibratory force or motion is imparted to its bearings as a result of centrifugal forces is called unbalance or the uneven distribution of mass about a rotor"s rotating centerline.



The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:
Static Unbalance – where the PIA is displaced parallel to the geometric centerline. (Shown above)
Couple Unbalance – where the PIA intersects the geometric centerline at the center of gravity. (CG)
Dynamic Unbalance – where the PIA and the geometric centerline do not coincide or touch.
The most common of these is dynamic unbalance.

In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. slight variation in the density of the material or
2. inaccuracies in the casting or
3. inaccuracies in machining of the parts.

1. A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2. As machines get bigger and go faster, the effect of the unbalance is much more severe.
3. The force caused by unbalance increases by the square of the speed.
4. If the speed is doubled, the force quadruples; if the speed is tripled the force increases

by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

alancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.

The objectives of balancing an engine are to ensure:

1. That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
2. That the couples involved in acceleration of the different moving parts balance each other.

a) Static Balancing:

i) Static balancing is a balance of forces due to action of gravity.

ii) A body is said to be in static balance when its centre of gravity is in the axis of rotation.

b)

i) Dynamic balance is a balance due to the action of inertia forces.

ii) A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.

iii) The conditions of dynamic balance are met, the conditions of static balance are also met.

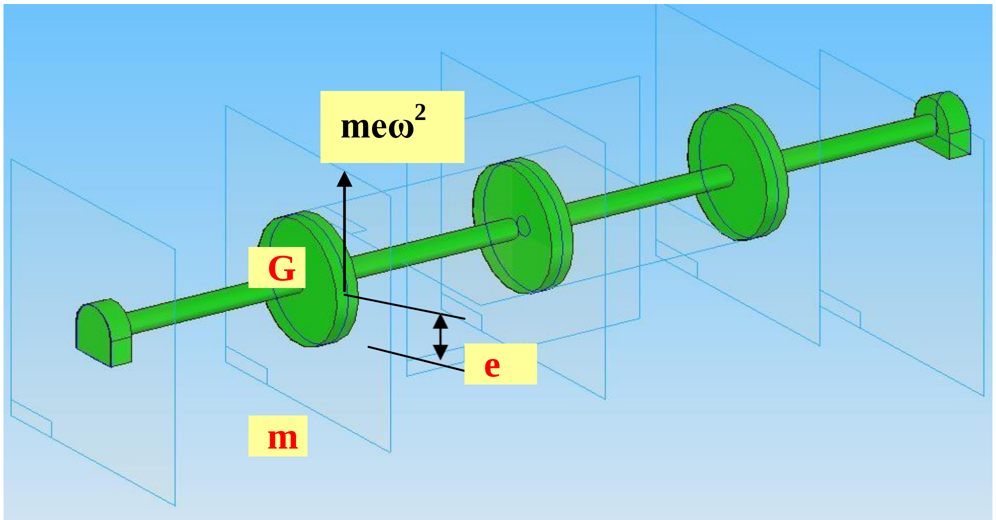
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.



The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members

Balancing of rotating masses can be of

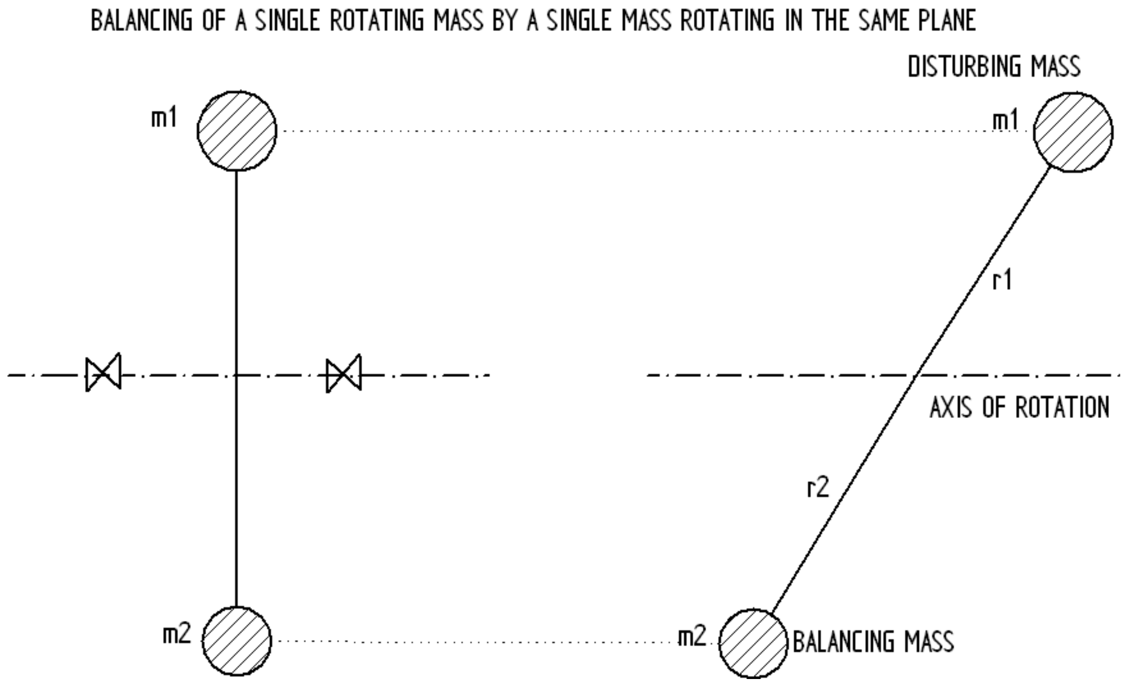
1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of several masses rotating in the same plane
4. Balancing of several masses rotating in different planes

STATIC BALANCING:

A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation

When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

BALANCING OF A SINGLE ROTATING MASS BY A SINGLE MASS ROTATING IN THE SAME PLANE



Consider a disturbing mass m_1 which is attached to a shaft rotating at ω rad/s.
Let

r_1 = *radius of rotation of the mass* m_1

= **distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1**

The centrifugal force exerted by mass m_1 on the shaft is given by,

$$\mathbf{F_{c1} = m_1 \omega^2 r_1} \text{ ------------------- } \mathbf{(1)}$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force F_{c1} , a balancing mass m_2 may be attached in the same plane of rotation of the disturbing mass m_1 such that the centrifugal forces due to the two masses are equal and opposite.

Let,

$\mathbf{r_2} = \mathbf{radius}$ *of rotation of the mass* m_2
$= \mathbf{distance\ between\ the\ axis\ of\ rotation\ of\ the\ shaft\ and}$
$\mathbf{the\ centre\ of\ gravity\ of\ the\ mass\ m_2}$

Therefore the centrifugal force due to mass m_2 will be,

$$\mathbf{F_{c2} = m_2\, \omega^2\, r_2} ------------------- \mathbf{(2)}$$

Equating equations (1) and (2), we get

$$F_{c1} = F_{c2}$$
$$m_1\, \omega^2\, r_1 = m_2\, \omega^2\, r_2 \quad \text{or} \quad m_1\, r_1 = m_2\, r_2 ----------------- (3)$$

The product $\mathbf{m}_2 \mathbf{r}_2$ can be split up in any convenient way. As for as possible the radius of rotation of mass m_2 that is r_2 is generally made large in order to reduce the balancing mass m_2.

BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.

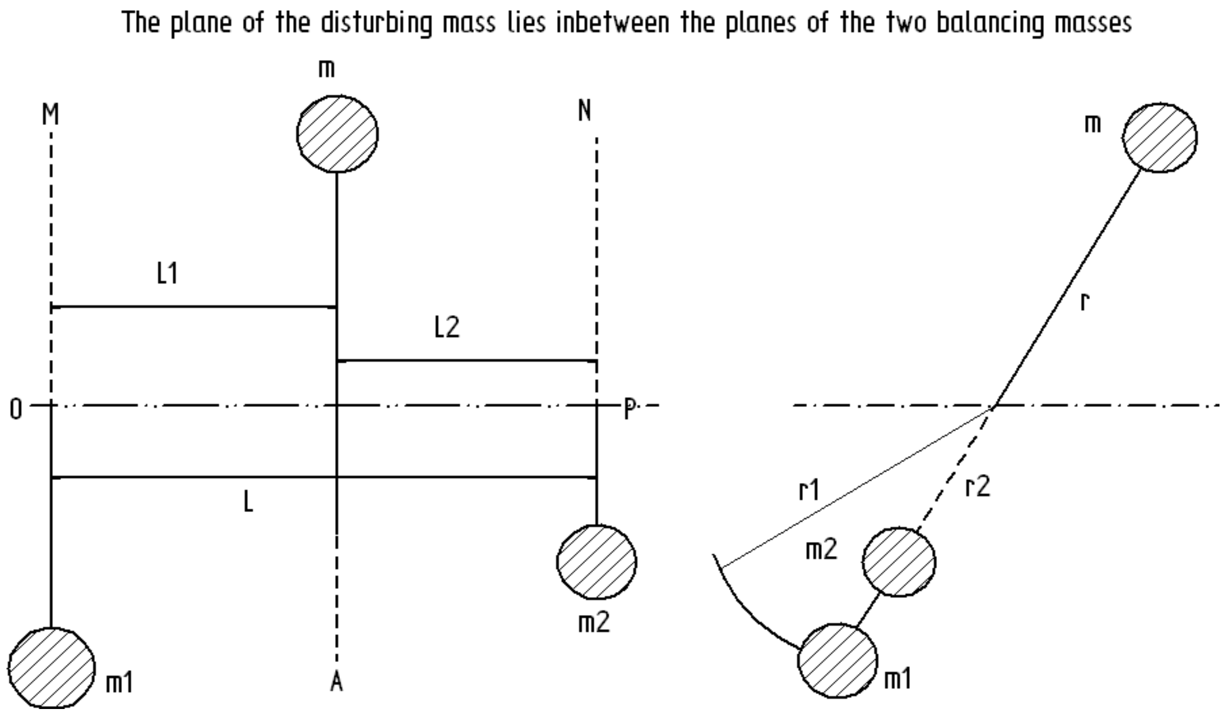
There are two possibilities while attaching two balancing masses:

1. *Τηε πλανε οφ τηε διστυρβινγ μασσ μαψ βε ιν βετωεεν τηε πλανεσ οφ τηε τω ο βαλανχινγ μασσεσ.*

2. The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.



Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses m_1 and m_2 lying in two different planes M and N which are parallel to the plane A as shown.

Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, M and N respectively.
Let L_1, L_2 and L be the distance between A and M, A and N, and M and N respectively.
Now,
The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\ \omega^2 r \text{ ------------------- } \mathbf{(1)}$$

Similarly,
The centrifugal force exerted by the mass m_1 in plane M will be,

$$F_{c1} = m_1 \omega^2 r_1 \text{ ------------------- } \mathbf{(2)}$$

And the centrifugal force exerted by the mass m_2 in plane N will be,

$$F_{c2} = m_2 \omega^2 r_2 \text{-------------------}(3)$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$
$$\text{or } m\omega^2 r = m_1 \omega^2 r_1 + m_2 \omega^2 r_2$$
$$\text{i.e. } mr = m_1 r_1 + m_2 r_2 \text{------------------}(4)$$

Now, to determine the magnitude of balancing force in the plane „M‟ or the dynamic force at the bearing „O‟ of a shaft, take moments about „ P ‟ which is the point of intersection of the plane N and the axis of rotation.

Therefore,

$$F_{c1} x L = F_c x L_2$$
$$\text{or } m_1 \omega^2 r_1 \text{ x } L = m\omega^2 r \text{ x} L_2$$

Therefore,

$$m_1 r_1 L = mrL_2 \quad \text{or } m_1 r_1 = mr\frac{L_2}{L} \text{---------}(5)$$

Similarly, in order to find the balancing force in plane „N‟ or the dynamic force at the bearing „P‟ of a shaft, take moments about „ O ‟ which is the point of intersection of the plane M and the axis of rotation.

Therefore,

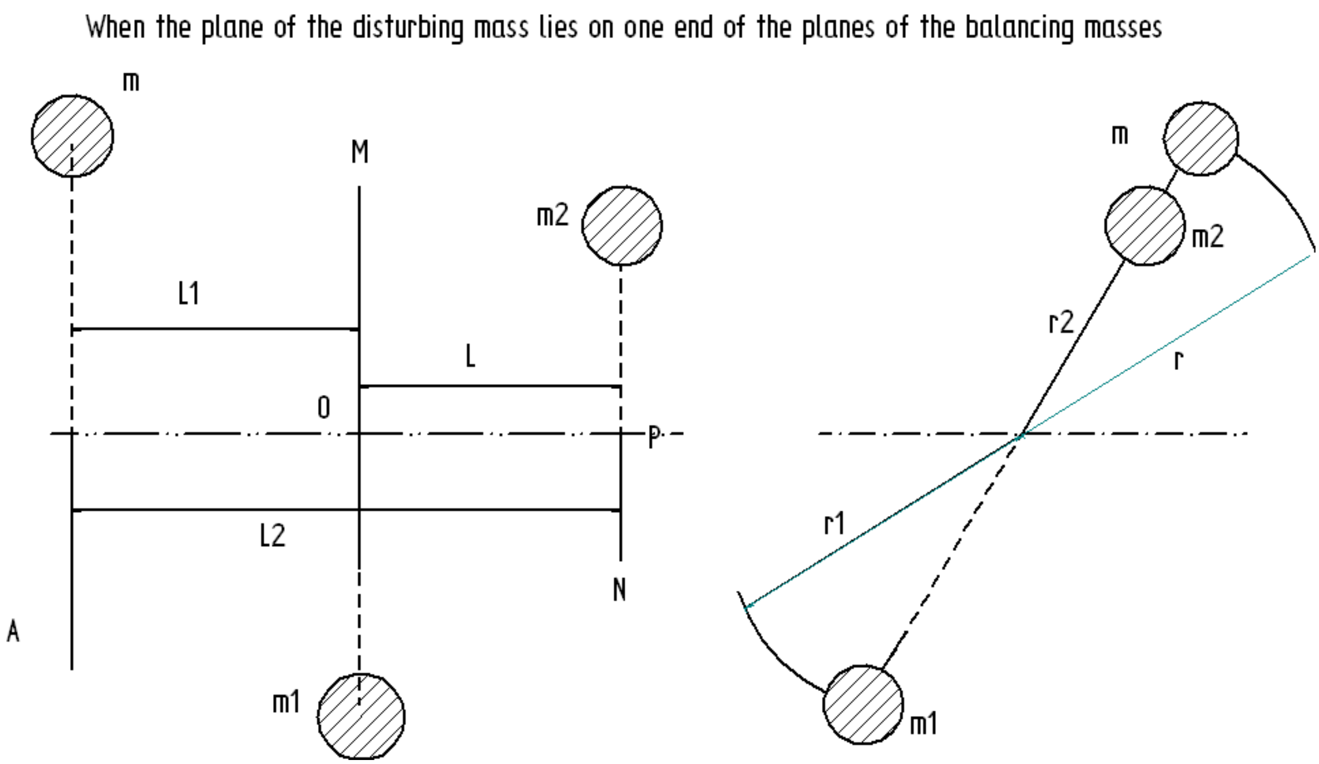
$$F_{c2} x L = F_c x L_1$$
$$\text{or } m_2 \omega^2 r_2 \text{ x } L = m\omega^2 r \text{ x} L_1$$

Therefore,

$$m_2 r_2 L = mrL_1 \quad \text{or } m_2 r_2 = mr\frac{L_1}{L} \text{---------}(6)$$

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.



For static balancing,

$$F_{c1} = F_c + F_{c2}$$

$$\text{or } m_1 \omega^2 r_1 = m\omega^2 r + m_2 \omega^2 r_2$$

$$\text{i.e. } m_1 r_1 = mr + m_2 r_2 \text{ ------------------ } (1)$$

For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.

To find the balancing force in the plane „M‟ or the dynamic force at the bearing „O‟ of a shaft, take moments about „P‟. i.e.

$$F_{c1} \times L = F_c \times L_2$$

$$\text{or} \quad m_1 \omega^2 r_1 \times L = m \omega^2 r \times L_2$$

Therefore,

$$m_1 r_1 L = m r L_2 \quad \text{or} \quad m_1 r_1 = m r \frac{L_2}{L} \text{ ---------(2)}$$

Similarly, to find the balancing force in the plane „N" , take moments about „O", i.e.,

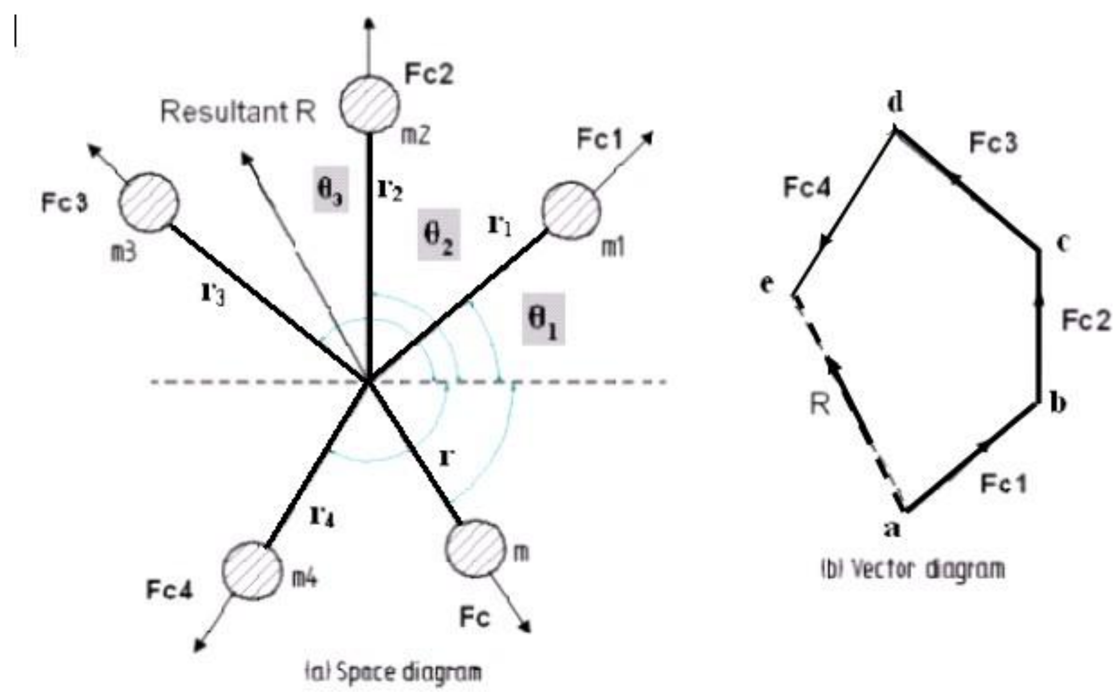
$$F_{c2} \times L = F_c \times L_1$$

$$\text{or} \quad m_2 \omega^2 r_2 \times L = m \omega^2 r \times L_1$$

Therefore,

$$m_2 r_2 L = m r L_1 \quad \text{or} \quad m_2 r_2 = m r \frac{L_1}{L} \text{ ---------(3)}$$

BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE



BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If m_1, m_2, m_3 and m_4 are the masses revolving at radii r_1, r_2, r_3 and r_4 respectively in the same plane.
The centrifugal forces exerted by each of the masses are F_{c1}, F_{c2}, F_{c3} and F_{c4} respectively.
Let F be the vector sum of these forces. i.e.

$$
\begin{aligned}
F &= F_{c1} + F_{c2} + F_{c3} + F_{c4} \\
&= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 ----------(1)
\end{aligned}
$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight (balance weight) of mass „m‟ at radius „r‟ to balance the rotor so that,

$$m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 + m\ \omega^2 r = 0 ---------- (2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m\ r = 0 ----------------- (3)$$

The magnitude of either „m‟ or „r‟ may be selected and the other can be calculated.
In general, if $\sum \mathbf{m_i\ r_i}$ is the vector sum of $\mathbf{m}_1 \mathbf{r}_1$, $\mathbf{m}_2 \mathbf{r}_2$, $\mathbf{m}_3 \mathbf{r}_3$, $\mathbf{m}_4 \mathbf{r}_4$ etc, then,

$$\sum m_i r_i + mr = 0 --------(4)$$

The above equation can be solved either analytically or graphically.

1. *Αναλψτιχαλ Μετηοδ:*

Procedure:
Step 1: Find out the centrifugal force or the product of mass and its radius of rotation exerted by each of masses on the rotating shaft, since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.
Step 2: Resolve these forces into their horizontal and vertical components and find their sums. i.e.,

$$\textit{Sum of the horizontal components}$$

$$= \sum_{i=1}^{n} \mathbf{m_i r_i \cos\theta_i} = \mathbf{m_1 r_1 \cos\theta_1} + \mathbf{m_2 r_2 \cos\theta_2} + \mathbf{m_3 r_3 \cos\theta_3} + --------$$

$$\textit{Sumof the vertical components}$$

$$= \sum_{i=1}^{n} \mathbf{m_i r_i \sin\theta_i} = \mathbf{m_1 r_1 \sin\theta_1} + \mathbf{m_2 r_2 \sin\theta_2} + \mathbf{m_3 r_3 \sin\theta_3} + --------$$

Step 3: Determine the magnitude of the resultant centrifugal force

$$R=\sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2+\left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

Step 4: If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta=\frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i}$$

Step 5: The balancing force is then equal to the resultant force, but in opposite direction.
Step 6: Now find out the magnitude of the balancing mass, such that

$$\boldsymbol{R = mr}$$

Where, m = balancing mass and r = its radius of rotation

2. *Γραπηιχαλ Μετηοδ:*

Step 1:
Draw the space diagram with the positions of the several masses, as shown.

Step 2:
Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:
Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.
Let ab, bc, cd, de represents the forces F_{c1}, F_{c2}, F_{c3} and F_{c4} on the vector diagram.
Draw „ab‟ parallel to force F_{c1} of the space diagram, at „b‟ draw a line parallel to force F_{c2}. Similarly draw lines cd, de parallel to F_{c3} and F_{c4} respectively.

Step 4:
As per polygon law of forces, the closing side „ae‟ represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:
The balancing force is then , equal and opposite to the resultant force.

Step 6:

Determine the magnitude of the balancing mass (m) at a given radius of rotation (r), such that,

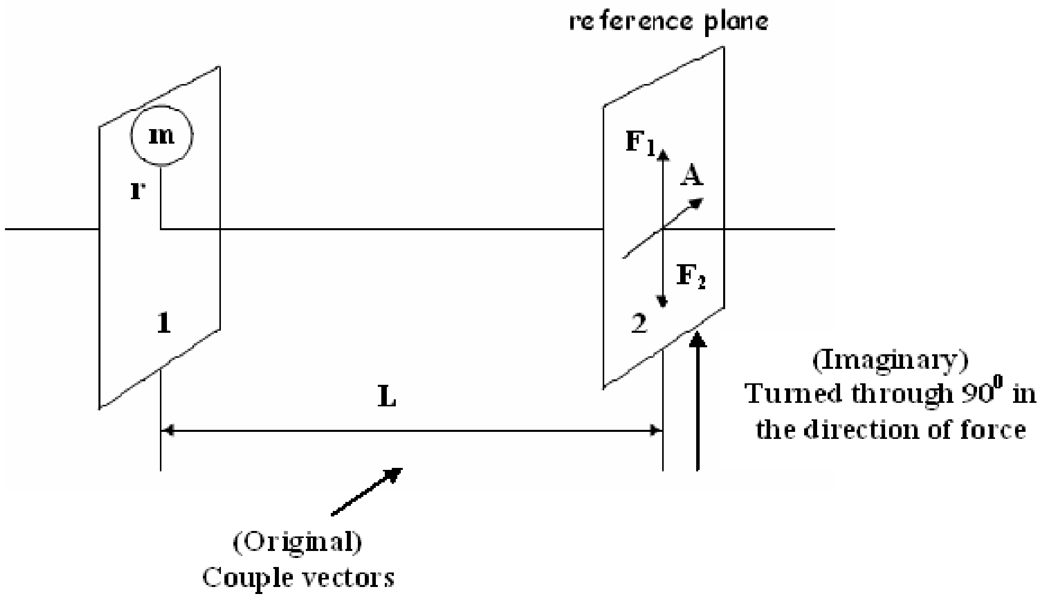
$$F_c = m\omega^2 r$$

or

$$mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.

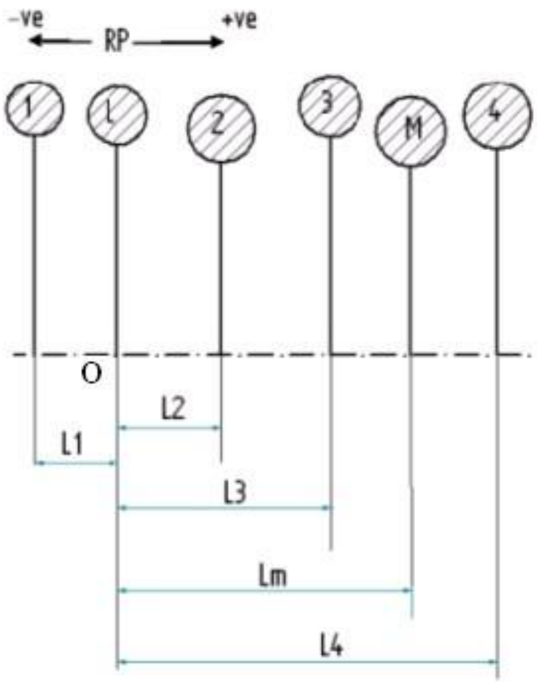


When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.
In order to have a complete balance of the several revolving masses in different planes,

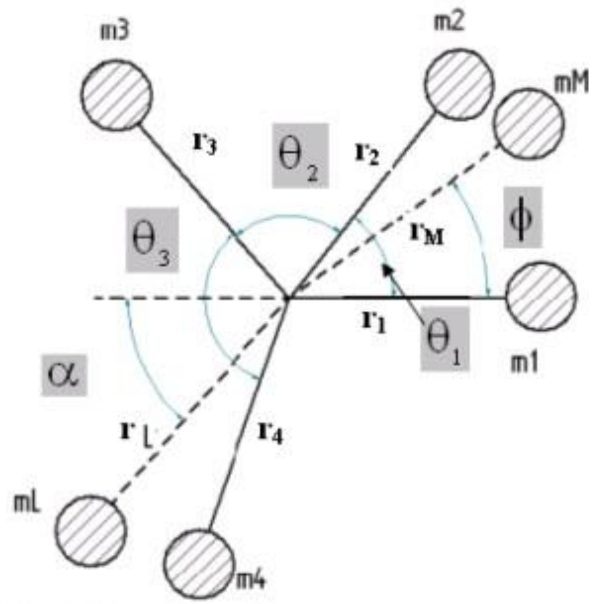
1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:
Consider four masses m_1, m_2, m_3 and m_4 attached to the rotor at radii r_1, r_2, r_3 and r_4 respectively. The masses m_1, m_2, m_3 and m_4 rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses



(b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at „O‟ so that the distance of the planes 1, 2, 3 and 4 from „O‟ are L_1, L_2 , L_3 and L_4 respectively. The reference plane chosen is plane „L‟. Choose another plane „M‟ between plane 3 and 4 as shown.

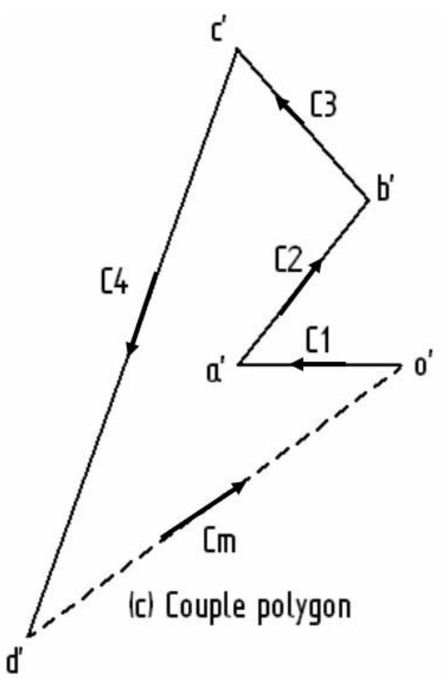
Plane „M‟ is at a distance of L_m from the reference plane „L‟. The distances of all the other planes to the left of „L‟ may be taken as negative(-ve) and to the right may be taken as positive (+ve).

The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained by following the steps given below.

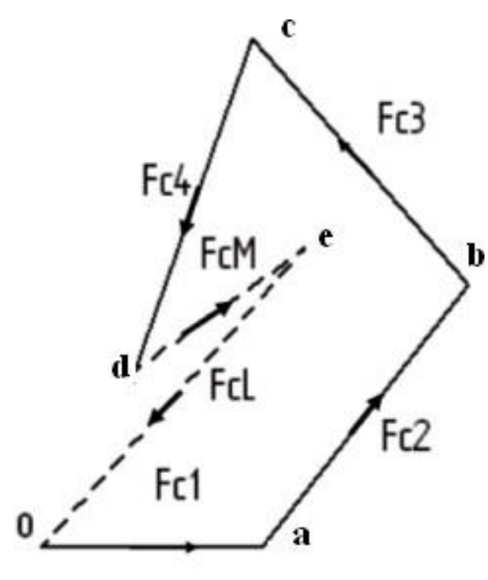
Step 1:
Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

Plane 1	Mass (m) 2	Radius (r) 3	Centrifugal force/ω^2 (m r) 4	Distance from Ref. plane „L‟(L) 5	Couple/ ω^2 (m r L) 6
1	m_1	r_1	$m_1\ r_1$	$-\ L_1$	$-\ m_1\ r_1\ L_1$
L	m_L	r_L	$m_L\ r_L$	0	0
2	m_2	r_2	$m_2\ r_2$	L_2	$m_2\ r_2\ L_2$
3	m_3	r_3	$m_3\ r_3$	L_3	$m_3\ r_3\ L_3$
M	m_M	r_M	$m_M\ r_M$	L_M	$m_M\ r_M\ L_M$
4	m_4	r_4	$m_4\ r_4$	L_4	$m_4\ r_4\ L_4$

Step 2:
Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)
Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be „$m_M\ r_M\ L_M$‟.



(c) Couple polygon



(d) Force polygon

The vector d ‟o‟ on the couple polygon represents the balanced couple. Since the balanced couple C_M is proportional to $m_M\ r_M\ L_M$, therefore,

$$C_M = m_M\, r_M\, L_M = \text{vector } d'o'$$

$$\text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M\, L_M}$$

From this the value of m_M in the plane M can be determined and the angle of inclination ϕ of this mass may be measured from figure (b).

Step 3:
Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with „$m_M\, r_M$". The closing vector will be „$m_L\, r_L$". This represents the balanced force. Since the balanced force is proportional to „$m_L\, r_L$" ,

$$m_L\, r_L = \text{vector eo}$$

$$\text{or} \quad m_L = \frac{\text{vector eo}}{r_L}$$

From this the balancing mass m_L can be obtained in plane „L" and the angle of inclination of this mass with the horizontal may be measured from figure (b).

Problems and solutions

Four masses A, B, C and D are attached to a shaft and revolve in the same plane. The masses are 12 kg, 10 kg, 18 kg and 15 kg respectively and their radii of rotations are 40 mm, 50 mm, 60 mm and 30 mm. The angular position of the masses B, C and D are 60^0 , 135^0 and 270^0 from mass A. Find the magnitude and position of the balancing mass at a radius of 100 mm.

Solution:
Given:

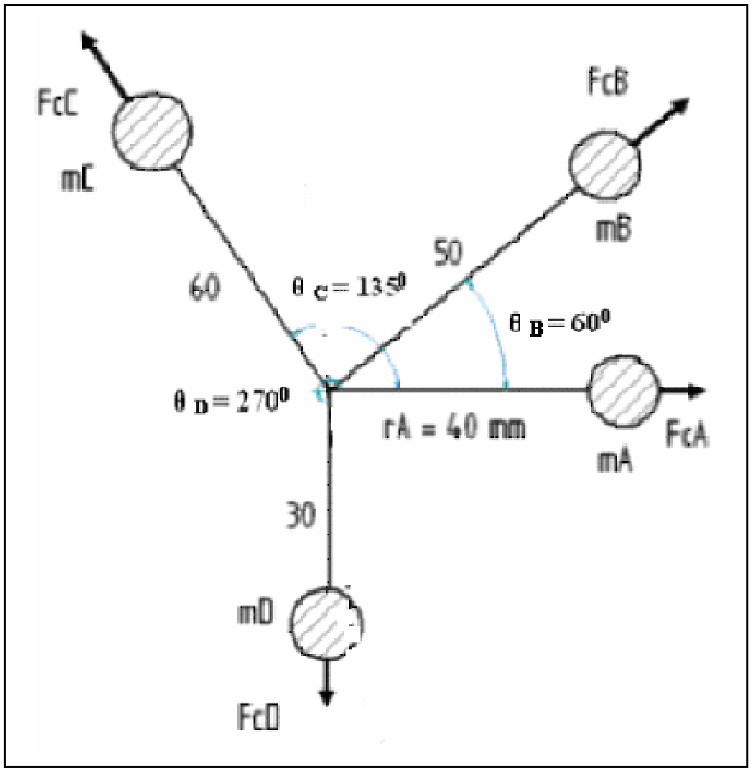
Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 12 kg (reference mass)	r_A = 0.04 m	$m_A r_A$ = 0.48 kg-m	$\theta_A = 0^0$
m_B = 10 kg	r_B = 0.05 m	$m_B r_B$ = 0.50 kg-m	$\theta_B = 60^0$
m_C = 18 kg	r_C = 0.06 m	$m_C r_C$ = 1.08 kg-m	$\theta_C = 135^0$
m_D = 15 kg	r_D = 0.03 m	$m_D r_D$ = 0.45 kg-m	$\theta_D = 270^0$

To determine the balancing mass „m" at a radius of r = 0.1 m.

The problem can be solved by either analytical or graphical method.

Analytical Method:

Step 1:
Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.



Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.

Resolve the centrifugal forces horizontally and vertically and find their sum.
Resolving m_Ar_A, m_Br_B, m_Cr_C and m_Dr_D horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 0.48 \text{ x } \cos 0^0 + 0.50 \text{ x } \cos 60^0 + 1.08 \text{ x } \cos 135^0 + 0.45 \text{ x } \cos 270^0$$

$$= 0.48 + 0.25 + (-0.764) + 0 = -\ 0.034 \text{ kg} - \text{m} ---------- (1)$$

Resolving m_Ar_A, m_Br_B, m_Cr_C and m_Dr_D vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 0.48 \text{ x } \sin 0^0 + 0.50 \text{ x } \sin 60^0 + 1.08 \text{ x } \sin 135^0 + 0.45 \text{ x } \sin 270^0$$

$$= 0 + 0.433 + 0.764 + (-0.45) = 0.747 \text{ kg}-\text{m} \quad ----------(2)$$

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(-0.034)^2 + (0.747)^2} = 0.748 \text{ kg} - \text{m}$$

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = mr = 0.748 \text{kg} - \text{m}$$

$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{0.748}{0.1} = 7.48 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Determine the position of the balancing mass „m".
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{0.747}{-0.034} = -21.97$$

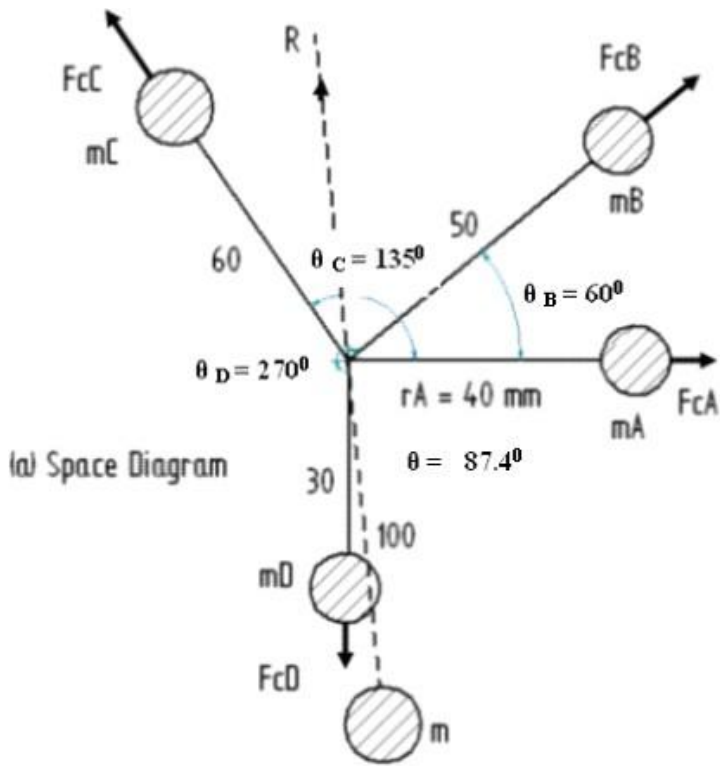
$$\text{and } \theta = -87.4^0 \text{ or } 92.6^0$$

Remember ALL STUDENTS TAKE COPY i.e. in first quadrant all angles **(sin θ, cos θ and tan θ)** are positive, in second quadrant only **sin** θ is positive, in third quadrant only **tan** θ is positive and in fourth quadrant only **cos** θ is positive.

Since numerator is positive and denominator is negative, the resultant force makes with the horizontal, an angle (measured in the counter clockwise direction)

$$\theta = 92.6^0$$

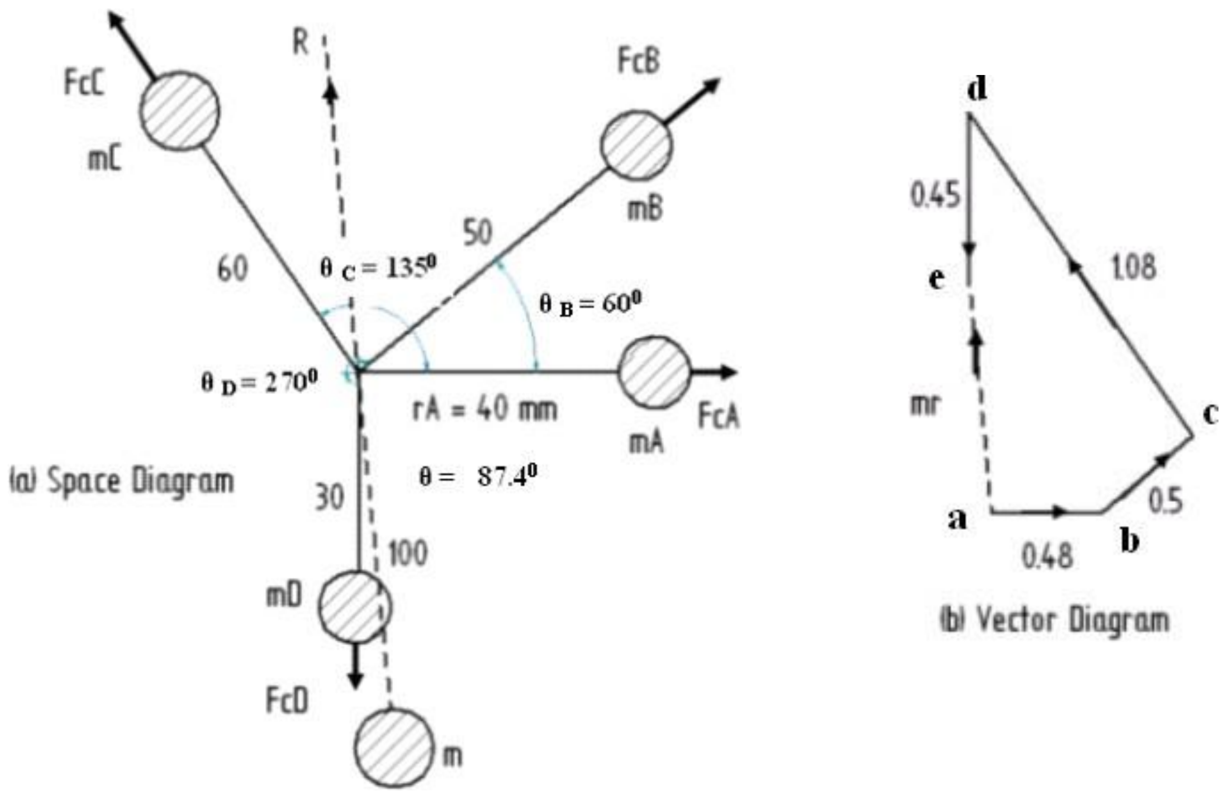
The balancing force is then equal to the resultant force, but in opposite direction.
The balancing mass „m" lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.



Graphical Method:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles(Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).



Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line „ab‟ parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At „b‟ draw a line „bc‟ parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines „cd‟, „de‟ parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side „ae‟ represents the resultant force „R‟ in magnitude and direction as shown on the vector diagram.

Στεπ 3:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

Therefore, $m = \frac{R}{r} = 7.48\,\text{kg}$ Ans

The balancing mass „m‟ lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.

The four masses A, B, C and D are 100 kg, 150 kg, 120 kg and 130 kg attached to a shaft and revolve in the same plane. The corresponding radii of rotations are 22.5 cm, 17.5 cm, 25 cm and 30 cm and the angles measured from A are 45^0, 120^0 and 255^0. Find the position and magnitude of the balancing mass, if the radius of rotation is 60 cm.

Solution:

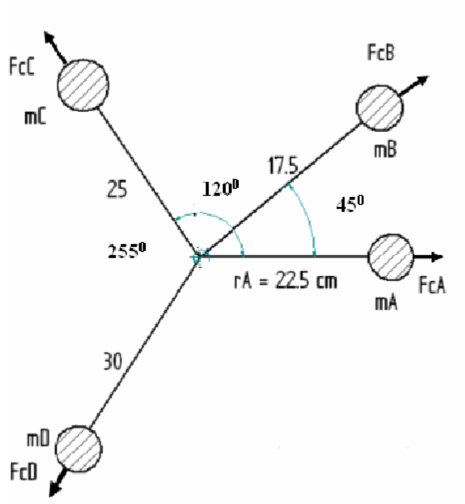
Analytical Method:

Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 100 kg (reference mass)	r_A = 0.225 m	$m_A r_A$ = 22.5 kg-m	$\theta_A = 0^0$
m_B = 150 kg	r_B = 0.175 m	$m_B r_B$ = 26.25 kg-m	$\theta_B = 45^0$
m_C = 120 kg	r_C = 0.250 m	$m_C r_C$ = 30 kg-m	$\theta_C = 120^0$
m_D = 130 kg	r_D = 0.300 m	$m_D r_D$ = 39 kg-m	$\theta_D = 255^0$
m =?	r = 0.60		θ = **?**

Step 1:
Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr‟ can be calculated and tabulated.



Resolve the centrifugal forces horizontally and vertically and find their sum.

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 22.5 \text{ x } \cos 0^0 + 26.25 \text{ x } \cos 45^0 + 30 \text{ x } \cos 120^0 + 39 \text{ x } \cos 255^0$$

$$= 22.5 + 18.56 + (-15) + (-10.1) = 15.97 \text{ kg} - \text{m} \quad ----------(1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 22.5 \text{ x } \sin 0^0 + 26.25 \text{ x } \sin 45^0 + 30 \text{ x } \sin 120^0 + 39 \text{ x } \sin 255^0$$

$$= 0 + 18.56 + 25.98 + (-37.67) = 6.87 \text{ kg} - \text{m} \quad ----------(2)$$

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(15.97)^2 + (6.87)^2} = 17.39 \text{ kg} - \text{m}$$

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = m\,r = 17.39 \text{ kg} - \text{m}$$

$$\textbf{Therefore,} \quad m = \frac{R}{r} = \frac{17.39}{0.60} = 28.98 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Determine the position of the balancing mass „m‟.
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{6.87}{15.97} = \mathbf{0.4302}$$

and

$$\boldsymbol{\theta = 23.28^{0}}$$

The balancing mass „m‟ lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203.28^{0}$ angle measured in the counter clockwise direction.

Graphical Method:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr‟ can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles (Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).

Draw a line „ab‟ parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At „b‟ draw a line „bc‟ parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines „cd‟, „de‟ parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side „ae‟ represents the resultant force „R‟ in magnitude and direction as shown on the vector diagram.

Στεπ 4:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore, } m = \frac{R}{r} = 29\,\text{kg Ans}$$

The balancing mass „m‟ lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\boldsymbol{\theta = 203^{0}}$ angle measured in the counter clockwise direction.

A rotor has the following properties.

Mass	magnitude	Radius	Angle	Axial distance from first mass
1	9 kg	100 mm	$\theta_A = 0^0$	-
2	7 kg	120 mm	$\theta_B = 60^0$	160 mm
3	8 kg	140 mm	$\theta_C = 135^0$	320 mm
4	6 kg	120 mm	$\theta_D = 270^0$	560 mm

If the shaft is balanced by two counter masses located at 100 mm radii and revolving in planes midway of planes 1 and 2, and midway of 3 and 4, determine the magnitude of the masses and their respective angular positions.

Solution:

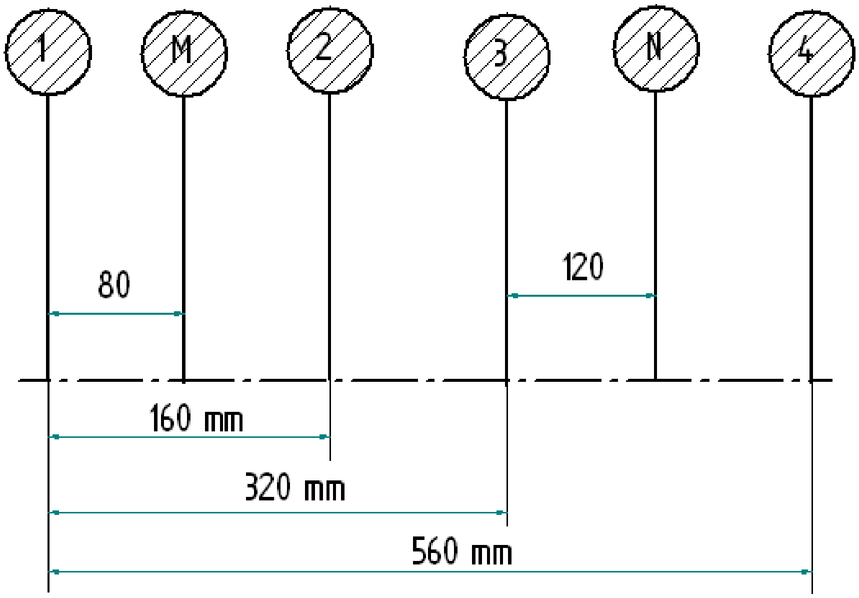
Analytical Method:

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „M" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
1	9.0	0.10	$m_1 r_1 = 0.9$	-0.08	-0.072	0^0
M	m_M = ?	0.10	$m_M r_M = 0.1 m_M$	0	0	θ_M =**?**
2	7.0	0.12	$m_2 r_2 = 0.84$	0.08	0.0672	60^0
3	8.0	0.14	$m_3 r_3 = 1.12$	0.24	0.2688	135^0
N	m_N = ?	0.10	$m_N r_N = 0.1 m_N$	0.36	$m_N r_N l_N = 0.036 m_N$	θ_N =**?**
4	6.0	0.12	$m_4 r_4 = 0.72$	0.48	0.3456	270^0

For dynamic balancing the conditions required are,

$\sum mr + m_M r_M + m_N r_N = 0$ ----------(I) for force balance

$\sum mrl + m_N r_N l_N = 0$ --------------(II) for couple balance



(a) Position of planes of masses

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_N r_N l_N \cos\theta_N = 0$$

On substitution we get

$$-0.072\cos 0^0 + 0.0672\cos 60^0 + 0.2688\cos 135^0$$
$$+0.3456\cos 270^0 + 0.036\, m_N \cos\theta_N = 0$$

i.e. $0.036 m_N \cos\theta_N = 0.2285 - - - - - (1)$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_N r_N l_N \sin\theta_N = 0$$

On substitution we get

$$-0.072\sin 0^0 + 0.0672\sin 60^0 + 0.2688\sin 135^0$$
$$+0.3456\sin 270^0 + 0.036\, m_N \sin\theta_N = 0$$

i.e. $0.036 m_N \sin\theta_N = 0.09733 - - - - - (2)$

Squaring and adding (1) and (2), we get

$$m_N r_N l_N = \sqrt{(0.2285)^2 + (0.09733)^2}$$

i.e., $0.036 m_N = 0.2484$

Therefore, $m_N = \frac{0.2484}{0.036} = 6.9\text{kg}$ Ans

Dividing (2) by (1), we get

$$\tan\theta_N = \frac{0.09733}{0.2285} \quad \text{and} \quad \theta_N = 23.07^0$$

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_M r_M \cos\theta_M + m_N r_N \cos\theta_N = 0$$

On substitution we get

$$0.9\cos 0^0 + 0.84\cos 60^0 + 1.12\cos 135^0 + 0.72\cos 270^0$$

$$+ m_M r_M \cos\theta_M + 0.1 x 6.9 x \cos 23.07^0 = 0$$

i.e. $m_M r_M \cos\theta_M = -1.1629 - - - - - (3)$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_M r_M \sin\theta_M + m_N r_N \sin\theta_N = 0$$

On substitution we get

$$0.9\sin 0^0 + 0.84\sin 60^0 + 1.12\sin 135^0 + 0.72\sin 270^0$$

$$+ m_M r_M \sin\theta_M + 0.1 x 6.9 x \sin 23.07^0 = 0$$

i.e. $m_M r_M \sin\theta_M = -1.0698 - - - - - (4)$

Squaring and adding (3) and (4), we get

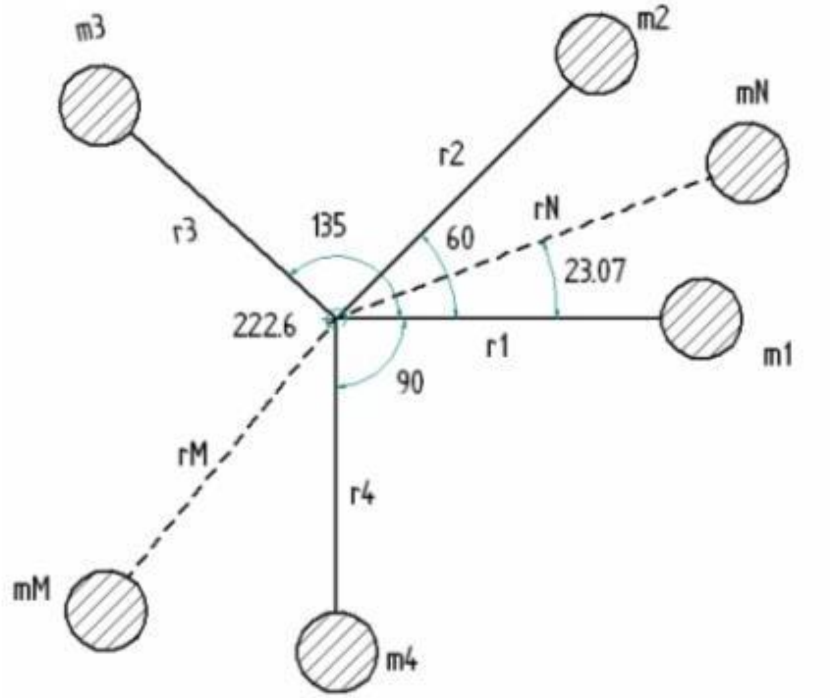
$$m_M r_M = \sqrt{(-1.1629)^2 + (-1.0698)^2}$$

i.e., $0.1 m_M = 1.580$

Therefore, $m_M = \frac{1.580}{0.1} = 15.8\text{kg}$ Ans

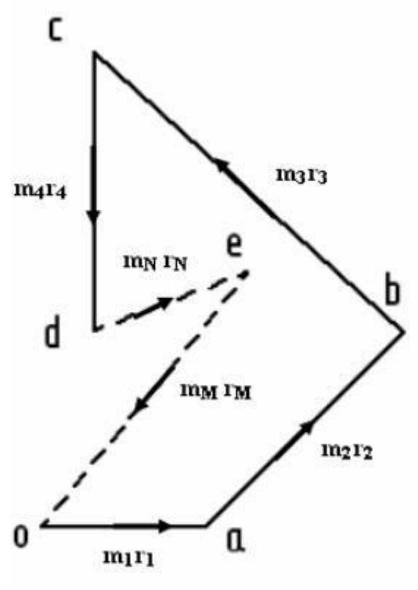
Dividing (4) by (3), we get

$$\tan\theta_M = \frac{-1.0698}{-1.1629} \quad \text{and} \quad \theta_M = 222.61^0 \text{ Ans}$$



(b) Angular position of masses

Graphical Solution:



(d) Force polygon

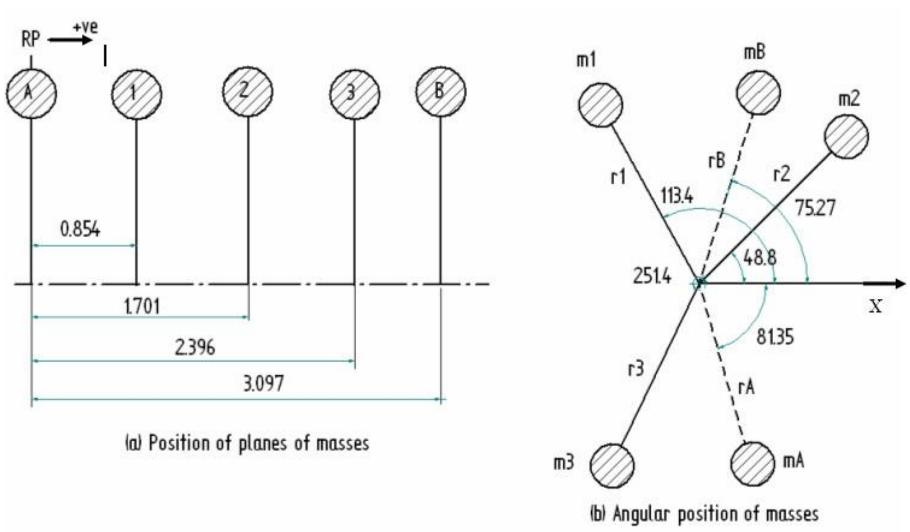
Problem 4:
The system has the following data.

$\mathbf{m_1} = 1.2$ **kg**	$\mathbf{r_1} = 1.135$ **m@**$\angle 113.4^0$
$\mathbf{m_1} = 1.8$ **kg**	$\mathbf{r_2} = 0.822$ **m@**$\angle 48.8^0$
$\mathbf{m_1} = 2.4$ **kg**	$\mathbf{r_3} = 1.04$ **m@**$\angle 251.4^0$

The distances of planes in metres from plane A are:

$$l_1 = 0.854,\ l_2 = 1.701, l_3 = 2.396, l_B = 3.097$$

Find the mass-radius products and their angular locations needed to dynamically balance the system using the correction planes A and B.



Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	m_A	r_A	$m_A r_A$ =?	0	0	$\boldsymbol{\theta_A}$ **=?**
1	1.2	1.135	1.362	0.854	1.163148	113.4^0
2	1.8	0.822	1.4796	1.701	2.5168	48.8^0
3	2.4	1.04	2.496	2.396	5.9804	251.4^0
B	m_B	r_B	$m_B r_B$ =?	3.097	3.097 $m_B r_B$	$\boldsymbol{\theta_B}$ **=?**

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_B r_B l_B \cos\theta_B = 0$$

On substitution we get

$$1.163148\cos 113.4^0 + 2.5168\cos 48.8^0 + 5.9804\cos 251.4^0$$
$$+ 3.097 m_B r_B \cos\theta_B = 0$$
$$\text{i.e. } m_B r_B \cos\theta_B = \frac{0.71166}{3.097} - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_B r_B l_B \sin\theta_B = 0$$

On substitution we get

$$1.163148\sin 113.4^0 + 2.5168\sin 48.8^0 + 5.9804\sin 251.4^0$$
$$+ 3.097 m_B r_B \sin\theta_B = 0$$
$$\text{i.e. } m_B r_B \sin\theta_B = \frac{2.7069}{3.097} - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$m_B r_B = \sqrt{\left(\frac{0.71166}{3.097}\right)^2 + \left(\frac{2.7069}{3.097}\right)^2}$$
$$\boldsymbol{= 0.9037 kg - m}$$

Dividing (2) by (1), we get

$$\tan\theta_B = \frac{2.7069}{0.71166} \quad \text{and} \quad \theta_B = 75.27^0 \qquad \text{Ans}$$

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_A r_A \cos\theta_A + m_B r_B \cos\theta_B = 0$$

On substitution we get

$$1.362\cos 113.4^0 + 1.4796\cos 48.8^0 + 2.496\cos 251.4^0$$
$$+ m_A r_A \cos\theta_A + 0.9037\cos 75.27^0 = 0$$

Therefore

$$m_A r_A \cos\theta_A = 0.13266 ---------- (3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_A r_A \sin\theta_A + m_B r_B \sin\theta_B = 0$$

On substitution we get

$$1.362\sin 113.4^0 + 1.4796\sin 48.8^0 + 2.496\sin 251.4^0$$
$$+ m_A r_A \sin\theta_A + 0.9037\sin 75.27^0 = 0$$

Therefore

$$m_A r_A \sin\theta_A = -0.87162 ---------- (4)$$

Squaring and adding (3) and (4), we get

$$m_A r_A = \sqrt{(0.13266)^2 + (-0.87162)^2}$$
$$= 0.8817 \text{ kg} - \text{m}$$

Dividing (4) by (3), we get

$$\tan\theta_A = \frac{-0.87162}{0.13266} \quad \text{and} \quad \theta_A = -81.35^0 \qquad \text{Ans}$$

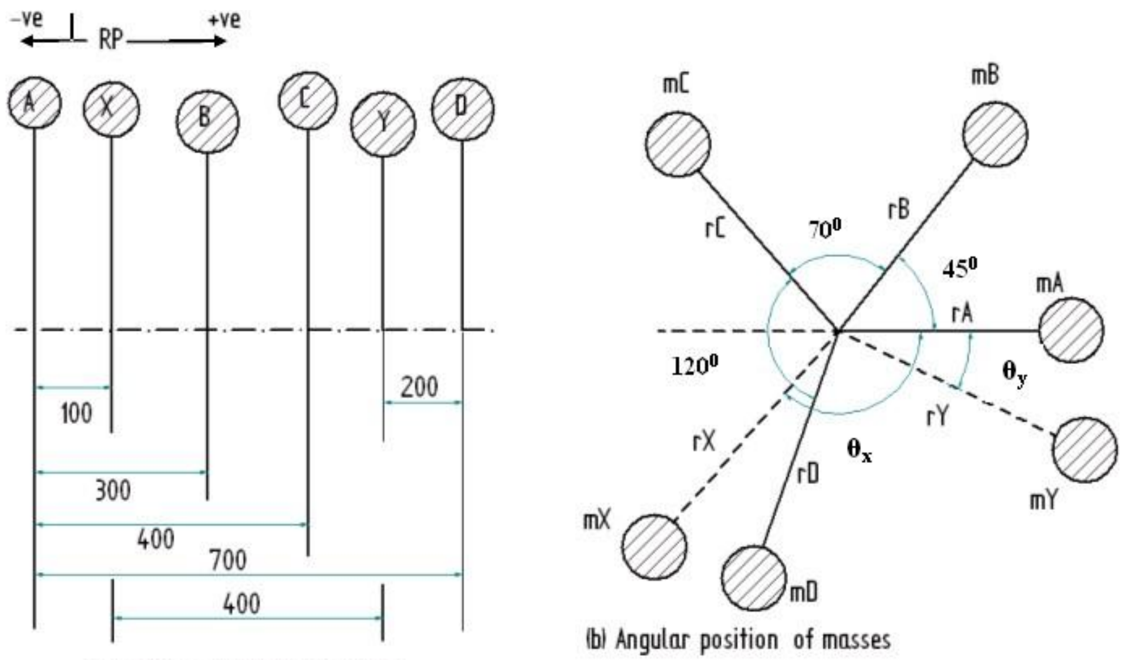
Problem 5:

A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45^0, B to C 70^0 and C to D 120^0. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

Graphical solution:

Let, m_X be the balancing mass placed in plane X and m_Y be the balancing mass placed in plane Y which are to be determined.

Draw the position of the planes as shown in figure (a).



(a) position of planes of masses

(b) Angular position of masses

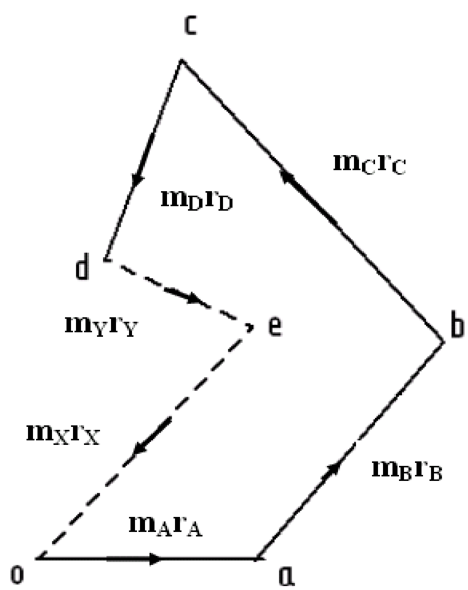
Let X be the reference plane (R.P.). The distances of the planes to the right of the plane X are taken as positive (+ve) and the distances of planes to the left of X plane are taken as negative(-ve). The data may be tabulated as shown

Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „m r" can be calculated and tabulated. Similarly the magnitude of the couples are proportional to the product of the mass , its radius and the axial distance from the reference plane, the product „m r l" can be calculated and tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „X" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	200	0.08	m_A r_A = 16	-0.10	-1.60	-
X	m_X =?	0.10	m_X r_X = 0.1 m_X	0	0	θ_X=**?**
B	300	0.07	m_B r_B = 21	0.20	4.20	A to B 45^0
C	400	0.06	m_C r_C = 24	0.30	7.20	B to C 70^0
Y	m_Y =?	0.10	m_Y r_Y = 0.1 m_Y	0.40	m_Y r_Y l_Y = 0.04 m_Y	θ_Y=**?**
D	200	0.08	m_D r_D = 16	0.60	9.60	C to D 120^0

Assuming the mass A as horizontal draw the sketch of angular position of masses as shown in figure (b).

Draw the couple polygon to some suitable scale by taking the values of „m r l" (column no. 6) of the table as shown in figure (c).



(d) Force polygon

Draw line o"a" parallel to the radial line of mass m_A.
At a" draw line a"b" parallel to radial line of mass m_B.
Similarly, draw lines b"c", c"d" parallel to radial lines of masses m_C and m_D respectively.
Now, join d" to o" which gives the balanced couple.

We get,

$$0.04\ m_Y = \text{vector } d'o' = 7.3\ kg - m_2$$

$$\text{or} \quad m_Y = 182.5\ kg \quad \text{Ans}$$

To find the angular position of the mass m_Y draw a line om_Y in figure (b) parallel to d"o" of the couple polygon.

By measurement we get $\theta_Y = 12^0$ in the clockwise direction from m_A.

Now draw the force polygon by considering the values of „m r" (column no. 4) of the table as shown in figure (d).
Follow the similar procedure of step 3. The closing side of the force polygon i.e. „e o" represents the balanced force.

$$m_X r_X = \text{vector } e o = 35.5\ kg - m$$

$$\text{or} \quad m_X = 355\ kg \quad \text{Ans}$$

The angular position of m_X is determined by drawing a line om_X parallel to the line „e o" of the force polygon in figure (b). From figure (b) we get,
$\theta_X = 145^0$, measured clockwise from m_A. Ans

A, B, C and D are four masses carried by a rotating shaft at radii 100 mm, 125 mm, 200 mm and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg and 4 kg respectively. Find the required mass A and relative angular settings of the four masses so that the shaft shall be in complete balance.
Solution:

Graphical Method:

Step 1:
Let, m_A be the balancing mass placed in plane A which is to be determined along with the relative angular settings of the four masses.
Let A be the reference plane (R.P.).
Assume the mass B as horizontal
Draw the sketch of angular position of mass m_B (line om_B) as shown in figure (b). The data may be tabulated as shown.

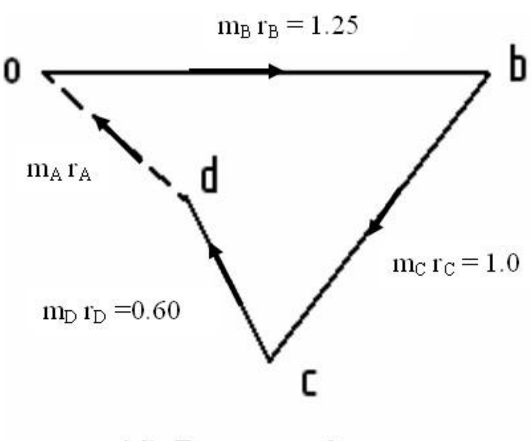
Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.1	$m_A r_A = 0.1 m_A$	0	0	θ_A =**?**
B	10	0.125	$m_B r_B = 1.25$	0.6	0.75	θ_B =0
C	5	0.2	$m_C r_C = 1.0$	1.2	1.2	θ_C =**?**
D	4	0.15	$m_D r_D = 0.6$	1.8	1.08	θ_D =**?**

Draw a line o"b" equal to 0.75 kg-m^2 parallel to the line om_B. At point o" and b" draw vectors o"c" and b"c" equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at point c".

Any one option can be used and relative to that the angular settings of mass C and D are determined.

$$\theta_D = 100^0 \quad \text{and} \quad \theta_C = 240^0 \quad \text{Ans}$$

In order to find m_A and its angular setting draw the force polygon as shown in figure (d).



(d) Force polygon

Closing side of the force polygon od represents the product m_A r_A . i.e.

$$m_A\ r_A = 0.70\,kg\text{-}m$$

Therefore, $$m_A = \frac{0.70}{r_A} = 7\ kg \quad Ans$$

Now draw line om_A parallel to od of the force polygon. By measurement, we get,

$$\theta_A = 155^0 \quad Ans$$

A shaft carries three masses A, B and C. Planes B and C are 60 cm and 120 cm from A. A , B and C are 50 kg, 40 kg and 60 kg respectively at a radius of 2.5 cm. The angular position of mass B and mass C with A are 90^0 and 210^0 respectively. Find the unbalanced force and couple. Also find the position and magnitude of balancing mass required at 10 cm radius in planes L and M midway between A and B, and B and C.

Case (i):

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	50	0.025	$m_A\ r_A = 1.25$	0	0	$\theta_A = 0^0$
B	40	0.025	$m_B\ r_B = 1.00$	0.6	0.6	$\theta_B = 90^0$
C	60	0.025	$m_C\ r_C = 1.50$	1.2	1.8	$\theta_C = 210^0$

Analytical Method

Step 1:
Determination of unbalanced couple
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mrl\ cos\theta} = \mathbf{0.6\ cos\ 90^0} + \mathbf{1.8cos210^0} = -\mathbf{1.559} - - - - - \mathbf{(1)}$$

Sum of the vertical components gives,

$$\sum \mathbf{mrl\ sin\theta} = \mathbf{0.6\ sin\ 90^0} + \mathbf{1.8\ sin\ 210^0} = -\mathbf{0.3} - - - - - \mathbf{(2)}$$

Squaring and adding (1) and (2), we get

$$\mathbf{C_{unbalanced}} = \sqrt{(\mathbf{-1.559})^2 + (\mathbf{-0.3})^2}$$
$$= \mathbf{1.588\ kg - m^2}$$

Determination of unbalanced force

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

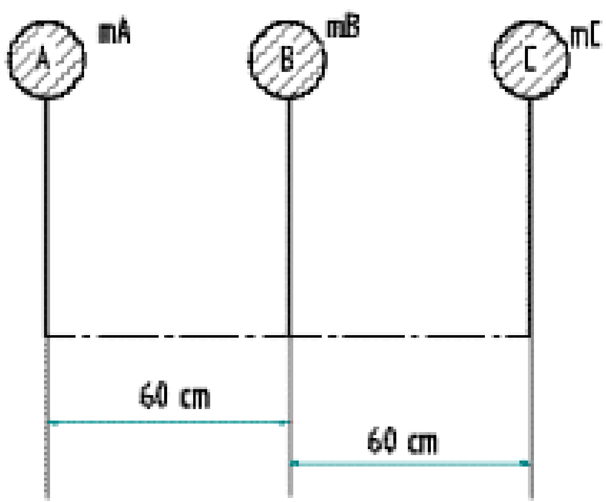
$$\sum \mathbf{mr\ \cos\theta = 1.25\cos 0^0 + 1.0\cos 90^0 + 1.5\cos 210^0}$$
$$\mathbf{= 1.25 + 0 + (-1.299) = -0.049 ---------- (3)}$$

Sum of the vertical components gives,

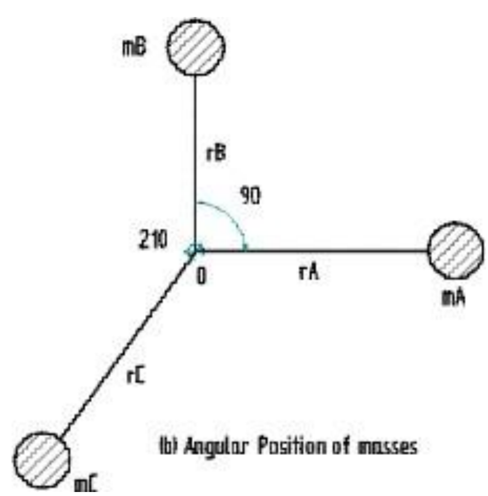
$$\sum \mathbf{mr\ \sin\theta = 1.25\sin 0^0 + 1.0\sin 90^0 + 1.5\sin 210^0}$$
$$\mathbf{= 0 + 1.0 + (-0.75) = 0.25 ---------- (4)}$$

Squaring and adding (3) and (4), we get

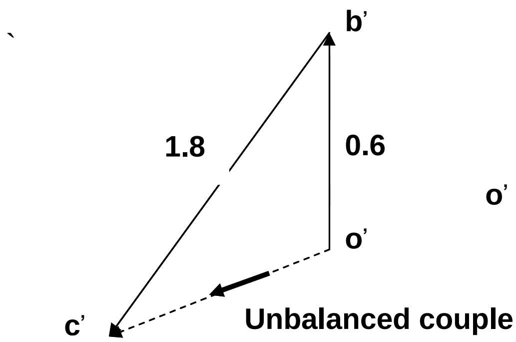
$$\mathbf{F_{unbalanced}} = \sqrt{(\mathbf{-0.049})^2 + (\mathbf{0.25})^2}$$
$$= \mathbf{0.2548\ kg - m}$$



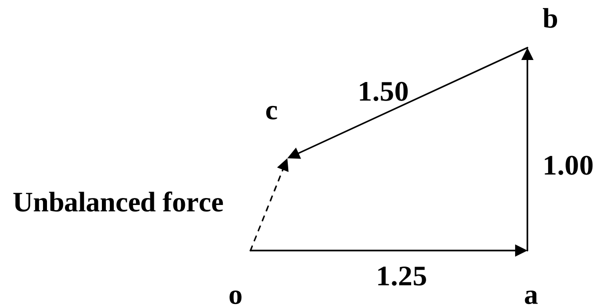
(a) Position of planes of masses



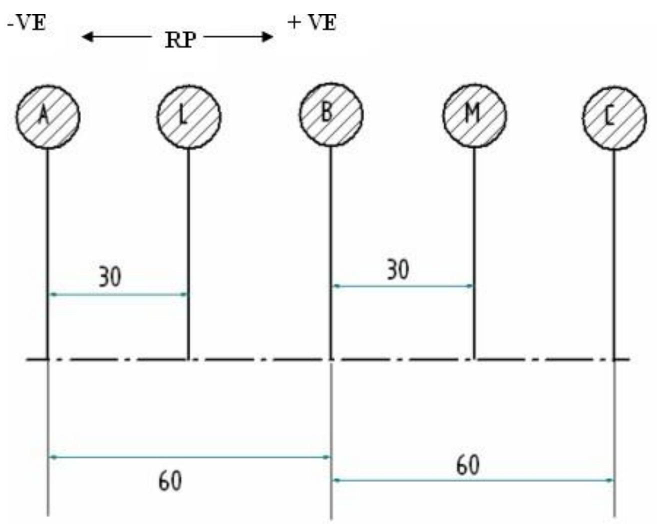
(b) Angular Position of masses



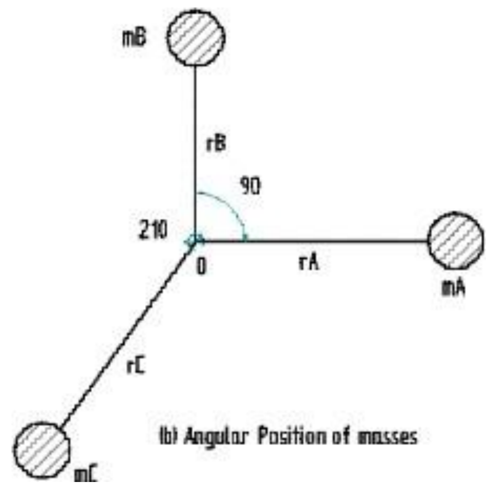
Couple polygon



Force polygon



(a) Position of planes of masses



(b) Angular Position of masses

To determine the magnitude and directions of masses m_M and m_L.

Let, m_L be the balancing mass placed in plane L and m_M be the balancing mass placed in plane M which are to be determined.

The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'L' m 5	Couple/ ω^2 (m r L) kg-m^2 6
A	50	0.025	m_A r_A = 1.25	-0.3	-0.375
L (R.P.)	m_L = ?	0.10	0.1 m_L	0	0
B	40	0.025	m_B r_B = 1.00	0.3	0.3
M	m_M = ?	0.10	0.1 m_M	0.6	0.06 m_M
C	60	0.025	m_C r_C = 1.50	0.9	1.35

Analytical Method:

Step 1:
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\,l\cos\theta + m_M r_M l_M \cos\theta_M = 0$$

On substituti on we get

$$-0.375\cos 0^\circ + 0.3\cos 90^\circ + 0.06\,m_M\cos\theta_M + 1.35\cos 210^\circ = 0$$

$$\text{i.e. } -0.375 + 0 + 0.06\,m_M\cos\theta_M + (-1.16913) = 0$$

$$0.06\,m_M\cos\theta_M = 1.54413$$

$$m_M\cos\theta_M = \frac{1.54413}{0.06} = 25.74 \quad -----(1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_M r_M l_M \sin\theta_M = 0$$

On substituti on we get

$$-0.375\sin 0^0 + 0.3\sin 90^0 + 0.06\, m_M \sin\theta_M + 1.35\sin 210^0 = 0$$

$$\text{i.e. } 0 + 0.3 + 0.06\, m_M \sin\theta_M + (-0.675) = 0$$

$$0.06\, m_M \sin\theta_M = 0.375$$

$$m_M \sin\theta_M = \frac{0.375}{0.06} = 6.25 \quad ----- (2)$$

Squaring and adding (1) and (2), we get

$$(m_M \cos\theta_M)^2 + (m_M \sin\theta_M)^2 = (25.74)^2 + (6.25)^2 = 701.61$$

$$\text{i.e. } m_M^2 = 701.61 \qquad \text{and} \qquad m_M = 26.5 \text{ kg} \qquad \text{Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_M = \frac{6.25}{25.74} \quad \text{and} \quad \theta_M = 13.65^0 \quad \text{Ans}$$

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr \cos\theta + m_L r_L \cos\theta_L + m_M r_M \cos\theta_M = 0$$

On substituti on we get

$$1.25 \cos 0^0 + 0.1 m_L \cos\theta_L + 1.0 \cos 90^0 + 2.649 \cos 13.65^0 + 1.5 \cos 210^0 = 0$$

$$1.25 + 0.1 m_L \cos\theta_L + 0 + 2.5741 + (-1.299) = 0$$

Therefore

$$0.1 m_L \cos\theta_L + 2.5251 = 0$$

$$\text{and} \qquad m_L \cos\theta_L = \frac{-2.5251}{0.1} = -25.251 \text{ ---------- } (3)$$

Sum of the vertical components gives,

$$\sum mr \sin\theta + m_L r_L \sin\theta_L + m_M r_M \sin\theta_M = 0$$

On substituti on we get

$$1.25 \sin 0^0 + 0.1 m_L \sin\theta_L + 1.0 \sin 90^0 + 2.649 \sin 13.65^0 + 1.5 \sin 210^0 = 0$$

$$0 + 0.1 m_L \sin\theta_L + 1 + 0.6251 + (-0.75) = 0$$

Therefore

$$0.1 m_L \sin\theta_L + 0.8751 = 0$$

$$\text{and} \qquad m_L \sin\theta_L = \frac{-0.8751}{0.1} = -8.751 \text{ ---------- } (4)$$

Squaring and adding (3) and (4), we get

$$(m_L \cos\theta_L)^2 + (m_L \sin\theta_L)^2 = (-25.251)^2 + (-8.751)^2 = 714.193$$

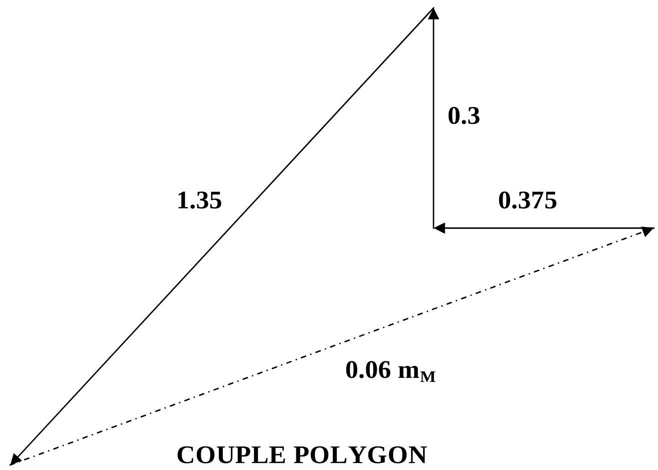
$$\text{i.e. } m_L^2 = 714.193 \qquad \text{and} \qquad m_L = 26.72 \text{ kg} \qquad \text{Ans}$$

Dividing (4) by (3), we get

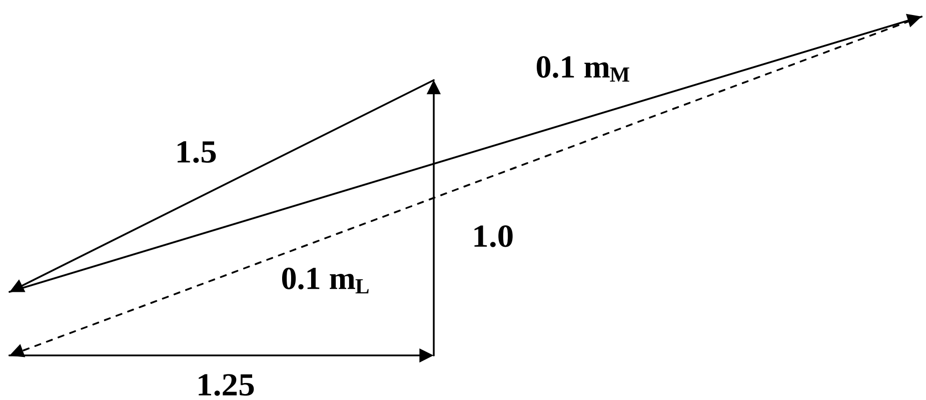
$$\tan\theta_L = \frac{-8.751}{-25.251} \quad \text{and} \quad \theta_L = 19.11^0 \quad \text{Ans}$$

The balancing mass m_L is at an angle $19.11^0 + 180^0 = 199.11^0$ measured in counter clockwise direction.

Graphical Method:



COUPLE POLYGON



FORCE POLYGON

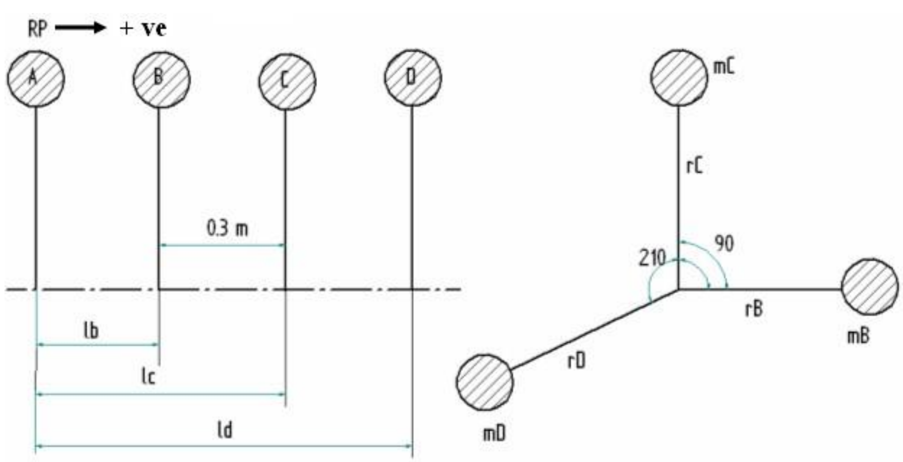
Problem 8:

Four masses A, B, C and D are completely balanced. Masses C and D make angles of 90^0 and 210^0 respectively with B in the same sense. The planes containing B and C are 300 mm apart. Masses A, B, C and D can be assumed to be concentrated at radii of 360 mm, 480 mm, 240 mm and 300 mm respectively. The masses B, C and D are 15 kg, 25 kg and 20 kg respectively. Determine i) mass A and its angular position ii) position of planes A and D.

Draw the space diagram or angular position of the masses. Since the angular position of the masses C and D are given with respect to mass B, take the angular position of mass B as $\theta_B = 0^0$.

Tabulate the given data as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.36	$m_A r_A$ = 0.36 m_A	0	0	θ_A =**?**
B	15	0.48	$m_B r_B$ = 7.2	l_B = ?	7.2 l_B	θ_B =0
C	25	0.24	$m_C r_C$ = 6.0	l_C = ?	6.0 l_C	$\theta_C = 90^0$
D	20	0.30	$m_D r_D$ = 6.0	l_D = ?	6.0 l_D	$\theta_D = 210^0$



(a) Position of planes of masses (Assumed)

(b) Angular position of masses

Mass m_A be the balancing mass placed in plane A which is to be determined along with its angular position.

Refer column 4 of the table. Since m_A is to be determined (which is the only unknown) ,resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D = 0$$

On substitution we get

$$0.36 m_A \cos\theta_A + 7.2\cos 0^0 + 6.0\cos 90^0 + 6.0\cos 210^0 = 0$$

Therefore

$$0.36\, m_A \cos\theta_A = -2.004 - - - - - - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D = 0$$

On substitution we get

$$0.36\, m_A \sin\theta_A + 7.2\sin 0^0 + 6.0\sin 90^0 + 6.0\sin 210^0 = 0$$

Therefore

$$0.36\, m_A \sin\theta_A = -3.0 - - - - - - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$0.36^2 (m_A)^2 = (-2.004)^2 + (-3.0)^2 = 13.016$$

$$m_A = \sqrt{\frac{13.016}{0.36^2}} = 10.02\ \text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_A = \frac{-3.0}{-2.004} \quad \text{and Resutltant makes an angle} = 56.26^0$$

$$\text{The balancing mass A makes an angle of } \theta_A = 236.26^0\ \text{Ans}$$

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\,l\cos\theta = m_A\,r_A\,l_A\cos\theta_A + m_B\,r_B\,l_B\cos\theta_B + m_C\,r_C\,l_C\cos\theta_C + m_D\,r_D\,l_D\cos\theta_D = 0$$

On substitution we get

$$0 + 7.2l_B\cos 0^0 + 6.0l_C\cos 90^0 + 6.0l_D\cos 210^0 = 0$$

$$7.2l_B - 5.1962l_D = 0 \text{ ---------------- } (3)$$

Sum of the vertical components gives,

$$\sum mr\,l\sin\theta = m_A\,r_A\,l_A\sin\theta_A + m_B\,r_B\,l_B\sin\theta_B + m_C\,r_C\,l_C\sin\theta_C + m_D\,r_D\,l_D\sin\theta_D = 0$$

On substitution we get

$$0 + 7.2l_B\sin 0^0 + 6.0l_C\sin 90^0 + 6.0l_D\sin 210^0 = 0$$

$$0 + 0 + 6.0l_C - 3l_D = 0 \text{ ---------------- } (4)$$

But from figure we have, $l_C = l_B + 0.3$

On substituting this in equation (4), we get

$$6.0(l_B + 0.3) - 3l_D = 0$$

$$\text{i.e. } 6.0l_B - 3l_D = 1.8 \text{ -------------- } (5)$$

Thus we have two equations (3) and (5), and two unknowns l_B, l_D

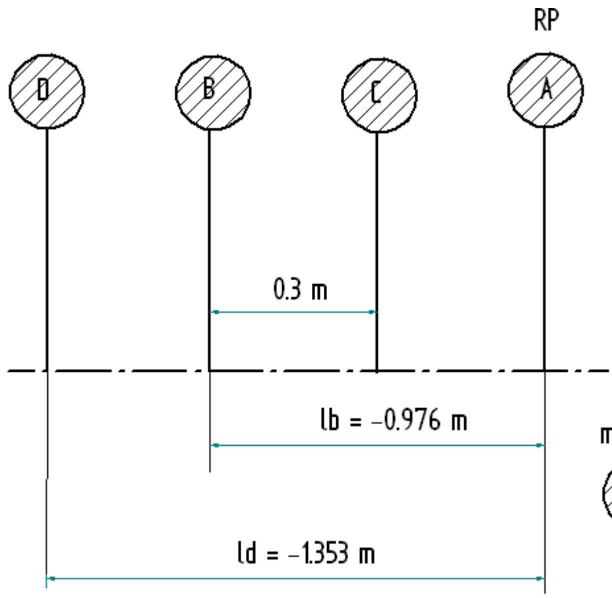
$$7.2l_B - 5.1962l_D = 0 \text{ ----------------- } (3)$$

$$6.0l_B - 3l_D = 1.8 \text{ - - - - - - - - - - } (5)$$

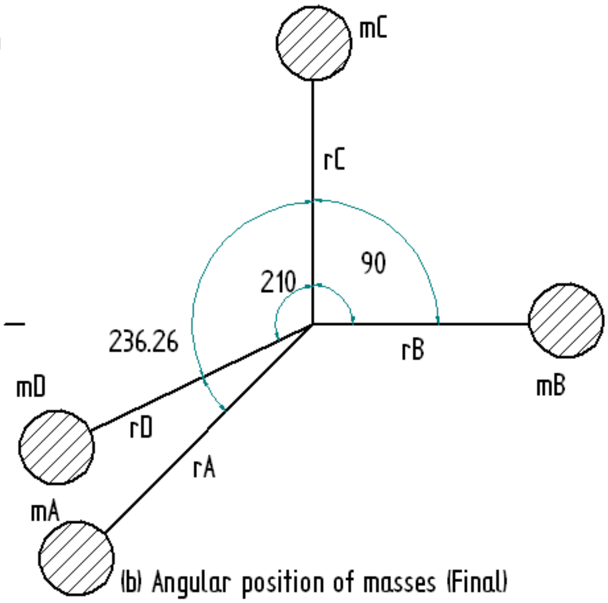
On solving the equations, we get

$l_D = -1.353\text{m}$ and $l_B = -0.976$ m

As per the position of planes of masses assumed the distances shown are positive (+ ve) from the reference plane A. But the calculated values of distances l_B and l_D are negative. The corrected positions of planes of masses is shown below.



(a) Position of planes of masses (Corrected)



(b) Angular position of masses (Final)