Chapter 1 Signal and Systems

1.1 Continuous-time and discrete-time Signals

1.1.1 Examples and Mathematical representation

Signals are represented mathematically as functions of one or more independent variables. Here we focus attention on signals involving a single independent variable. For convenience, this will generally refer to the independent variable as ***time***.

There are two types of signals: *continuous-time signals* and *discrete-time* signals.

Continuous-time signal: the variable of time is continuous. A speech signal as a function of time is a continuous-time signal.

Discrete-time signal: the variable of time is discrete. The weekly Dow Jones stock market index is an example of discrete-time signal.

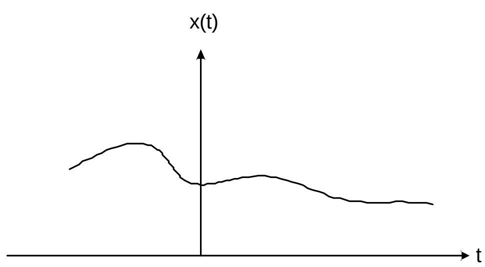


Fig. 1.1 Graphical representation of continuous-time signal.

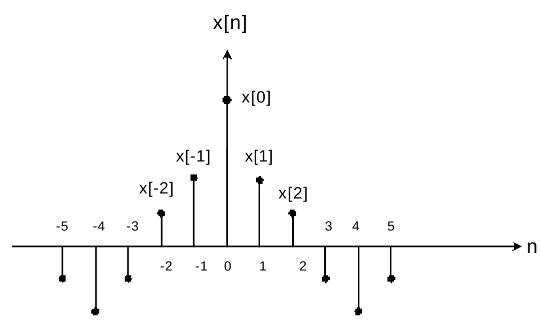


Fig. 1.2 Graphical representation of discrete-time signal.

To distinguish between continuous-time and discrete-time signals we use symbol t to denote the continuous variable and n to denote the discrete-time variable. And for continuous-time signals we will enclose the independent variable in parentheses (•), for discrete-time signals we will enclose the independent variable in bracket [•].

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. A discrete-time signal $x[n]$ may represent successive samples of an underlying phenomenon for which the independent variable is continuous. For example, the processing of speech on a digital computer requires the use of a discrete time sequence representing the values of the continuous-time speech signal at discrete points of time.

1.1.2 Signal Energy and Power

If $v(t)$ and $i(t)$ are respectively the voltage and current across a resistor with resistance R, then the instantaneous power is

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t). \tag{1.1}$$

The total energy expended over the time interval $t_1 \le t \le t_2$ is

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt, \tag{1.2}$$

and the average power over this time interval is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt. \tag{1.3}$$

For any continuous-time signal $x(t)$ or any discrete-time signal $x[n]$, the total energy over the time interval $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt, \tag{1.4}$$

where $|x|$ denotes the magnitude of the (possibly complex) number x. The time-averaged power is $\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} |x(t)|^2 dt$. Similarly the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \le n \le n_2$ is defined as

$$\sum_{n_1}^{n_2} |x[n]|^2 \tag{1.5}$$

The average power is $\frac{1}{n_2 - n_1 + 1}\sum_{n_1}^{n_2} |x[n]|^2$

In many systems, we will be interested in examining the power and energy in signals over an infinite time interval, that is, for $-\infty \le t \le +\infty$ or $-\infty \le n \le +\infty$. The total energy in continuous time is then defined

$$E_\infty = \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt, \tag{1.6}$$

and in discrete time

$$E_\infty = \lim_{N\to\infty} \sum_{-N}^{+N} |x[n]|^2 = \sum_{-\infty}^{+\infty} |x[n]|^2 . \tag{1.7}$$

For some signals, the integral in Eq. (1.6) or sum in Eq. (1.7) might not converge, that is, if $x(t)$ or $x[n]$ equals a nonzero constant value for all time. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

The time-averaged power over an infinite interval

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{1.8}$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} |x[n]|^2 \tag{1.9}$$

Three classes of signals:

- Class 1: signals with finite total energy, $E_\infty < \infty$ and zero average power, (Energy Signal)

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0 \tag{1.10}$$

- Class 2: with finite average power P_∞. If $P_\infty > 0$, then $E_\infty = \infty$. An example is the signal $x[n] = 4$, it has infinite energy, but has an average power of P_∞=16. (Power Signal)

Class 3: signals for which neither P_∞ and E_∞ are finite. An example of this signal is $x(t) = t$.

1.2 Transformations of the independent variable

In many situations, it is important to consider signals related by a modification of the independent variable. These modifications will usually lead to reflection, scaling, and shift.

1.2.1 Examples of Transformations of the Independent Variable

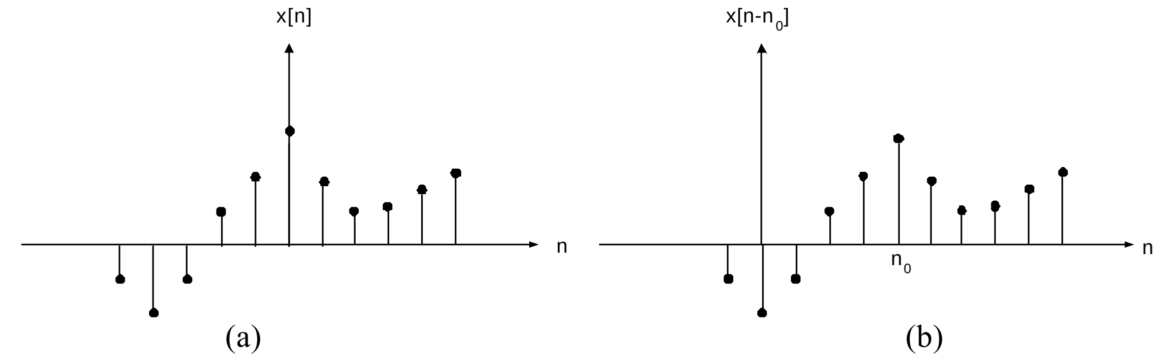


Fig.1.3 Discrete-time signals related by a time shift.

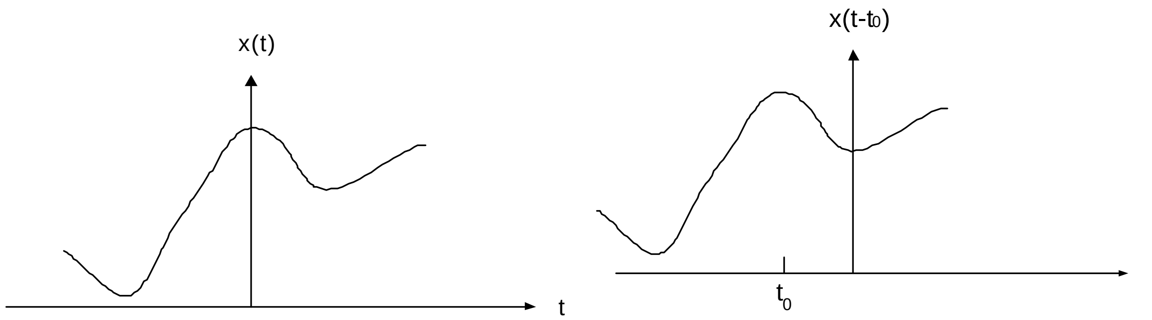


Fig. 1.4 Continuous-time signals related by a time shift.

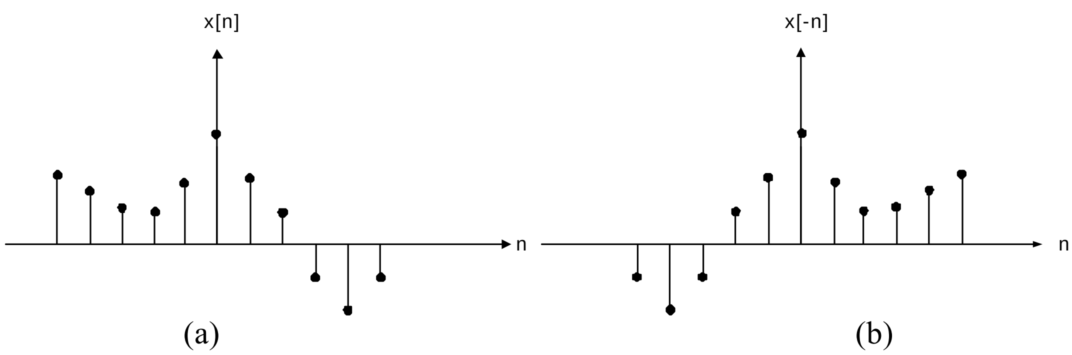


Fig. 1.5 (a) A discrete-time signal $x[n]$; (b) its reflection, $x[-n]$ about $n = 0$.

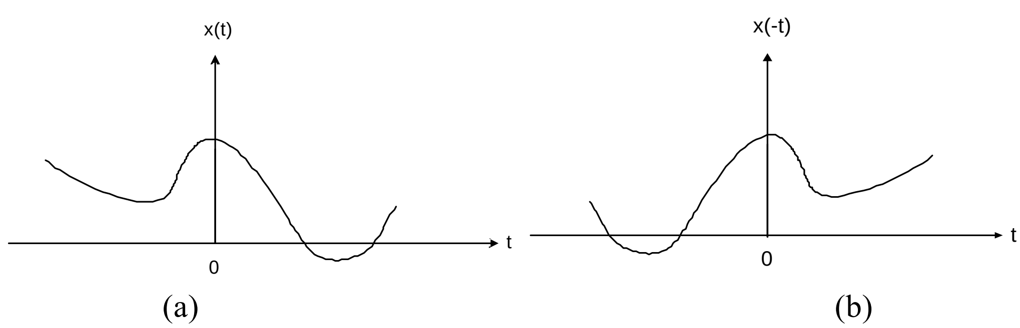


Fig. 1.6 (a) A continuous-time signal $x(t)$; (b) its reflection, $x(-t)$ about $t = 0$.

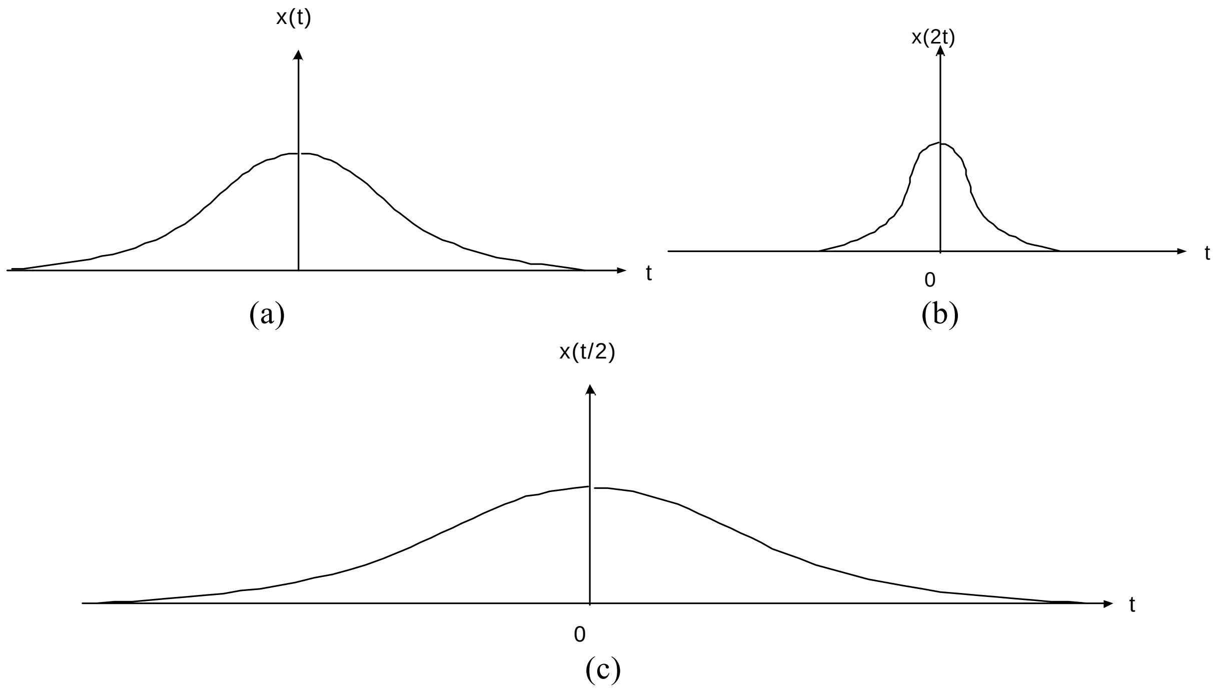


Fig. 1.7 Continuous-time signals related by time scaling.

1.2.2 Periodic Signals

A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which

$$x(t) = x(t+T) \text{ for all } t \tag{1.11}$$

From Eq. (1.11), we can deduce that if $x(t)$ is periodic with period T, then $x(t) = x(t+mT)$ for all t and for all integers m. Thus, $x(t)$ is also periodic with period $2T$, $3T$, …. The fundamental period T_0 of $x(t)$ is the smallest positive value of T for which Eq. (1.11) holds.

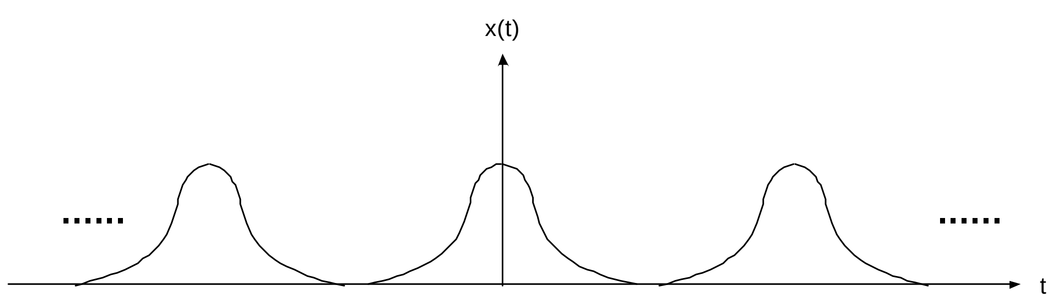


Fig. 1.8 Continuous-time periodic signal.

A discrete-time signal $x[n]$ is periodic with period N, where N is an integer, if it is unchanged by a time shift of N,

$$x[n] = x[n+N] \tag{1.12}$$

for all values of n. If Eq. (1.12) holds, then $x[n]$ is also periodic with period $2N$, $3N$, …. The fundamental period N_0 is the smallest positive value of N for which Eq. (1.12) holds.

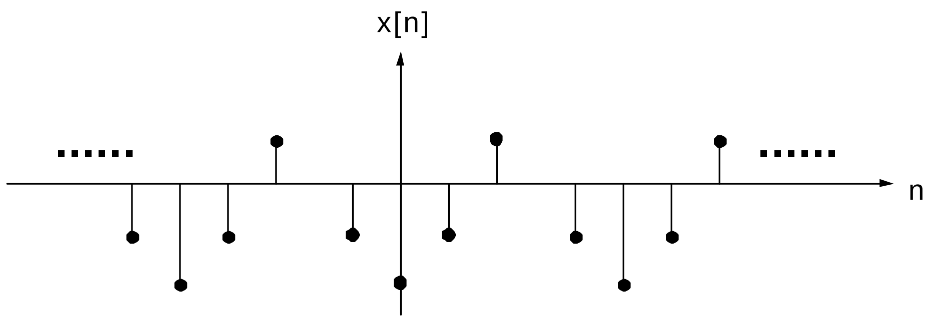


Fig. 1.9 Discrete-time periodic signal.

1.2.3 Even and Odd Signals

In addition to their use in representing physical phenomena such as the time shift in a radar signal and the reversal of an audio tape, transformations of the independent variable are extremely useful in examining some of the important properties that signal may possess.

Signal with these properties can be even or odd signal, periodic signal:

An important fact is that any signal can be decomposed into a sum of two signals, one of which is even and one of which is odd.

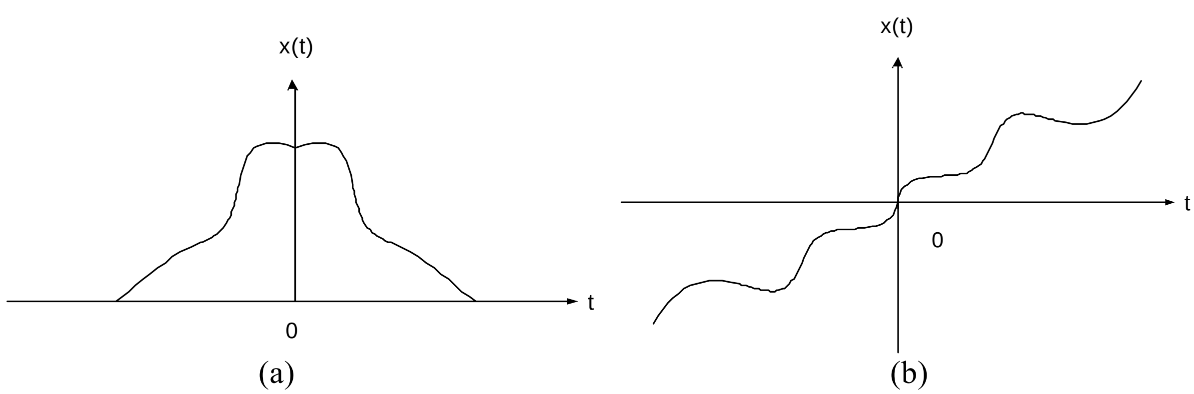


Fig. 1.10 An even continuous-time signal; (b) an odd continuous-time signal.

$$EV\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \qquad (1.13)$$

which is referred to as the even part of $x(t)$. Similarly, the odd part of $x(t)$ is given by

$$OD\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \qquad (1.14)$$

Exactly analogous definitions hold in the discrete-time case.

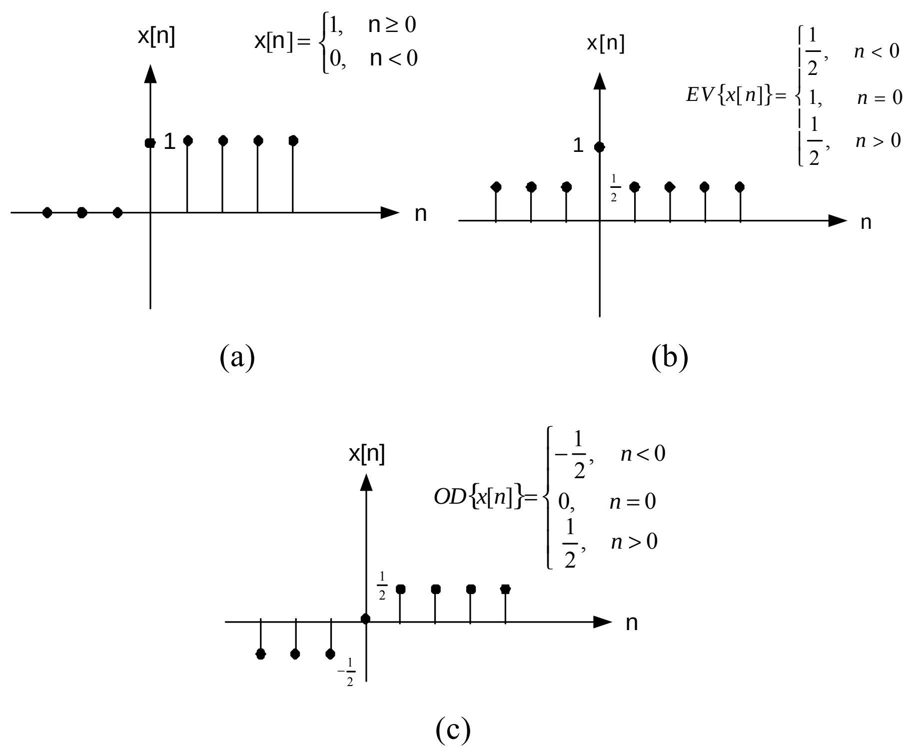


Fig.1.11 The even-odd decomposition of a discrete-time signal.

1.3 Exponential and sinusoidal signals

1.3.1 Continuous-time complex exponential and sinusoidal signals

The continuous-time complex exponential signal

$$x(t) = Ce^{at} \qquad (1.\ 15)$$

where C and a are in general complex numbers.

Real exponential signals

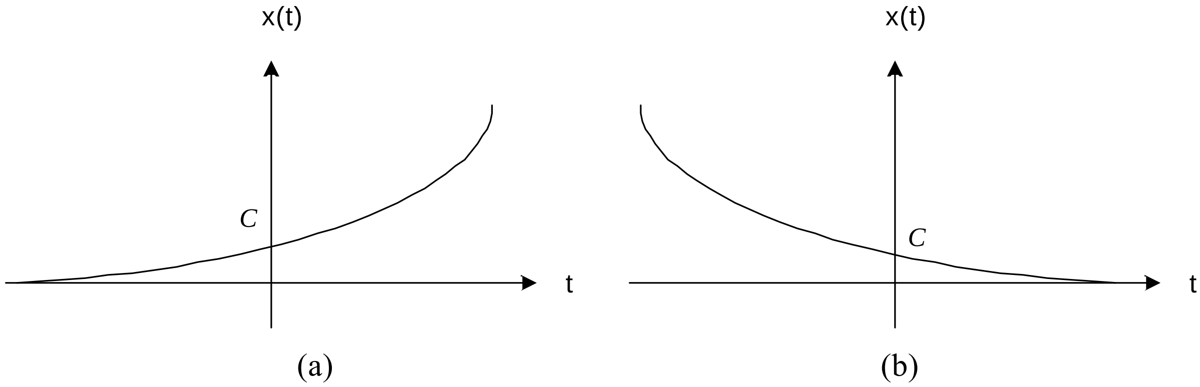


Fig. 1.12 The continuous-time complex exponential signal $x(t) = Ce^{at}$, (a) $a > 0$; (b) $a < 0$.

Periodic complex exponential and sinusoidal signals

If a is purely imaginary, we have

$$x(t) = e^{j\omega_0 t} \tag{1.16}$$

An important property of this signal is that it is periodic. We know $x(t)$ is periodic with period T if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \tag{1.17}$$

For periodicity, we must have

$$e^{j\omega_0 T} = 1 \tag{1.18}$$

For $w_0 \neq 0$, the fundamental period T_0 is

$$T_0 = \frac{2p}{|w_0|} \tag{1.19}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period.

A signal closely related to the periodic complex exponential is the *sinusoidal signal*

$$x(t) = A\cos(\omega_0 t + \phi) \tag{1.20}$$

With seconds as the unit of *t*, the units of f and w_0 are radians and radians per second. It is also known $w_0 = 2pf_0$, where f_0 has the unit of circles per second or Hz.

The sinusoidal signal is also a periodic signal with a fundamental period of T_0.

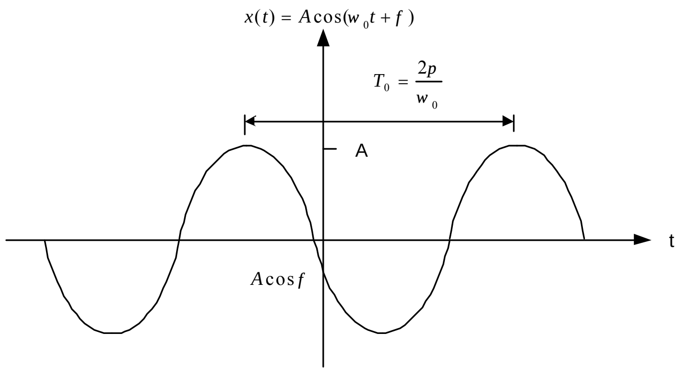


Fig. 1.13 Continuous-time sinusoidal signal.

Using Euler's relation, a complex exponential can be expressed in terms of sinusoidal signals with the same fundamental period:

$$e^{jw_0t} = \cos w_0 t + j\sin w_0 t \tag{1.21}$$

Similarly, a sinusoidal signal can also be expressed in terms of periodic complex exponentials with the same fundamental period:

$$A\cos(w_0 t + f) = \frac{A}{2} e^{jf} e^{jw_0 t} + \frac{A}{2} e^{-jf} e^{-jw_0 t} \tag{1.22}$$

A sinusoid can also be expresses as

$$A\cos(w_0 t + f) = A\,\mathrm{Re}\left\{e^{j(w_0 t + f)}\right\} \tag{1.23}$$

and

$$A\sin(w_0 t + f) = A\,\mathrm{Im}\left\{e^{j(w_0 t + f)}\right\} \tag{1.24}$$

Periodic signals, such as the sinusoidal signals provide important examples of signal with infinite total energy, but finite average power. For example:

$$E_{period} = \int_0^{T_0} \left|e^{jw_0 t}\right| dt = \int_0^{T_0} 1 dt = T_0 \tag{1.25}$$

$$P_{period} = \frac{1}{T_0}\int_0^{T_0} \left|e^{jw_0 t}\right| dt = \int_0^{T_0} 1 dt = 1 \tag{1.26}$$

Since there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. The average power is finite since

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|e^{jw_0 t}\right|^2 dt = 1 \qquad (1.27)$$

Harmonically related complex exponentials:

$$f_k(t) = e^{jkw_0 t},\quad k = 0, \pm 1, \pm 2, ... \qquad (1.28)$$

w_0 is the fundamental frequency.

Example:

Signal $x(t) = e^{j2t} + e^{j3t}$ can be expressed as $x(t) = e^{j2.5t}(e^{-j0.5t} + e^{j0.5t}) = 2e^{j2.5t}\cos(0.5t)$, the magnitude of $x(t)$ is $|x(t)| = 2|\cos(0.5t)|$, which is commonly referred to as a full-wave rectified sinusoid, shown in Fig. 1.14.

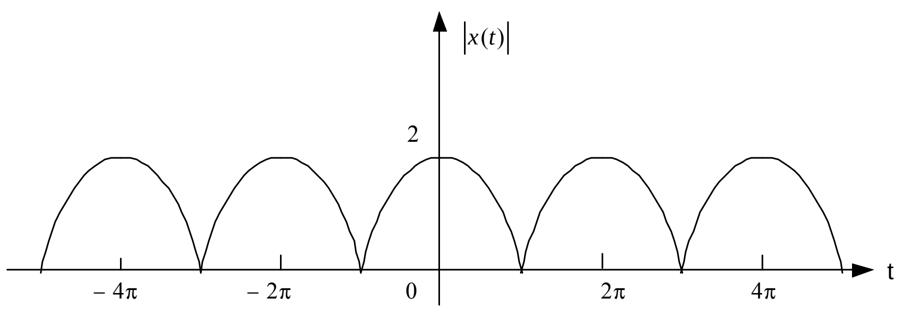


Fig. 1.14 Full-wave rectified sinusoid.

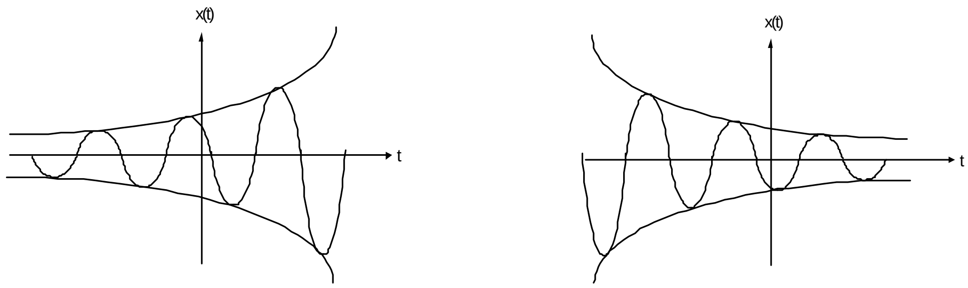
General complex Exponential signals

Consider a complex exponential Ce^{at}, where $C = |C|e^{jq}$ is expressed in polar and $a = r + jw_0$ is expressed in rectangular form. Then

$$Ce^{at} = |C|e^{jq}e^{(r+jw_0)t} = |C|e^{rt}e^{j(w_0 t+q)} = |C|e^{rt}\cos(w_0 t + q) + j|C|e^{rt}\sin(w_0 t + q)\,. \qquad (1.29)$$

Thus, for $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal.
For $r > 0$, sinusoidal signals multiplied by a growing exponential.
For $r < 0$, sinusoidal signals multiplied by a decaying exponential.

Damped signal – Sinusoidal signals multiplied by decaying exponentials are commonly refereed to as damped signal.



(a) (b)

Fig. 1.15 (a) Growing sinusoidal signal; (b) decaying sinusoidal signal.

1.3.2 Discrete-time complex exponential and sinusoidal signals

A discrete complex exponential or sequence is defined by

$$x[n] = Ca^n, \tag{1.30}$$

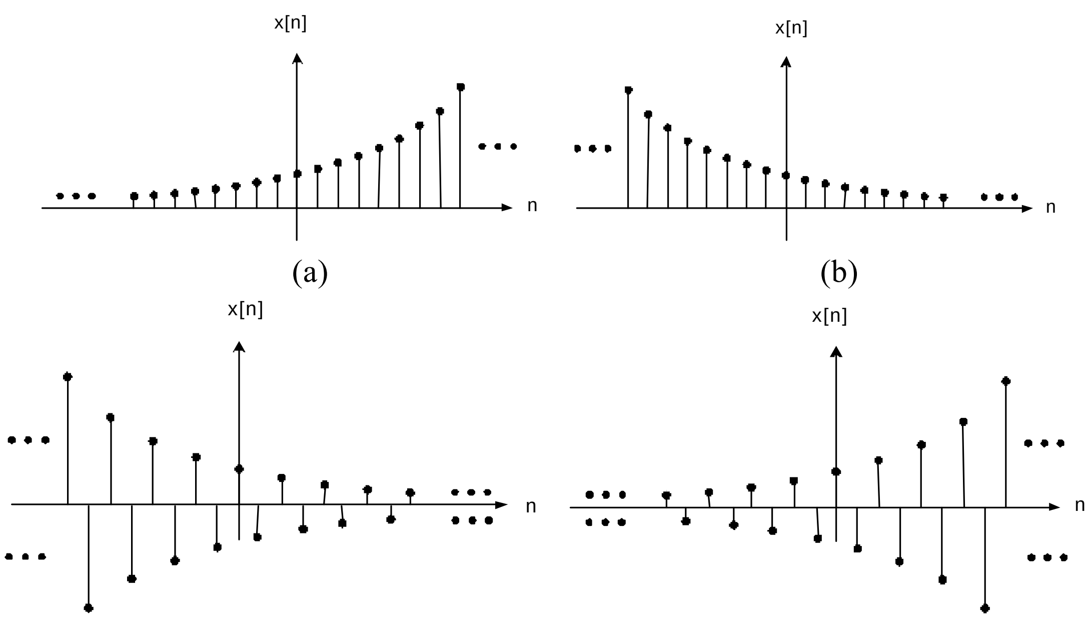
where C and a are in general complex numbers. This can be alternatively expressed

$$x[n] = Ce^{bn}, \tag{1.31}$$

where $a = e^b$.

Real Exponential Signals

If C and a are real, we have the real exponential signals.



(a) (b)

(c) (d)

Fig. 1.16 Real Exponential Signal $x[n] = Ca^n$: (a) a >1; (b) 0<a <1; (c) –1<a <0; (d) a <-1.

Sinusoidal Signals

$$x[n] = e^{j\omega_0 n} \tag{1.32}$$

$$e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n \tag{1.33}$$

Similarly, a sinusoidal signal can also be expresses in terms of periodic complex exponentials with the same fundamental period:

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n} \tag{1.34}$$

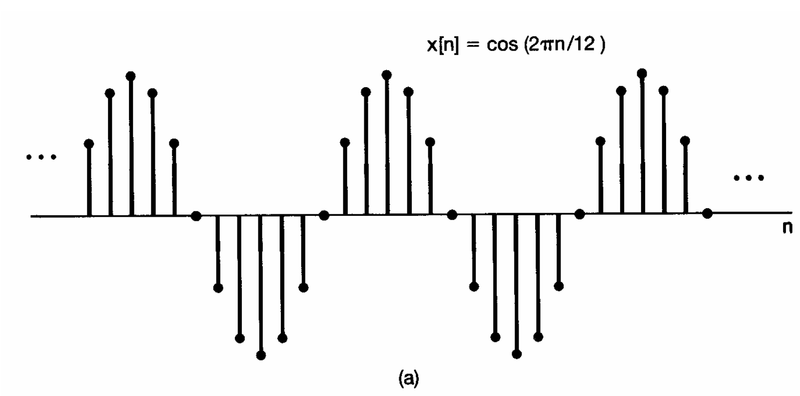
A sinusoid can also be expresses as

$$A\cos(\omega_0 n + \phi) = A\mathrm{Re}\left\{e^{j(\omega_0 n + \phi)}\right\} \tag{1.35}$$

and

$$A\sin(\omega_0 n + \phi) = A\mathrm{Im}\left\{e^{j(\omega_0 n + \phi)}\right\} \tag{1.36}$$

The above signals are examples of discrete signals with infinite total energy, but finite average power. For example: every sample of $x[n] = e^{j\omega_0 n}$ contributes 1 to the signal's energy. Thus the total energy $-\infty < n < +\infty$ is infinite, while the average power is equal to 1.



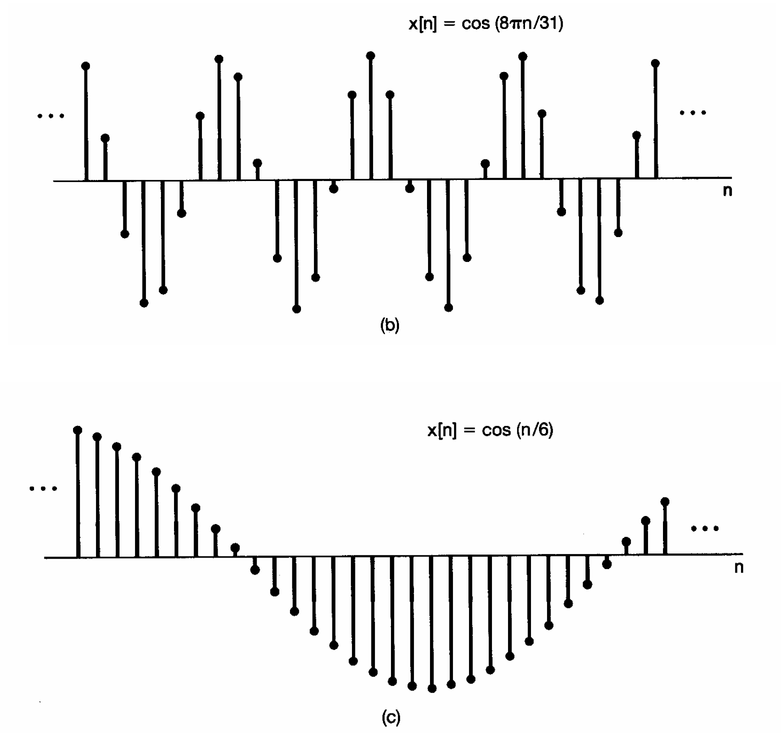


Fig.1.17 Discrete-time sinusoidal signal.

General Complex Exponential Signals

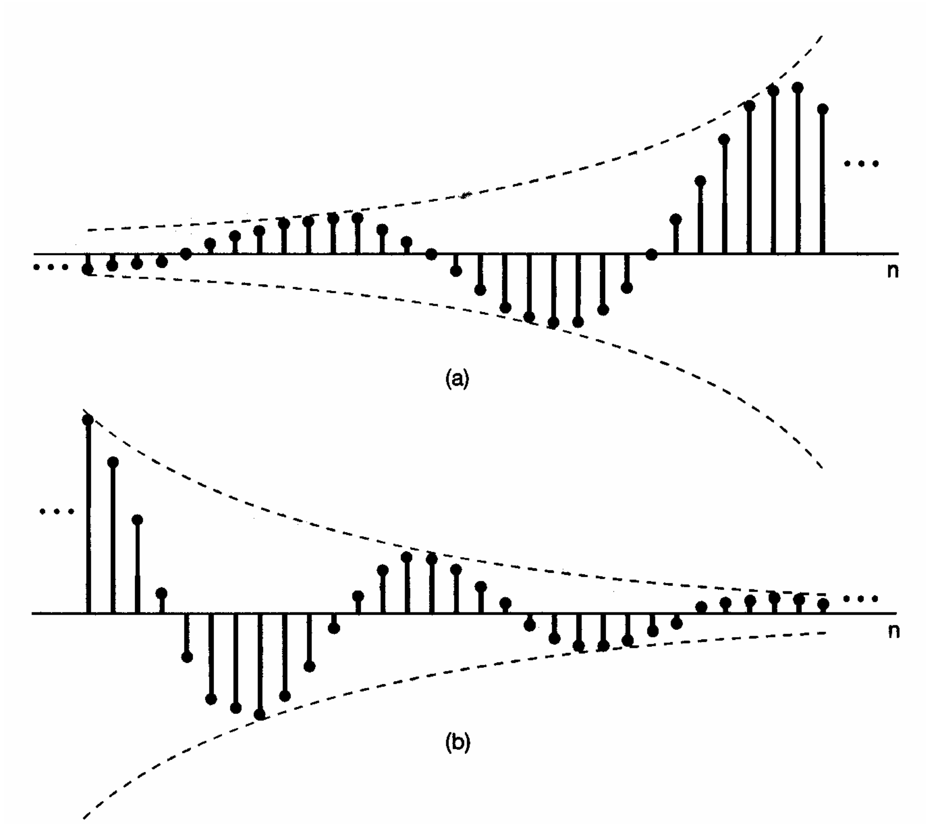
Consider a complex exponential Ca^n, where $C = |C|e^{jq}$ and $\alpha = |\alpha|e^{j\omega_0}$, then

$$Ca^n = |C||a|^n \cos(w_0 n + q) + j|C||a|^n \sin j(w_0 n + q). \tag{1.37}$$

Thus, for $|a| = 1$, the real and imaginary parts of a complex exponential are sinusoidal.

For $|a| < 1$, sinusoidal signals multiplied by a decaying exponential.

For $|a| > 1$, sinusoidal signals multiplied by a growing exponential.



(a) (b)

Fig. 1.18 (a) Growing sinusoidal signal; (b) decaying sinusoidal signal.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

There are a number of important differences between continuous-time and discrete-time sinusoidal signals. The continuous-time signals $e^{j\omega_0 t}$ are distinct for distinct values of ω_0. For discrete-time signals, however, these values are not distinct because the signal with ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on,

$$e^{j(\omega_0 \pm 2\pi)n} = e^{j(\omega_0 \pm 4\pi)n} = e^{j\omega_0 n}. \qquad (1.38)$$

In considering discrete-time exponentials, we need only consider a frequency interval of 2π. In most occasions, we will use the interval $0 \le \omega_0 < 2\pi$ or $-\pi \le \omega_0 < \pi$.

The discrete-time signal $x[n] = e^{j\omega_0 n}$ does not have a continuously increasing rate of oscillation as ω_0 is increased in magnitude, but as ω_0 is increased from 0, the signal oscillates more and more rapidly until ω_0 reaches π, and when ω_0 is continuously increased, the rate of oscillation decreases until ω_0 reaches 2π. We conclude that the low-frequency discrete-time exponentials have values of ω_0 near 0, 2π, and any other even multiple of π, while the high-frequencies are located near $\omega_0 = \pm\pi$ and other odd multiples of π.

In order for the signal $x[n] = e^{j\omega_0 n}$ to be periodic with period $N > 0$, we must have

$$e^{j\omega_0 (n+N)} = e^{j\omega_0 n}, \tag{1.39}$$

or equivalently

$$e^{j\omega_0 N} = 1. \tag{1.40}$$

For Eq. (1.40) to hold, $\omega_0 N$ must be a multiple of 2π. That is, there must be an integer m such that

$$\omega_0 N = 2\pi m, \tag{1.41}$$

or equivalently

$$\frac{\omega_0}{2\pi} = \frac{m}{N}. \tag{1.42}$$

From Eq. (1.40), $x[n] = e^{j\omega_0 n}$ is a periodic if $\omega_0 / 2\pi$ is a *rational number* and is not periodic otherwise.

The fundamental frequency of the discrete-time signal $x[n] = e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m}, \tag{1.43}$$

and the fundamental period of the signal can be

$$N = m\left(\frac{2\pi}{\omega_0}\right). \tag{1.44}$$

The comparison of the continuous-time and discrete-time signals are summarized in the table below:

Table 1 Comparison of the signals $e^{jw_0 t}$ and $e^{jw_0 n}$.

$e^{jw_0 t}$	$e^{jw_0 n}$
Distinct signals for distinct values of w_0	Identical signals for values of w_0 separated by multiples of $2p$
Periodic for any choice of w_0	Periodic only if $w_0 = 2pm / N$ for some integers $N > 0$ and m.
Fundamental frequency w_0	Fundamental frequency w_0 / m
Fundamental period $w_0 = 0$: undefined $w_0 \neq 0$: $\dfrac{2p}{w_0}$	Fundamental period $w_0 = 0$: undefined $w_0 \neq 0$: $m\left(\dfrac{2p}{w_0}\right)$

Example: Suppose that we wish to determine the fundamental period of the discrete-time signal

$$x[n] = e^{j(2p/3)n} + e^{j(3p/4)n} \tag{1.45}$$

Solution:

The first exponential on the right hand side has a fundamental period of 3. The second exponential has a fundamental period of 8.

For the entire signal to repeat, each of the terms in Eq. (1.45) must go through an integer number of its own fundamental period. The smallest increment of n the accomplished this is 24. That is, over an interval of 24 points, the first term will have gone through 8 of its fundamental periods, and the second term through three of its fundamental periods, and the overall signal through exactly one of its fundamental periods.

Harmonically related periodic exponentials

$$f_k[n] = e^{jk(2p/N)n},\ k = 0, \pm 1, ... \tag{1.46}$$

In the continuous-time case, all of the harmonically related complex exponentials $e^{jk(2p/N)t}$, $k = 0, \pm 1, ...$, are distinct. But this is not the case for discrete-time signals:

$$f_{k+N}[n] = e^{j(k+N)(2p/N)n} = e^{j(k2p/N)n} e^{j2pn} = f_k[n] \tag{1.47}$$

There are only N distinct period exponentials in the set given in Eq. (1.46).

1.4 The Unit Impulse and Unit Step Functions

The unit impulse and unit step functions in continuous and discrete time are considerably important in signal and system analysis.

1.4.1 The discrete-Time Unit Impulse and Unit Step Sequences

Discrete-time unit impulse is defined as

$$d[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}, \tag{1.48}$$

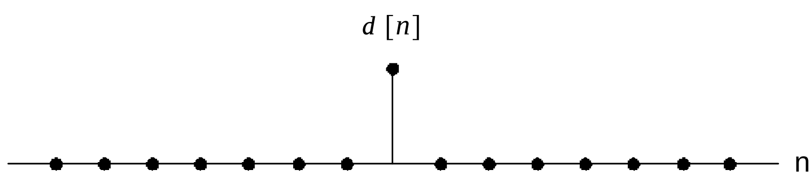


Fig. 1.19 Discrete-time unit impulse.

Discrete-time unit step is defined as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}, \tag{1.49}$$

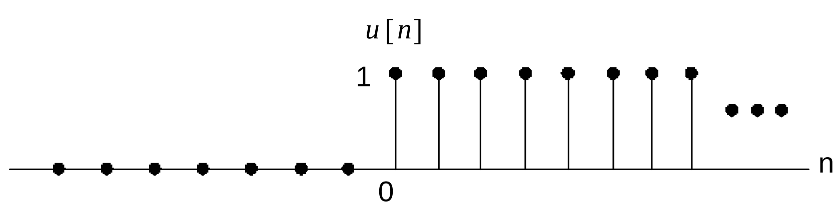


Fig. 1.20 Discrete-time unit step sequence.

The discrete-time impulse unit is the *first difference* of the discrete-time step

$$d[n] = u[n] - u[n-1], \tag{1.50}$$

The discrete-time unit step is the *running sum* of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} d[m], \tag{1.51}$$

It can be seen that for $n < 0$, the running sum is zero, and for $n \geq 0$, the running sum is 1.

If we change the variable of summation from m to $k = n - m$ we have, $u[n] = \sum_{k=0}^{\infty} d[n-k]$.

The unit impulse sequence can be used to ***sample*** the value of a signal at $n = 0$. Since $d[n]$ is nonzero only for $n = 0$, it follows that

$$x[n]d[n] = x[0]d[n]. \tag{1.52}$$

More generally, a unit impulse $d[n-n_0]$, then

$$x[n]d[n-n_0] = x[n_0]d[n-n_0] \tag{1.53}$$

This ***sampling*** property is very important in signal analysis.

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

Continuous-time unit step is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}, \tag{1.54}$$

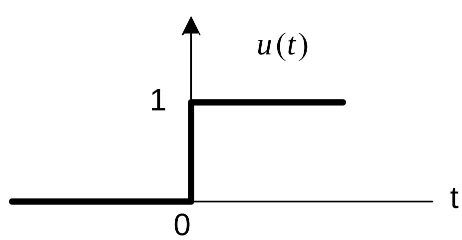


Fig. 1.21 Continuous-time unit step function.

The continuous-time unit step is the running integral of the unit impulse

$$u(t) = \int_{-\infty}^{t} d(t)dt. \tag{1.55}$$

The continuous-time unit impulse can also be considered as the first derivative of the continuous-time unit step,

$$d(t) = \frac{du(t)}{dt}. \tag{1.56}$$

Since $u(t)$ is discontinuous at $t = 0$ and consequently is formally not differentiable. This can be interpreted, however, by considering an approximation to the unit step $u_\Delta(t)$, as illustrated in the figure below, which rises from the value of 0 to the value 1 in a short time interval of length Δ.

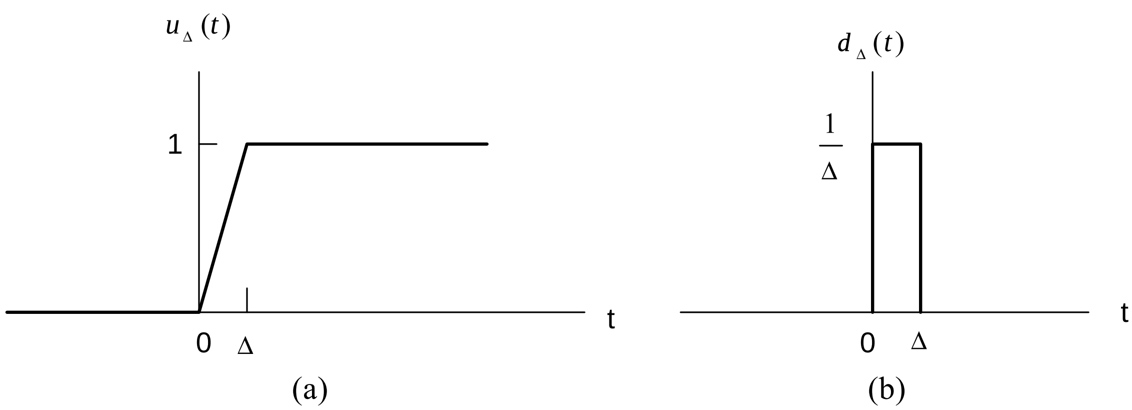


Fig. 1.22 (a) Continuous approximation to the unit step $u_\Delta(t)$; (b) Derivative of $u_\Delta(t)$.

The derivative is

$$d_\Delta(t) = \frac{du_\Delta(t)}{dt}, \tag{1.57}$$

$$d_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases}, \tag{1.58}$$

as shown in Fig. 1.22.

Note that $d_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $d_\Delta(t)$ becomes narrower and higher, maintaining its unit area. At the limit,

$$d(t) = \lim_{\Delta \to 0} d_\Delta(t), \tag{1.59}$$

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t), \tag{1.60}$$

and

$$d(t) = \frac{du(t)}{dt}. \tag{1.61}$$

Graphically, $d(t)$ is represented by an arrow pointing to infinity at $t = 0$, "1" next to the arrow represents the area of the impulse.

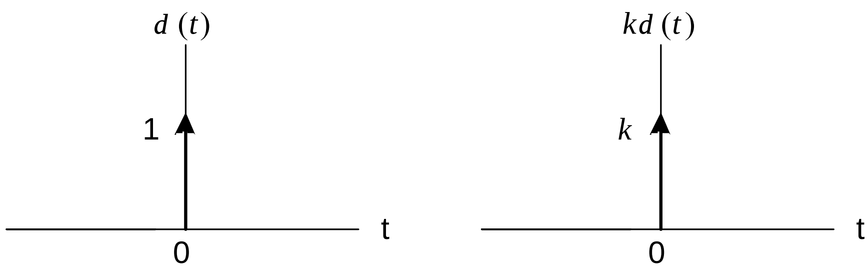


Fig. 1.23 Continuous-time unit impulse.

Sampling property of the continuous-time unit impulse:

$$x(t)d(t) = x(0)d(t), \tag{1.62}$$

Or more generally,

$$x(t)d(t - t_0) = x(t_0)d(t - t_0) \tag{1.63}$$

Example:

Consider the discontinuous signal $x(t)$

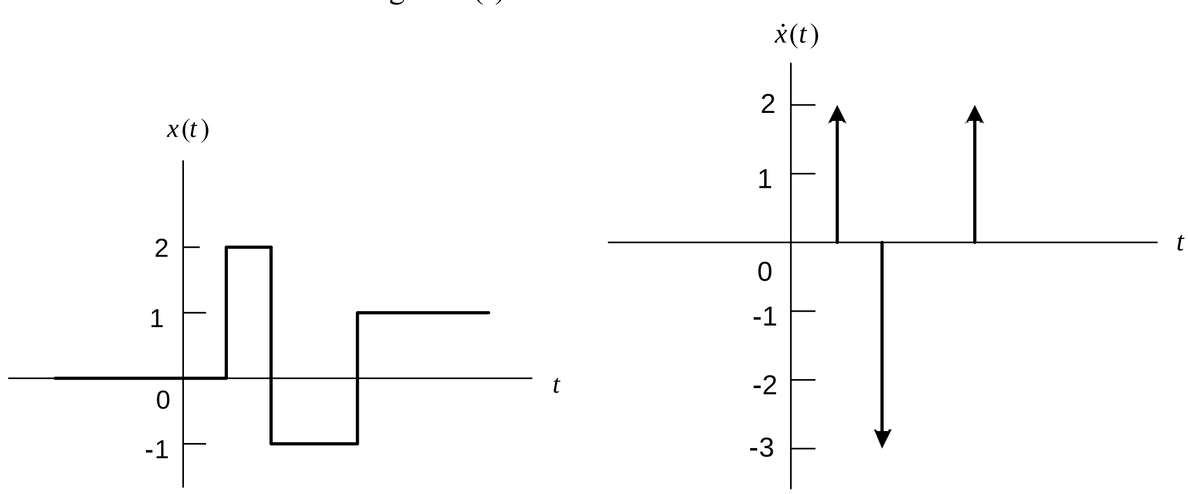


Fig. 1.24 The discontinuous signal and its derivative.

Note that the derivative of a unit step with a discontinuity of size of k gives rise to an impulse of area k at the point of discontinuity.

1.5 Continuous-Time and Discrete-Time Systems

A ***system*** can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

Examples

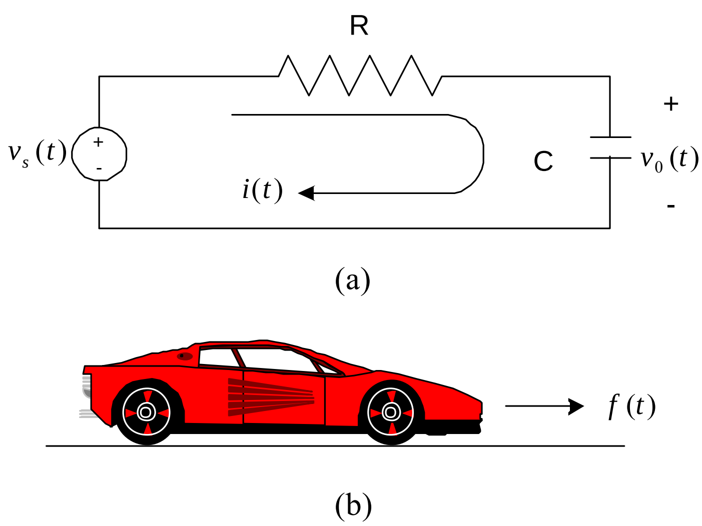
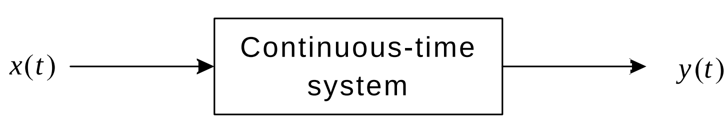
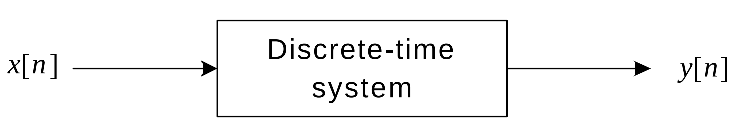


Fig. 1. 25 Examples of systems. (a) A system with input voltage $v_s(t)$ and output voltage $v_0(t)$. (b) A system with input equal to the force $f(t)$ and output equal to the velocity $v(t)$.

A ***continuous-time system*** is a system in which continuous-time input signals are applied and results in continuous-time output signals.



A ***discrete-time system*** is a system in which discrete-time input signals are applied and results in discrete-time output signals.



1.5.1 Simple Examples of Systems

Example 1: Consider the RC circuit in Fig. 25 (a).

The current $i(t)$ is proportional to the voltage drop across the resistor:

$$i(t) = \frac{v_s(t) - v_C(t)}{R}. \tag{1.64}$$

The current through the capacitor is

$$i(t) = C\frac{dv_C(t)}{dt}. \tag{1.65}$$

Equating the right-hand sides of Eqs. 1.64 and 1.65, we obtain a differential equation describing the relationship between the input and output:

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_s(t), \tag{1.66}$$

Example 2: Consider the system in Fig. 25 (b), where the force $f(t)$ as the input and the velocity $v(t)$ as the output. If we let m denote the mass of the car and rv the resistance due to friction. Equating the acceleration with the net force divided by mass, we obtain

$$\frac{dv(t)}{dt} = \frac{1}{m}\left[f(t) - rv(t)\right] \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{r}{m}v(t) = \frac{1}{m}f(t). \tag{1.67}$$

Eqs.1.66 and 1.77 are two examples of ***first-order linear differential equations*** of the form:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{1.66}$$

Example 3: Consider a simple model for the balance in a bank account from month to month. Let $y[n]$ denote the balance at the end of nth month, and suppose that $y[n]$ evolves from month to month according the equation:

$$y[n] = 1.01y[n-1] + x[n], \tag{1.67}$$

or

$$y[n] - 1.01y[n-1] = x[n], \tag{1.68}$$

where $x[n]$ is the net deposit (deposits minus withdraws) during the nth month $1.01y[n-1]$ models the fact that we accrue 1% interest each month.

Example 4: Consider a simple digital simulation of the differential equation in Eq. (1.67), in which we resolve time into discrete intervals of length Δ and approximate $dv(t)/d(t)$ at $t = n\Delta$ by the first backward difference, i.e.,

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

Let $v[n] = v(n\Delta)$ and $f[n] = f(n\Delta)$, we obtain the following discrete-time model relating the sampled signals $v[n]$ and $f[n]$,

$$v[n] - \frac{m}{(m + r\Delta)} v[n-1] = \frac{\Delta}{(m + r\Delta)} f[n]. \tag{1.69}$$

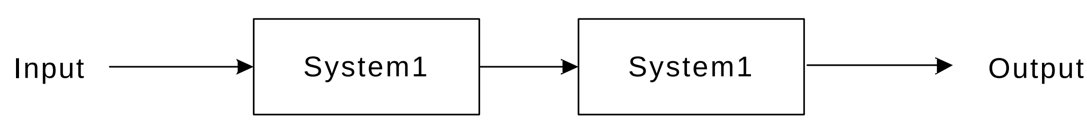
Comparing Eqs. 1.68 and 1.69, we see that they are two examples of the ***first-order linear difference equation***, that is,

$$y[n] + ay[n-1] = bx[n]. \tag{1.70}$$

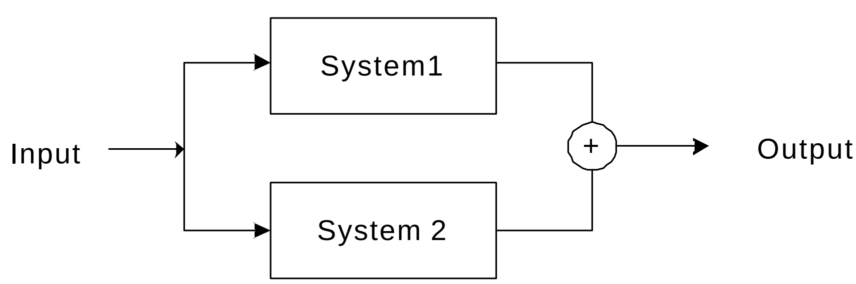
Some conclusions:

- Mathematical descriptions of systems have great deal in common;
- A particular class of systems is referred to ***as linear, time-invariant*** systems.
- Any model used in describing and analyzing a physical system represents an ***idealization*** of the system.

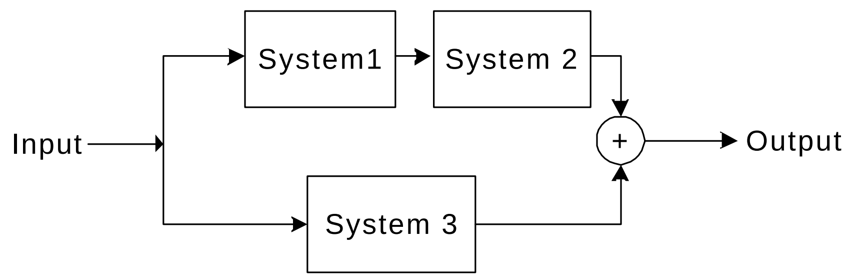
1.5.2 Interconnects of Systems



(a)



(b)



(c)

Fig. 1.26 Interconnection of systems. (a) A series or cascade interconnection of two systems; (b) A parallel interconnection of two systems; (c) Combination of both series and parallel systems.

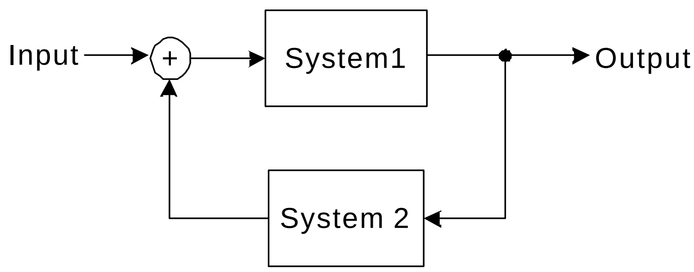
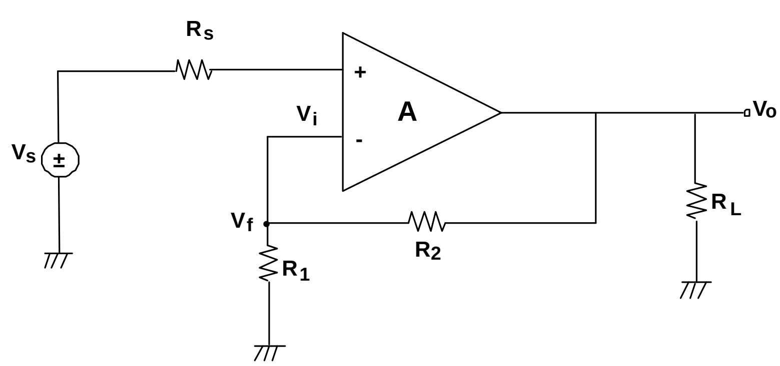
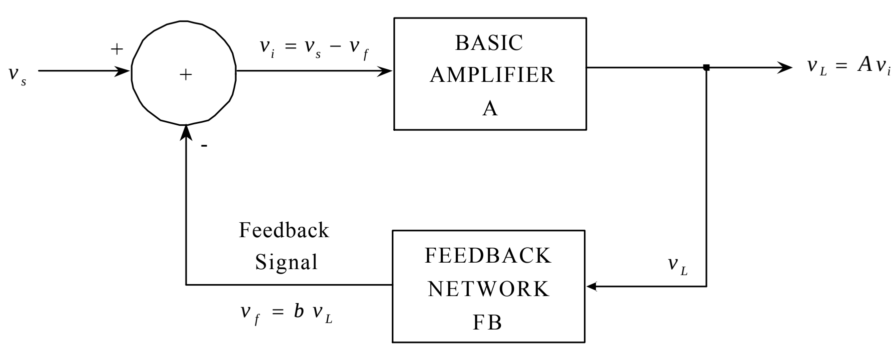


Fig. 1.27 Feedback interconnection.



(a)



(b)

Fig. 1.28 A feedback electrical amplifier.

1.6 Basic System Properties

1.6.1 Systems with and without Memory

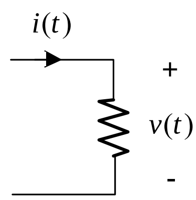
A system is ***memoryless*** if its output for each value of the independent variable as a given time is dependent only on the input at the same time. For example:

$$y[n] = (2x[n] - x^2[n])^2 , \tag{1.71}$$

is memoryless.

A resistor is a memoryless system, since the input current and output voltage has the relationship:

$$v(t) = R\,i(t) , \tag{1.72}$$



where R is the resistance.

One particularly simple memoryless system is the ***identity system***, whose output is identical to its input, that is

$y(t) = x(t)$, or $y[n] = x[n]$

An example of a discrete-time system with memory is an ***accumulator*** or ***summer***.

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n] \text{, or} \tag{1.73}$$

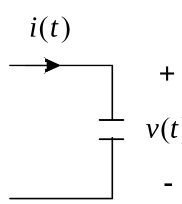
$$y[n] - y[n-1] = x[n] . \tag{1.74}$$

Another example is a ***delay***

$$y[n] = x[n-1] . \tag{1.75}$$

A capacitor is an example of a continuous-time system with memory,

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt , \tag{1.76}$$



where C is the capacitance.

1.6.2 Invertibility and Inverse System

A system is said to be ***invertible*** if distinct inputs leads to distinct outputs.

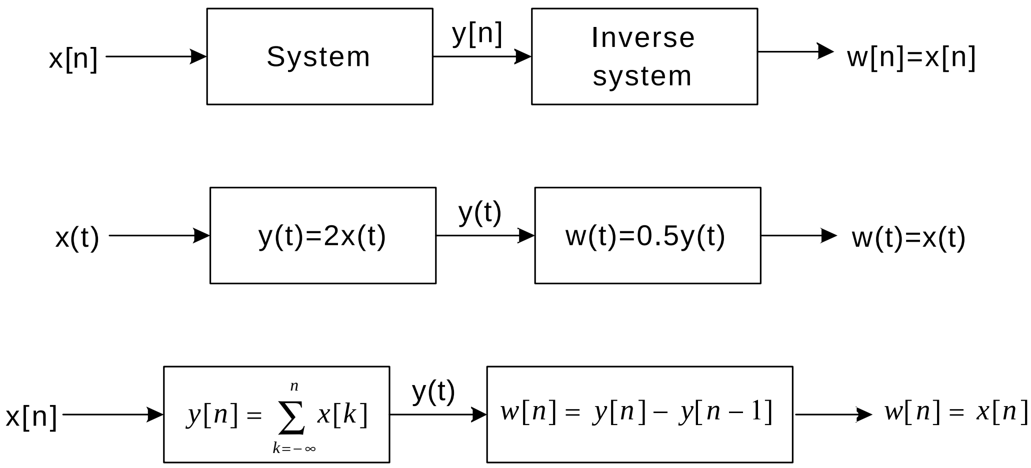


Fig. 1.29 Concept of an inverse system.

Examples of ***non-invertible systems***:

$y[n] = 0$,

the system produces zero output sequence for any input sequence.

$y(t) = x^2(t)$,

in which case, one cannot determine the sign of the input from the knowledge of the output.

Encoder in communication systems is an example of invertible system, that is, the input to the encoder must be exactly recoverable from the output.

1.6.3 Causality

A system is ***causal*** if the output at any time depends only on the values of the input at present time and in the past. Such a system is often referred to as being ***nonanticipative***, as the system output does not anticipate future values of the input.

The RC circuit in Fig. 25 (a) is causal, since the capacitor voltage responds only to the present and past values of the source voltage. The motion of a car is causal, since it does not anticipate future actions of the driver.

The following expressions describing systems that are not causal:

$$y[n] = x[n] - x[n+1], \tag{1.77}$$

and

$$y(t) = x(t+1). \tag{1.78}$$

All memoryless systems are causal, since the output responds only to the current value of input.

Example: Determine the Causality of the two systems:

(1) $y[n] = x[-n]$
(2) $y(t) = x(t)\cos(t+1)$

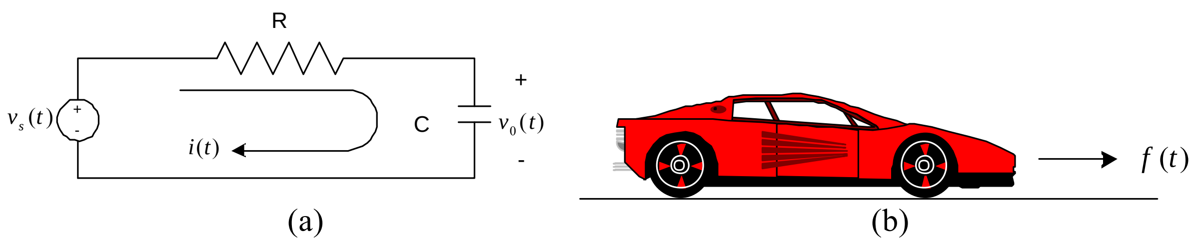
Solution: System (1) is not causal, since when $n < 0$, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of input.

System (2) is causal. The output at any time equals the ***input at the same time*** multiplied by a number that varies with time.

1.6.4 Stability

A ***stable system*** is one in which small inputs leads to responses that ***do not diverge***. More formally, if the input to a stable system is bounded, then the output must be also bounded and therefore cannot diverge.

Examples of stable systems and unstable systems:



The above two systems are stable system.

The ***accumulator*** $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not stable, since the sum grows continuously even if $x[n]$ is bounded.

Check the stability of the two systems:

- S1; $y(t) = tx(t)$;
- S2: $y(t) = e^{x(t)}$

- S1 is not stable, since a constant input $x(t) = 1$, yields $y(t) = t$, which is not bounded – no matter what finite constant we pick, $|y(t)|$ will exceed the constant for some *t*.

- S2 is stable. Assume the input is bounded $|x(t)| < B$, or $-B < x(t) < B$ for all *t*. We then see that $y(t)$ is bounded $e^{-B} < y(t) < e^{B}$.

1.6.5 Time Invariance

A system is ***time invariant*** if a time shift in the input signal results in an identical time shift in the output signal. Mathematically, if the system output is $y(t)$ when the input is $x(t)$, a time-invariant system will have an output of $y(t - t_0)$ when input is $x(t - t_0)$.

Examples:

- The system $y(t) = \sin[x(t)]$ is time invariant.

- The system $y[n] = nx[n]$ is not time invariant. This can be demonstrated by using ***counterexample***. Consider the input signal $x_1[n] = d[n]$, which yields $y_1[n] = 0$. However, the input $x_2[n] = d[n-1]$ yields the output $y_2[n] = nd[n-1] = d[n-1]$. Thus, while $x_2[n]$ is the shifted version of $x_1[n]$, $y_2[n]$ is not the shifted version of $y_1[n]$.

- The system $y(t) = x(2t)$ is not time invariant. To check using counterexample. Consider $x_1(t)$ shown in Fig. 1.30 (a), the resulting output $y_1(t)$ is depicted in Fig. 1.30 (b). If the input is shifted by 2, that is, consider $x_2(t) = x_1(t-2)$, as shown in Fig. 1.30 (c), we obtain the resulting output $y_2(t) = x_2(2t)$ shown in Fig. 1.30 (d). It is clearly seen that $y_2(t) \neq y_1(t-2)$, so the system is not time invariant.

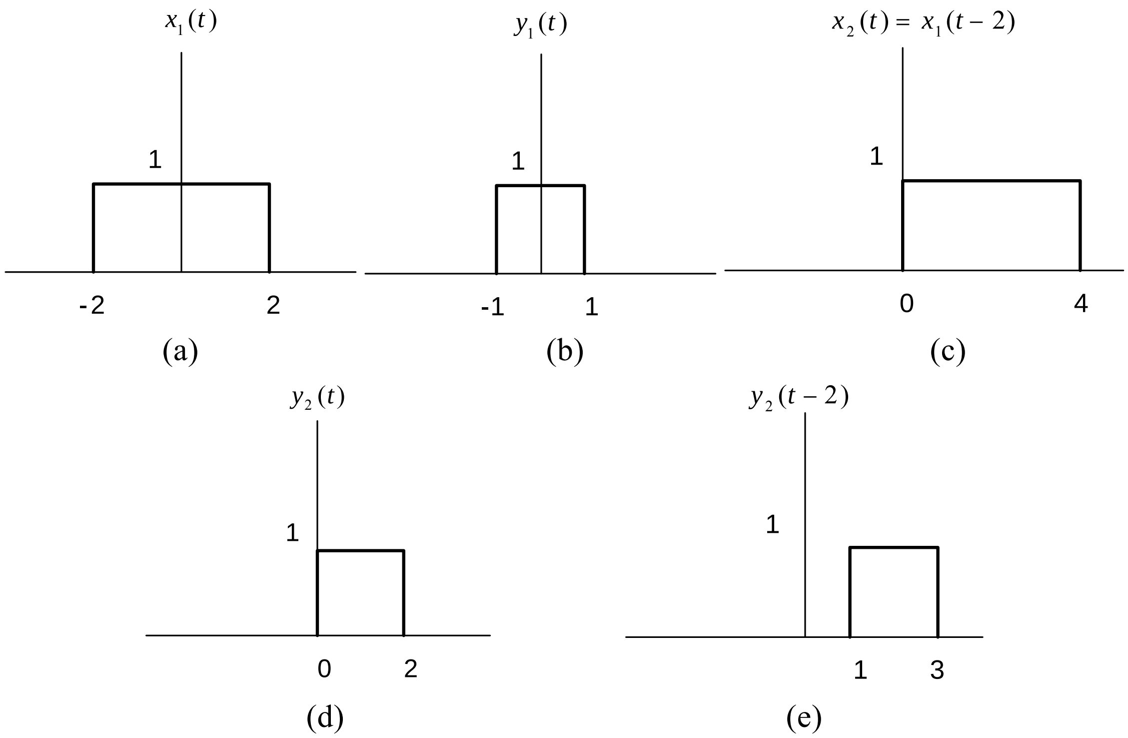


Fig. 1.30 Inputs and outputs of the system $y(t) = x(2t)$.

1.6.6 Linearity

The system is linear if

- The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$ - ***additivity*** property
- The response to $ax_1(t)$ is $ay_1(t)$ - ***scaling*** or ***homogeneity*** property.

The two properties defining a linear system can be combined into a single statement:

- Continuous time: $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$,
- Discrete time: $ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$.

Here a and b are any complex constants.

Superposition property: If $x_k[n],\ k = 1, 2, 3, \ldots$ are a set of inputs with corresponding outputs $y_k[n],\ k = 1, 2, 3, \ldots$, then the response to a linear combination of these inputs given by

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + a_3 x_3[n] + \ldots, \quad (1.79)$$

is

$$y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + a_3 y_3[n] + \dots, \tag{1.80}$$

which holds for linear systems in both continuous and discrete time.

For a linear system, ***zero input leads to zero output***.

Examples:

- The system $y(t) = tx(t)$ is a linear system.
- The system $y(t) = x^2(t)$ is not a liner system.
- The system $y[n] = \mathrm{Re}\{x[n]\}$, is additive, but does not satisfy the homogeneity, so it is not a linear system.
- The system $y[n] = 2x[n] + 3$ is not linear. $y[n] = 3$ if $x[n] = 0$, the system violates the "zero-in/zero-out" property. However, the system can be represented as the sum of the output of a linear system and another signal equal to the zero-input response of the system. For system $y[n] = 2x[n] + 3$, the linear system is

  $x[n] \to 2x[n]$,

  and the zero-input response is

  $y_0[n] = 3$

  as shown in Fig. 1.31.

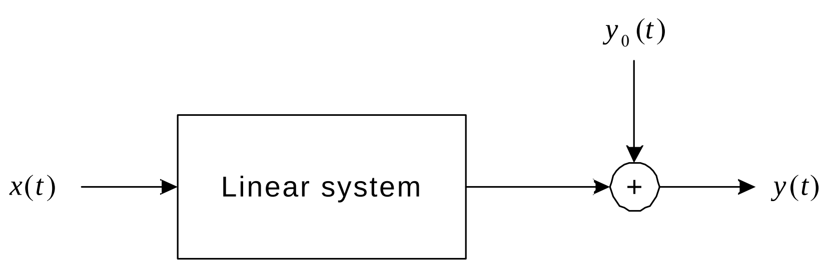


Fig. 1.31 Structure of an incrementally linear system. $y_0(t)$ is the zero-input response of the system.

The system represented in Fig. 1.31 is called incrementally linear system. The system responds linearly to the changes in the input.

The overall system output consists of the superposition of the response of a linear system with a zero-input response.