QUESTION BANK SOLUTION

MODULE-I

DIFFERENTIAL CALCULUS-I

1) Find the nth derivative of $y = e^{-3x}\cos^3 x$ (July 2016)

Soln: Let y = $e^{-3x} \cos^3 x$

We know that $\cos^3 x = \frac{1}{4}\cos 3x + 3\cos x$

$\therefore$ y = $e^{-3x}\left[\cos^3 x\right] = \frac{e^{-3x}}{4}\cos 3x + 3\cos x$

$\therefore y = \frac{1}{4}\left[e^{-3x}\cos 3x + 3e^{-3x}\cos x\right]$

Hence, $y_n = \frac{1}{4}\left[r_1^n e^{-3x}\cos\ 3x + n\theta_2\ + 3r_2^n e^{-3x}\cos\ x + n\theta_3\ \right]$

where $r_1 = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$; $r_3 = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$

$\theta_1 = \tan^{-1}\left(\frac{3}{-3}\right) = -\frac{\pi}{4}$; $\theta_2 = \tan^{-1}\left(\frac{1}{-3}\right)$

2) Find the angle of intersection between the curves soln:

$r = a\ 1 + \sin\theta\ ,\ r = a\ 1 - \cos\theta$ (July 2016)

Sol: Consider

$r = a\ 1 + \sin\theta$

$r = b(1 - \cos\theta)$

$Diff\ \ w.r.t\ \ \theta$

$Diff\ \ w.r.t\ \ \theta$

$$\frac{dr}{d\theta} = a\cos\theta$$

$$r\frac{d\theta}{dr} = \frac{a\;1+\sin\theta}{a\cos\theta}$$

$$\tan\varphi_1 = \frac{\left[\sin\,{}^{\theta}\!/_{2} + \cos\,{}^{\theta}\!/_{2}\right]^2}{\cos^2\,{}^{\theta}\!/_{2} - \sin^2\,{}^{\theta}\!/_{2}}$$

$$= \frac{\left[\sin\,{}^{\theta}\!/_{2} + \cos\,{}^{\theta}\!/_{2}\right]}{\cos\,{}^{\theta}\!/_{2} - \sin\,{}^{\theta}\!/_{2}}$$

$$\tan\varphi_1 = \frac{1+\tan\,{}^{\theta}\!/_{2}}{1-\tan\,{}^{\theta}\!/_{2}}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

i.e $\Rightarrow \varphi_1 = {}^{\pi}\!/_{4} + {}^{\theta}\!/_{2}$

$$\frac{dr}{d\theta} = b\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta} = \tan{}^{\theta}\!/_{2}$$

$$\tan\varphi_1 = \tan{}^{\theta}\!/_{2} \Rightarrow \varphi_2 = {}^{\theta}\!/_{2}$$

Angle between the curves

$$|\varphi_1 - \varphi_2| = \left|\,{}^{\pi}\!/_{4} + {}^{\theta}\!/_{2} - {}^{\theta}\!/_{2}\right| = {}^{\pi}\!/_{4}$$

3) Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$ (July 2016)

Sol: *Let* $x^3 + y^3 = 3axy$

$Diff\ \ w.r.t\ \ 'x'$ we get

$3x^2 + 3y^2 y_1 = 3a\ \ y + xy_1$

$\Rightarrow y_1 = \dfrac{ay - x^2}{y^2 - ax}$

$y_2 = \dfrac{ay_1 - 2x \quad y^2 - ax \ - \ ay - x^2 \quad 2yy_1 - a}{y^2 - ax \quad ^2}$

$at\ the\ point\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$

$y_1 = -1\ and\ \ y_2 = -\dfrac{32}{3a}$

$\therefore \quad \rho = \left|\dfrac{\{1+y_1^{\,2}\}^{\frac{3}{2}}}{y_2}\right| = \dfrac{3a}{8\sqrt{2}}$

4) If $y = \sin\log(x^2 + 2x + 1)$ *prove that* $(x+1)^2 y_{n+2} + (2n+1)(x+1)y_n + (n^2 + 4)y_n = 0$

(July 2016, Jan 2014, Jan 2015)

Sol: $y = \sin\log(x^2 + 2x + 1)$

$y = \sin 2\log(x+1)$

$y_1 = \dfrac{2\cos 2\log(x+1)}{x+1}$

$x+1\ \ y_1 = 2\cos\ \ 2\log(x+1)$

$x+1\ \ y_2 + y_1 = -\dfrac{2\sin\ \ 2\log(x+1)}{x+1}$

$x+1\ ^2\ y_2 + \ \ x+1\ \ y_1 + 4y = 0$

Differentiating each term n times

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_n + (n^2 + 4)y_n = 0$

5) Find the pedal equation of the curve $r^m \cos m\theta = a^m$ (July 2016)

Sol:

$Let \quad r^m \cos m\theta = a^m$..........(1)

$taking \log on b.s$

$m\log r + \log \cos m\theta = m\log a$

$$\frac{m}{r}\frac{dr}{d\theta} - m\frac{\sin m\theta}{\cos m\theta} = 0$$

$$\cot\phi = \tan m\theta \Rightarrow \phi = \frac{\pi}{2} - m\theta$$

$$p = r\sin\varphi = r\sin\left(\frac{\pi}{2} - m\theta\right) = r\cos m\theta$$

$$= r\left(\frac{a^m}{r^m}\right) = a^m r^{1-m}$$

6) Find the radius of curvature of $x^4 + y^4 = 2$ at the point (1,1) (July 2016)

Soln: $Let \quad x^4 + y^4 = 2$

$Diff \ w.r.t \ 'x'$

$$4x^3 + 4y^3 y_1 = 0 \Rightarrow y_1 = -\frac{x^3}{y^3} \text{ and } y_2 = -\frac{y^3 \; 3x^2 \; - x^3 \; 3y^2 y_1}{y^6}$$

$At \quad (1,1)$

$y_1 = -1$ and $y_2 = 6$

$$\therefore \quad \rho = \left|\frac{\{1+y_1^2\}^{\frac{3}{2}}}{y_2}\right| = \frac{(2)^{3/2}}{6} = \frac{\sqrt{2}}{3}$$

7) Let $y = \dfrac{x^2}{2x^2 + 7x + 6}$ **(Jan 2016)**

Sol: This is an improper function. We make it proper fraction by actual division and later spilt that into partial fractions.

$$\text{i.e} \quad x^2 \div (2x^2 + 7x + 6) = \frac{1}{2} + \frac{(-7/2\, x - 3)}{2x^2 - 7x + 6}$$

$$\therefore \ y = \frac{1}{2} + \frac{-7/2\, x - 3}{(2x+3)(x+2)}$$ Resolving this proper fraction into partial fractions,

we get

$y = \frac{1}{2} + \left[\frac{A}{(2x+3)} + \frac{B}{(x+2)} \right]$. Following the above examples for finding A & B, we get

$$y = \frac{1}{2} + \left[\frac{9/2}{2x+3} + \frac{(-4)}{x+2} \right]$$

Hence, $y_n = 0 + \frac{9}{2}\left[\frac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n \right] - 4\left[\frac{(-1)^n n!}{(x+2)^{n+1}}(1)^n \right]$

i.e $y_n = (-1)^n n!\left[\frac{9/2\,(2)^n}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}} \right]$

8) Find the angle of intersection between the curves (Jan 2016)

Sol: $r^2 \sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$

$$r^2 \sin 2\theta = 4$$

$$2\log r + \log\sin 2\theta = \log 4$$

Differentiatingthis w.r.t θ, *we get*

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$$

$$ie\ \frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta$$

$$\frac{1}{r}dr = -\cot 2\theta d\theta$$

$$ie,\ \cot\phi_1 = \cot(-2\theta)$$

$$\phi_1 = -2\theta$$

$$r^2 = 16\sin 2\theta$$

$$2\log r = \log\sin 2\theta + \log 16$$

Differentiatingthis w.r.t θ, *we get*

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

$$ie \frac{1}{r}\frac{dr}{d\theta} = \cot 2\theta$$

$$\frac{1}{r}dr = \cot 2\theta d\theta$$

$$ie, \cot\phi_2 = \cot(2\theta)$$

$$\phi_2 = 2\theta$$

$$\therefore \ |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta.............(1)$$

Now consider

Substituting $\theta = \frac{\pi}{12}$ in (1) we get $|\phi_1 - \phi_2| = \frac{\pi}{3}$

$\therefore$ *angle of* intersection= $\frac{\pi}{3} = 60^0$

9) Find the radius of curvature represented by

$x = a\ \theta + \sin\theta\ ,\ y = a\ 1 - \cos\theta$ (Jan 2016)

Sol: $x = a\ \theta + \sin\theta\ ,\ y = a\ 1 - \cos\theta$

$$\frac{dx}{d\theta} = a\ 1 + \cos\theta\ ,\ \frac{dy}{d\theta} = a\ \sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{a\ \sin\theta}{a\ 1 + \cos\theta}$$

$$= \frac{2\sin(\theta/2)\cos\ \theta/2}{2\cos^2(\theta/2)}$$

$$\therefore \quad y_1 = \tan\ \theta/2$$

Differentiating w.r.t x we get

$$y_2 = \sec^2\ \theta/2\ \frac{1}{2}\frac{d\theta}{dx}$$

$$= \sec^2\ \theta/2\ \frac{1}{2}\frac{1}{a(1 + \cos\theta)}$$

$$= \frac{\sec^2\ \theta/2}{4a\cos^2\ \theta/2}$$

$$\therefore \quad y_2 = \frac{1}{4a}\sec^2(\theta/2)$$

$$we\,have\;\rho = \frac{1+y_1^{\,2}}{y_2}$$

$$= \frac{\left(1+\tan^2(\theta/2)\right)^{3/2} 4a}{\sec^4(\theta/2)}$$

$$= \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

$$Thus \qquad \rho = \cos(\theta/2)$$

10) $y^{1/m} + y^{-1/m} = 2x$, or $y = \left[x+\sqrt{x^2-1}\right]^m$ or $y = \left[x-\sqrt{x^2-1}\right]^m$

Show that $\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0$ **(**Jan 2016, **July 2015)**

Sol: Consider $y^{1/m} + y^{-1/m} = 2x \quad \Rightarrow \quad y^{1/m} + \frac{1}{y^{1/m}} = 2x$

$\Rightarrow \left(y^{1/m}\right)^2 - 2x\left(y^{1/m}\right) + 1 = 0$ Which is quadratic equation in $y^{1/m}$

$$\therefore\; y^{1/m} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2} = \left(x \pm \sqrt{x^2-1}\right) \Rightarrow y^{1/m} = \left(x \pm \sqrt{x^2-1}\right)$$

$$\therefore\; y = \left(x \pm \sqrt{x^2-1}\right)^m$$

so, we can consider $y = \left[x+\sqrt{x^2-1}\right]^m$ or $y = \left[x-\sqrt{x^2-1}\right]^m$

Let us take $y = \left[x+\sqrt{x^2-1}\right]^m$

$$\therefore\; y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(1+\frac{1}{2\sqrt{x^2-1}}(2x)\right)$$

$$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$$

$\left(\sqrt{x^2-1}\right)y_1 = my$. On squaring

$(x^2-1)y_1^2 = m^2y^2$.

Again differentiating w.r.t x,

$$(x^2-1)2y_1y_2 + y_1^2(2x) = m^2(2yy_1)$$

or $\quad (x^2-1)y_2 + xy_1 = m^2y \quad (\div 2y_1)$

or $\quad (x^2-1)y_2 + xy_1 - m^2y = 0$ ------------(*)

Differentiating (*) n- times using Leibnitz's theorem and simplifying, we get

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

11) Find the pedal equation of $r^n = a(1+\cos n\theta)$ (Jan 2016)

$Sol: Let \quad r^n = a(1+\cos n\theta)$

$taking \log on b.s$

$n\log r = \log a + \log(1+\cos n\theta)$

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{1+\cos n\theta} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin n\theta}{1+\cos n\theta}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\varphi = \frac{1}{r^2}\left(1+\frac{\sin^2 n\theta}{(1+\cos n\theta)^2}\right)$$

$$= \frac{1}{r^2}\left(\frac{2(1+\cos n\theta)}{(1+\cos n\theta)^2}\right) = \frac{1}{r^2}\left(\frac{2}{1+\cos n\theta}\right)$$

$$= \frac{1}{r^2}\left(\frac{2a}{r^n}\right) = \frac{2a}{r^{n+2}}$$

12) Find the radius of curvature of $r^n = a^n \sin n\theta$ (Jan 2016)

$Sol: Let \quad r^n = a^n \sin n\theta$..........(2)

$taking \log on b.s$

$n\log r = n\log a + \log\sin n\theta$

$$\frac{n}{r}\frac{dr}{d\theta}=n\frac{\cos n\theta}{\sin n\theta}$$

$$\cot\phi=\cot n\theta\Rightarrow\phi=n\theta$$

$$p=r\sin\varphi=r\sin\ n\theta\ =r\left(\frac{r^n}{a^n}\right)=\frac{r^{1+n}}{a^n}$$

$$\frac{dp}{dr}=\frac{1+n\ r^n}{a^n}$$

$$\rho=r/\frac{dp}{dr}=\frac{a^n}{1+n\ r^{n-1}}$$

13) If y=cos(mlogx), prove that $x^2y_{n+2}-\ 2n+1\ xy_{n+1}+\ m^2+n^2\ \ y_n=0$ (Jan 2015)

$$sol:\ consider\ y=\cos(\text{mlogx})$$

$$\Rightarrow\ y_1=-\sin(\text{mlogx})\frac{m}{x}$$

$$\Rightarrow\ x\,y_1=-m\sin(\text{mlogx})$$

$$\Rightarrow x^2\,y_2+x\,y_1+m^2y=0$$

$$differentiating\ n\,times\ we\,get$$

$$\Rightarrow x^2\,y_{n+2}+2nx\,y_{n+1}+n(n-1)y_n+x\,y_{n+1}+ny_n+m^2y_n=0$$

$$\Rightarrow x^2y_{n+2}-\ 2n+1\ xy_{n+1}+\ m^2+n^2\ \ y_n=0$$

14) Find the pedal equation for the curve $r^m=a^m\sin m\theta+b^m\cos m\theta$ **(July 2015)**

Sol: Consider $r^m=a^m\sin m\theta+b^m\cos m\theta$

Diff. w.r.t θ

$$mr^{m-1}\frac{dr}{d\theta}=a^m(m\cos m\theta)+b^m(-m\sin m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta}=a^m\cos m\theta-b^m\sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{a^m\cos m\theta-b^m\sin m\theta}{a^m\sin m\theta+b^m\cos m\theta}$$

$$\cot\phi=\frac{a^m\cos m\theta-b^m\sin m\theta}{a^m\sin m\theta+b^m\cos m\theta}$$

Consider $p=r\sin\phi,\frac{1}{p}=\frac{1}{r}\operatorname{cosec}\phi$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$= \frac{1}{r^2}\left[1+\left(\frac{a^m\cos m\theta - b^m\sin m\theta}{a^m\sin m\theta + b^m\cos m\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[\frac{\left(a^m\sin m\theta + b^m\cos m\theta\right)^2 + \left(a^m\cos m\theta - b^m\sin m\theta\right)^2}{\left(a^m\sin m\theta + b^m\cos m\theta\right)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2m}+b^{2m}}{r^{2m}}\right]$$

$\Rightarrow p^2 = \dfrac{r^{2(m+1)}}{a^{2m}+b^{2m}}$ is the required *p*-*r* equation

15) Find the angle of intersection between the curves

$r = a\log\theta$ and $r = a/\log\theta$ **(Jan 2015)**

Sol: Consider

$r = a\log\theta$ and $r = a/\log\theta$

Diff w.r.t θ

$$\frac{dr}{d\theta} = a/\theta$$

$$r\frac{d\theta}{dr} = a\log\theta\left(\theta/a\right)$$

$$\tan\phi_1 = \theta\log\theta.........(i)$$

Diff w.r.t θ

$$\frac{dr}{d\theta} = -a/\left(\log\theta\right)^2 . 1/\theta$$

$$r\frac{d\theta}{dr} = -\left(a/\log\theta\right)\left(\frac{\left(\log\theta\right)^2\theta}{a}\right)$$

$$\tan\phi_2 = -\theta\log\theta.........(ii)$$

We know that

$$\tan\left(\phi_1-\phi_2\right) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$$= \frac{\theta\log\theta - \left(-\theta\log\theta\right)}{1+\left(\theta\log\theta\right)\left(-\theta\log\theta\right)}$$

i.e $\tan\left(\phi_1-\phi_2\right) = \dfrac{2\theta\log\theta}{1-\left(\theta\log\theta\right)^2}$...........$(iii)$

From the data: $a\log\theta = r = a/\log\theta \Rightarrow \left(\log\theta\right)^2 = 1 \quad or \log\theta = \pm 1$

As θ is acute, we take by θ =1$\Rightarrow \theta = e$ $\|NOTE\|$

Substituting $\theta = e$ in (iii), we get

$$\tan(\phi_1 - \phi_2) = \frac{2e\log e}{1-(\log e)^2} = \left(\frac{2e}{1-e^2}\right) \quad (\because \log_e^e = 1)$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

16) Find the nth derivative of $\sin^3 x\cos^3 x$ (July 2015)

Sol: Let $y = \sin^3 x\cos^3 x = \left(\frac{\sin 2x}{2}\right)^3 = \frac{\sin^3 2x}{8} = \frac{1}{8}\left[\frac{-\sin 6x + 3\sin 2x}{4}\right]$

$$= \frac{1}{32}[3\sin 2x - \sin 6x]$$

$$y_n = \frac{1}{32}\left[3.2^n \sin\left(2x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right)\right]$$

17) Find the radius of curvature of the curve

$x = a\log(\text{sect} + \text{tant}),\ \ y = \text{asect}$ (July 2015)

$sol: \Rightarrow x = a\log(\text{sect} + \text{tant})$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t}\sec t\tan t + \sec^2 t = \frac{a\sec t(\text{seet} + \text{tant})}{(\text{seet} + \text{tant})}$$

$$\therefore \frac{dx}{dt} = a\sec t$$

Also $y = a\sec t$ *gives*

$$\frac{dy}{dt} = a\sec t\tan t$$

$$Now,\ y_1 = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a\sec t\tan t}{a\sec t}$$

$$y_1 = \tan t$$

Differentiating w.r.t x we get

$$y_2 = \sec^2 t\frac{dt}{dx}$$

$$\therefore y_2 = \frac{\sec t}{a}$$

$$we\,have\ \rho=\frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$$\rho=\frac{a\left(1+\tan^2 t\right)^{3/2}}{\sec t}$$

$$\rho=a\sec^2 t$$

18) Show that the curves $r=a(1+\cos\theta)$ and $r=b(1-\cos\theta)$ (July 2015)

Sol: Consider

$r=a(1+\cos\theta)$

Diff w.r.t θ

$$\frac{dr}{d\theta}=-a\sin\theta$$

$$r\frac{d\theta}{dr}=\frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$\tan\phi_1=-\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$=-\cot\theta/2$$

i.e $\tan\phi_1=\tan\left(\pi/2+\theta/2\right)\Rightarrow\phi_1=\left(\pi/2+\theta/2\right)$

$r=b(1-\cos\theta)$

Diff w.r.t θ

$$\frac{dr}{d\theta}=b\sin\theta$$

$$r\frac{d\theta}{dr}=\frac{b(1-\cos\theta)}{b\sin\theta}$$

$$\tan\phi_1=-\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$=\tan\theta/2$$

$\tan\phi_1=\tan\theta/2\Rightarrow\phi_1=\phi_2$

Angle between the curves

$$|\phi_1-\phi_2|=\left|\left(\pi/2+\theta/2\right)-\theta/2\right|=\pi/2$$

Hence, the given curves intersect orthogonally.

19) Find the radius of curvature of $a^2y=x^3-a^3$ at the point where the curve cuts x-axis.

(July 2014)

Soln: Since the curve cuts X-axis, gives the point (a,0)

$Let \quad a^2 y = x^3 - a^3$

$Diff \;\; w.r.t \;\; 'x'$

$a^2 y_1 = 3x^2 \Rightarrow y_1 = \frac{3x^2}{a^2}$ and $y_2 = \frac{6x}{a^2}$

$At \quad (a,0)$

$y_1 = 3$ and $y_2 = \frac{6}{a}$

$$\therefore \quad \rho = \left| \frac{\{1+y_1^2\}^{\frac{3}{2}}}{y_2} \right| = \frac{(1+9)^{3/2}}{(6/a)} = \frac{5\sqrt{10}a}{3}$$

20) **Find the radius of curvature at any point t of the curve**

$$\boldsymbol{x = a(cost + logtant/2) \qquad y = asint.}$$

(July 2014)

Soln: For the given curve we have

$$\frac{dx}{dt} = a[-sint + \frac{1}{\tan\left(\frac{t}{2}\right)}.sec^2\left(\frac{t}{2}\right).1/2]$$

$$=a[-sint + \frac{1}{\text{sint}}]$$

$\frac{dx}{dt}$ =a.$\frac{cos^2 t}{sint}$

$\frac{dx}{dt}$ =a$cos^2 t$ cosect and also $\frac{dy}{dt}$ =a.cost.

Now $y_1 = \frac{dy}{dt} = \frac{dy}{dt} / \frac{dx}{dt}$ =tant.

Hence $\quad y_2 = \frac{sec^4 t \; sint}{a}$.

We have, $\quad \rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2} = \frac{\boldsymbol{a\,sec^3 t}}{\boldsymbol{sec^4 t \;\; sint}}$

ρ =a cost.

21) If x = sint and y = sinpt Prove that

$$(1-x^2)y_{n+2}-(2n+1)\,\mathrm{x}\,y_{n+1}+\ \ p^2-n^2\ \ y_n=0 \qquad July 2014$$

Solution: $y=\sin\ \ p\sin^{-1}x$

$$y_1=\cos\ \ p\sin^{-1}x\ \ \frac{p}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1=p\cos\ \ p\sin^{-1}x$$

:

$$1-x^2\ \ y_2-2xy_1+p^2y=0$$

Differentiating each term n times

$$(1-x^2)y_{n+2}-(2n+1)\,\mathrm{x}\,y_{n+1}+\ \ p^2-n^2\ \ y_n=0$$

MODULE-II

DIFFERENTIAL CALCULUS-II

1) Obtain a Taylor's expansion for $f(x) = sin\ x$ in the ascending powers of $\left(x-\frac{\pi}{2}\right)$ up to the fourth degree term. (July 2016, Jan 2016)

Soln: The Taylor's expansion for $f(x)$ about $\frac{\pi}{2}$ is

$$f(x) = f(\frac{\pi}{2}) + (x-\frac{\pi}{2}) f'(\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^2}{\lfloor 2} f''(\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{\lfloor 3} f'''(\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^4}{\lfloor 4} f^{(4)}(\frac{\pi}{2}) \ldots. \rightarrow (1)$$

$$f(x) = sin\ x \Rightarrow f\left(\frac{\pi}{2}\right) = sin\frac{\pi}{2} = 1\ ; \quad f'(x) = cos\ x \Rightarrow f'\left(\frac{\pi}{2}\right) = cos\frac{\pi}{2} = 0$$

$$f''(x) = -\,sin\ x \Rightarrow f''\left(\frac{\pi}{2}\right) = -sin\frac{\pi}{2} = -1$$

$$f'''(x) = -\,cos\ x \Rightarrow f'''\left(\frac{\pi}{2}\right) = -cos\frac{\pi}{2} = 0$$

$$f^{(4)}(x) = sin\ x \Rightarrow f^{(4)}\left(\frac{\pi}{2}\right) = sin\frac{\pi}{2} = 1$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x) = 1 + (x-\frac{\pi}{2})(0) + \frac{(x-\frac{\pi}{2})^2}{\lfloor 2}(-1) + \frac{(x-\frac{\pi}{2})^3}{\lfloor 3}(0) + \frac{(x-\frac{\pi}{2})^4}{\lfloor 4}(1) \ldots.$$

$$f(x) = \left[1 - \frac{(x-\frac{\pi}{2})^2}{\lfloor 2} + \frac{(x-\frac{\pi}{2})^4}{\lfloor 4} + \ldots\right]$$

2) Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ **(July 2016)**

$$Let\ A=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}\ (1^{\infty}\ \ form)$$

Taking log on both sides to write

$$\log_e A=\lim_{x\to 0}\log\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)\quad (\infty\times 0\ form)$$

$$=\lim_{x\to 0}\frac{\log(\frac{\tan x}{x})}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sin x.\cos x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{\frac{2}{\sin 2x}-\frac{1}{x}}{2x}=\lim_{x\to 0}\frac{2x-\sin 2x}{2x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2-2\cos 2x}{4x\sin 2x+4x^2\cos 2x}\left(\frac{0}{0}\right)==\lim_{x\to 0}\frac{-4\sin 2x}{4\sin 2x+16x\cos 2x-8x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x}{24\cos 2x-48x\sin 2x-16x^2\cos 2x}=\frac{-8}{24}=\frac{-1}{3}$$

$$\log_e A=-\frac{1}{3}\Rightarrow A=e^{-\frac{1}{3}}\quad \therefore \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}=e^{-\frac{1}{3}}$$

3) $If\ u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$ **(July 2016)**

$Sol: Let\quad f=\tan u=\frac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1

Therefore $x\frac{\partial f}{dx}+y\frac{\partial f}{dy}=nf=\tan u$

$$\Rightarrow x\left[\sec^2 u\frac{\partial u}{dx}\right]+y\left[\sec^2 u\frac{\partial u}{dy}\right]=\tan u$$

Dividing throut by '$\sec^2 u$' *we get*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\tan u}{\sec^2 u}=\frac{\sin u}{\cos u}\times\cos^2 u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\sin 2u$$

4) Expand $\log(1+e^x)$ by Maclaurian series upto 3rd degree terms. (July2016)

$$sol: y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + ...(1)$$

$$y = \log(1+e^x) \qquad \Rightarrow y(0) = \log 2$$

$$y_1 = \frac{e^x}{1+e^x} \qquad \Rightarrow y_1(0) = \frac{1}{2}$$

$$(1+e^x)y_2 + e^x y_1 = e^x \qquad \Rightarrow y_2(0) = \frac{1}{4}$$

$$(1+e^x)y_3 + 2e^x y_2 + e^x y_1 = e^x \qquad \Rightarrow y_3(0) = 0$$

Thus by substituting these values in (1) we have

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8}.......$$

5) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. (July 2016)

$$Let \quad p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}$$

$$then\ u = f(p, q, r)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{x}{y}\right) - \frac{\partial u}{\partial r}\left(\frac{z}{x}\right).......(1)$$

Similarly $y\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q}\left(\frac{y}{z}\right) - \frac{\partial u}{\partial p}\left(\frac{x}{y}\right).......(2)$

$$z\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\left(\frac{z}{x}\right) - \frac{\partial u}{\partial q}\left(\frac{y}{z}\right).......(3)$$

Adding (1), (2) and (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

6) If $x = rsin\theta cos\varphi, y = rsin\theta sin\varphi, z = rcos\theta,$ find the Jacobian of (x, y, z) with respect to r, θ, φ. (July 2016, July 2014)

Soln:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$J = \begin{vmatrix} sin\theta cos\varphi & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

After expanding, we get

J= $r^2 sin\theta\ cos^2\theta.1 + r^2 sin^3\theta.1$

J= $r^2 sin\theta\ (cos^2\theta + sin^3\theta)$

J=$\boldsymbol{r^2 sin\theta}$.

7) Evaluate $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$ **(Jan 2016)**

$$\lim_{x\to 0}\frac{x\,e^x - \log(1+x)}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{e^x + x\,e^x - \frac{1}{1+x}}{2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{e^x + e^x + x\,e^x + \frac{1}{1+x^{\ 2}}}{2}$$

$$= \frac{1+1+0+1}{2} = \frac{3}{2}$$

$$\lim_{x\to 0}\frac{x\,e^x - \log(1+x)}{x^2} = \frac{3}{2}$$

8) $u = x + \text{y} + \text{z}, uv = y + \text{z}, uvw = z$ then find $\frac{\partial\ \ x,y,z}{\partial\ \ u,v,w}$ (Jan 2016)

Solving given we get

$x = u - v;\ y = v - uvw;\ z = uvw$

Soln: Now, $$\frac{\partial\ x, y, z}{\partial\ u, v, w} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= uv$$

9) Find Maclaurin's series expansion of secx up to x^4. (Jan 2016)

Sol:

The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots. \to (1)$$

Here
$$f(\text{x}) = \text{secx} \Rightarrow f(0) = 1$$
$$f'(x) = \sec x \tan x \Rightarrow f'(0) = 0$$
$$f''(x) = \sec^3 x + \sec x \tan^2 x \Rightarrow f''(0) = 1$$

$$f'''(x) = 3\ \ f(x)\ ^2\ f'(x) + f'(x)\sec^2 x + f''(\text{x})\,\text{tanx} \Rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = 6\ f(x)\ \ f'(x)\ ^2 + 3\ \ f(x)\ ^2 f''(\text{x}) + 2f'(x)\sec^2 x \tan x + \sec^2 x f''(\text{x})$$
$$\Rightarrow f^{(4)}(0) = 4$$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = \sec x$ as

$$f(x) = \sec x = 1 + x\ (0) + \frac{x^2}{\lfloor 2}(1) + \frac{x^3}{\lfloor 3}(0) + \frac{x^4}{\lfloor 4}(4) + \frac{x^5}{\lfloor 5}(0)\ldots.$$

$$\Rightarrow \tan x = 1 + \frac{x^2}{2} + \frac{x^4}{6} + \ldots.$$

10) If $V(\text{x}, \text{y}) = \ 1 - 2xy + y^2\ ^{-1/2}$, and $x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = y^2 v^k$ then find k *Jan* 2016

$$then\ u = f\ \ p, q, r$$

$$\frac{\partial v}{\partial x} = \frac{1}{2}\ \ 1 - 2xy + y^2\ ^{-3/2}\ \ -2y$$

$$x\frac{\partial v}{\partial x} = xyv^3 .....(1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{2}\ \ 1 - 2xy + y^2\ ^{-3/2}\ \ -2x + 2y$$

$$y\frac{\partial v}{\partial y} = \ \ xy - y^2\ \ v^3 ......(2)$$

$from$ (1) & (2)

$$x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = \mathrm{v}^3\ \ y^2$$

$\therefore$ k=3

11) $If\ u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ Jan 2016

$Sol: Let\quad f = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8}$ is a homogeneous function of degree -7

Therefore $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = -7\sin u$

$$\Rightarrow x\left[\cos u\frac{\partial u}{dx}\right] + y\left[\cos u\frac{\partial u}{dy}\right] + z\left[\cos u\frac{\partial u}{dz}\right] = -7\sin u$$

Dividing throut by '$\cos u$' *we get*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u$$

12) Obtain a Maclaurin's series for $f(x) = \tan x$ up to the term containing x^5. (July 2015)

Sol:
The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) .... \to (1)$$

Here $f'(x) = \sec^2 x = 1 + f^2(\mathrm{x}) \Rightarrow f'(0) = 1 + f(0) = 1$

$f''(x) = 2f(x)\ f'(x) \Rightarrow f''(0) = 0 \qquad f'''(x) = 2\ f'(x)f''(x) + 2f'^2(x) \Rightarrow f'''(0) = 2$

$f^{(4)}(x) = 2\ f(x)f'''(x) + 6f'(x)\ f''(x) \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = 2ff^4 + 8ff''' + 6f''^2 \Rightarrow f^{(5)}(0) = 16$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = sin\ x$ as

$$f(x) = \tan x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(2) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(16)\ ....$$

$$\Rightarrow \tan x = x + \frac{x^3}{3} + 2\frac{x^5}{15}\ ....$$

13) Find first four non zero terms in the expansion of $f(x) = \dfrac{x}{e^{x-1}}$ (Jan 2015)

$$sol : y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + ...(1)$$

$$y = \frac{x}{e^{x-1}} = \frac{x}{e^x}e \qquad \Rightarrow y\ \ 0\ \ = 0$$

$$e^x y_1 + e^x y = e \qquad \Rightarrow y_1(0) = \mathrm{e}$$

$$e^x y_2 + e^x y_1{}^2 + e^x y = o \qquad \Rightarrow y_2(0) = -2\,\mathrm{e}$$

$$e^x y_3 + 3e^x y_2 + 3e^x y_1 + e^x y = o \Rightarrow y_3(0) = 3\,\mathrm{e}$$

$$e^x y_4 + 4e^x y_3 + 6e^x y_2 + 4e^x y_1 + e^x y = o \Rightarrow y_4(0) = -4\,\mathrm{e}$$

Thus by substituting these values in (1) we have

$$\frac{x}{e^{x-1}} = e(\mathrm{x} - \mathrm{x}^2 + \frac{x^3}{2} - \frac{x^4}{6} + ...)$$

14) If $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{\cot u}{2}$**.** **(Jan 2015)**

$Sol : Let \quad f = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function

of degree $\dfrac{1}{2}$

Therefore $x\frac{\partial f}{dx}+y\frac{\partial f}{dy}=nf\ =\frac{1}{2}\cos u$

$\Rightarrow x\left[-\sin u\frac{\partial u}{dx}\right]+y\left[-\sin u\frac{\partial u}{dy}\right]=\frac{1}{2}\cos u$

Dividing throut by '$-\sin u$ 'we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{\cot u}{2}$$

15) If $\log u=\frac{x^3+y^3}{3x+4y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$. **(July 2015)**

Soln:

Let $f=\log u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function

of degree 2

Therefore $x\frac{\partial f}{dx}+y\frac{\partial f}{dy}=nf\ =2\log u$

$\Rightarrow x\left[\frac{1}{u}\frac{\partial u}{dx}\right]+y\left[\frac{1}{u}\frac{\partial u}{dy}\right]=2\log u$

DMultiply throut by 'u 'we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

16) Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u=x^2+\mathrm{y}^2+\mathrm{z}^2, v=xy+yz+zx, w=x+y+z$ (Jan 2015)

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}2x & 2y & 2z\\ y+z & x+z & x+y\\ 1 & 1 & 1\end{vmatrix}$$

i.e $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=2x(\mathrm{z}-\mathrm{y})-2\,\mathrm{y}(\mathrm{z}-\mathrm{x})+2\,\mathrm{z}(\mathrm{y}-\mathrm{x})$$

$$=0$$

17) If $u=\frac{yz}{x}, v=\frac{zx}{y}, w=\frac{xy}{z}$, Show that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=4$ (July2014, July 2015)

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

i.e $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \left(-\frac{yz}{x^2}\right)\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right)\right\} - \left(\frac{z}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{y}{z}\right)\left(\frac{x}{y}\right)\right\}$$
$$+\left(\frac{y}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{x}{z}\right) - \left(\frac{y}{z}\right)\left(\frac{-zx}{y^2}\right)\right\}$$

$$\frac{\partial\ u,v,w}{\partial\ x,y,z} = 4, \text{ as desired.}$$

18) Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ **(Jan 2015)**

$Sol: Let\ A = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}\ (1^{\infty}\ \ form)$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log\left(\frac{\sin x}{x}\right)\ (\infty\times 0\ form)$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x}\cdots\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\frac{x}{\sin x}\cdot\frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}\frac{x\cos x - \sin x}{2x^3}$$

$$= \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^3}\cdots\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{6x^2}$$

$$= \lim_{x\to 0} \frac{-1}{6}\frac{x}{\sin x} = \frac{-1}{6}$$

$$\text{i.e } \log A = \frac{-1}{6}$$

$$A = e^{\frac{-1}{6}}$$

19) Evaluate $\lim_{x\to 0}\left\{\frac{e^x \sin x - x - x^2}{x^2 + x\log x(1-x)}\right\}$ (July 2015)

$$Sol: \; let \;\; A = \lim_{x\to 0}\left\{\frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}\right\}....\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left\{\frac{e^x \;\; \sin x + \cos x \;\; -1-2x}{2x + \log(1-x) + \dfrac{x}{1-x}}\right\}....\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left\{\frac{2e^x \sin x - 2}{2 + \dfrac{2}{1-x} + \dfrac{x}{1-x \;\; ^2}}\right\} = \frac{-1}{2}$$

$$A = \frac{-1}{2}$$

20) If $w = f(x, y)$,where $x = r\cos\theta$, $y = r\sin\theta$ show that (Jan 2015)

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

Sol: As $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial x}{\partial r} = \cos\theta \, , \frac{\partial x}{\partial \theta} = -r\sin\theta \, ; \frac{\partial y}{\partial r} = \sin\theta \;\; \& \;\; \frac{\partial y}{\partial \theta} = r\cos\theta \, .$$

Using Chain rule (6) & (7) we have

$$\left(\frac{\partial w}{\partial r}\right) = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x} \;\; \cos\theta \;\; + \frac{\partial w}{\partial y} \;\; \sin\theta$$

$$\left(\frac{\partial w}{\partial \theta}\right)=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial w}{\partial x}\;-r\sin\theta\;+\frac{\partial w}{\partial y}\;r\cos\theta$$

Squaring on both sides, the above equations, we get

$$\left(\frac{\partial w}{\partial r}\right)^2=\left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta+2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\sin\theta\cos\theta$$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta-2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\sin\theta\cos\theta$$

Adding the above equations , we get

$$\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2=\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}\;\cos^2\theta+\sin^2\theta$$

$$=\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2 \text{ as desired.}$$

$$\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2=\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

21) Expand $\sqrt{1+\sin 2x}$ by Maclaurian series. (July2014)

Sol : Maclaurin's series is given by

$$y(x)=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+\frac{x^4}{4!}y_4(0)+............$$

$$y=\sqrt{1+\sin 2x}=\cos x+\sin x\Rightarrow y\;\;0\;\;=1$$

$$y_1 = -\sin x + \cos x \Rightarrow y_1\ \ 0\ \ = 1$$
$$y_2 = -\sin x - \cos x \Rightarrow y_2\ \ 0\ \ = -1$$
$$y_3 = -y_1 \Rightarrow y_3\ \ 0\ \ = -1$$
$$y_4 = -y_2 \Rightarrow y_4\ \ 0\ \ = 1$$
$$\therefore\ y = \sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots\dots\dots\dots\dots\dots$$

22) Expand $f(x) = \sin(e^x - 1)$ **in powers of 'x' upto the terms containing x^4. (July2014)**

Soln: **Let** $y = \sin(e^x - 1) \Rightarrow y(0) = 0$

$$y_1 = e^x \sin(e^x - 1) \Rightarrow y_1(0) = 1$$
$$y_2 = e^x \cos(e^x - 1) - e^{2x}\sin(e^x - 1) \Rightarrow y_2(0) = 1$$
$$y_3 = e^x \cos(e^x - 1) - 3e^{2x}\sin(e^x - 1) - e^{3x}\cos(e^x - 1) \Rightarrow y_3(0) = 0$$
$$y_4 = e^x \cos(e^x - 1) - 7e^{2x}\sin(e^x - 1) - 6e^{3x}\cos(e^x - 1) + e^{4x}\sin(e^x - 1) \Rightarrow y_3(0) = -5$$

Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \dots\dots\dots\dots$$
$$\therefore\ y(x) = x + \frac{x^2}{2!} - \frac{5x^4}{4!} + \dots\dots\dots\dots$$

23) Evaluate $\underset{x\to 0}{Lt}\left(\frac{1}{x} - \cot x\right)$. **(July 2014)**

Soln:

$$K = \lim_{x\to 0}\left(\frac{1}{x} - \cot x\right)\dots\dots\dots(\infty - \infty)\ form$$
$$= \lim_{x\to 0}\frac{\sin x - x\cot x}{x\sin x}\dots\dots\dots\left(\frac{0}{0}\right) form$$

Apply L' Hospital's rule

$$= \lim_{x\to 0}\frac{x\sin x}{x\cos x + \sin x}\dots\dots\dots\left(\frac{0}{0}\right) form$$

Again applying L' Hospital's rule

$$= \lim_{x\to 0}\frac{x\cos x + \sin x}{\cos x - x\sin x + \cos x}\dots\dots\dots\left(\frac{0}{0}\right) form$$

Hence K =0

24) Evaluate $\underset{x\to 0}{Lim}\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$. **(July 2015)**

Soln:

Let $K = \lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$

Taking log on both sides

$\log K = \lim_{x \to 0} \frac{1}{x} \log \left(\frac{a^x + b^x + c^x}{3} \right)$......$0 \times \infty$

$= \lim_{x \to 0} \frac{\log(a^x + b^x + c^x) - \log 3}{x}$......$\left(\frac{0}{0}\right)$

By L'Hospital rule

$= \lim_{x \to 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$

$= \frac{1}{3} \log abc = \log(abc)^{1/3}$

K=$(abc)^{\frac{1}{3}}$

25) **Evaluate** $\lim_{x\to 0}[\frac{a^x+b^x+c^x+d^x}{4}]^{1/x}$ (July 2014)

Soln: Let K=$\lim_{x\to 0}[\frac{a^x+b^x+c^x+d^x}{4}]^{\frac{1}{x}}$ ... ... (1^∞)

log K=$\lim_{x\to 0} 1/x log\,[\frac{a^x+b^x+c^x+d^x}{4}]$ ... ... $\left(\frac{0}{0}\right)$

Applying L'Hospital's rule we get

logk = $\lim_{x\to 0} \frac{\left(\frac{4}{a^x+b^x+c^x+d^x}\right)(a^x loga+b^x logb+c^x logc+d^x logd)\cdot 1/4}{1}$

= ¼ log(abcd)

logk = $\log(abcd)^{\frac{1}{4}}$

K = $(abcd)^{\frac{1}{4}} = \sqrt[4]{abcd}.$

26) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, find $x^2 \frac{\partial u}{\partial x}$. **(July 2014)**

Soln:

Let $s = \frac{y-x}{xy}, t = \frac{z-x}{xz}$

$$then\ u = f\ \ s,t$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial f}{\partial s}\left[\frac{-y^2}{x^2y^2}\right] + \frac{\partial f}{\partial t}\left[\frac{-z^2}{x^2z^2}\right]$$

$$x\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}\left[\frac{-1}{x}\right] + \frac{\partial f}{\partial t}\left[\frac{-1}{x}\right]$$

$$= \frac{-1}{x}\left[\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right]$$

MODULE III
VECTOR CALCULUS

1) A particle moves along the curve $x=2t^2, y=t^2-4t, z=3t-5$, where t is the time, find the component of its velocity and acceleration in the direction of the vector $i-3j+2k$ at $t=1$.

(July 2016)

Sol: We have $\vec{r} = 2t^2\, I + (t^2-4t)\, J + (3t-5)\, K$ being the position vector of a particle at time t.

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\, I + (2t-4)\, J + 3K$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4I + 2J$$

$$(\vec{v})_{t=1} = 4I - 2J + 3K = \vec{V}\ (\text{say})$$

$$(\vec{a})_{t=1} = 4I + 2J = \vec{A}\ (\text{say})$$

Now the unit vector in the given direction

$$\vec{D} = I-3J+2K \text{ is } \hat{n} = \frac{I-3J+2K}{\sqrt{14}}$$

$\therefore$ The required velocity component in the direction of $\vec{D}$ is given by

$$\vec{V}.\,\hat{n} = (4I-2J+3K).\frac{I-3J+2K}{\sqrt{14}}$$

$$= \frac{4+6+6}{\sqrt{14}} = 8\sqrt{2/7}$$

Also the required acceleration component in the direction of $\vec{D}$ is given by

$$\vec{A}.\,\hat{n} = (4I+2J).\frac{I-3J+2K}{\sqrt{14}}$$

$$= \frac{4-6}{\sqrt{14}} = -\sqrt{\frac{2}{7}}$$

Thus the required velocity and acceleration components are $8\sqrt{2/7}$ and $-\sqrt{\frac{2}{7}}$.

2) Show that $\vec{F} = (6xy+z^3)I + (3x^2-z)J - (3xz^2-y)K$ is irrotational, find ϕ such that $F = \nabla\phi$.

(July 2016)

Sol: We have to shoe that $\text{curl}\,\vec{F} = 0$

$$\text{curl}\,\vec{F}=\begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$=I\{-1-(-1)\}+J\{3z^2-3z^2\}+K\{6x-6x\}$$

i.e.,

$$=0I+0J+0K=\vec{0}$$

Hence $\vec{F}$ is irrotational.

Now we have to find ϕ such that $F=\nabla\phi$

$$(6xy+z^3)I+(3x^2-z)J+(3xz^2-y)K=\frac{\partial\phi}{\partial x}I+\frac{\partial\phi}{\partial y}J+\frac{\partial\phi}{\partial z}K$$

$$\Rightarrow\frac{\partial\phi}{\partial x}=6xy+z^3,\ \frac{\partial\phi}{\partial y}=3x^2-z,\ \frac{\partial\phi}{\partial z}=3xz^2-y$$

$$we\ have\ d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$$

$$=(6xy+z^3)dx+(3x^2-z)dy+(3xz^2-y)dz$$

$$=d(3x^2y)+d(z^3x)-d(yz)$$

$$\therefore\phi=3x^2y+z^3x-yz+\text{constant}$$

This is the required solution.

3) Prove that $\text{div}(\text{curl}\,u)=0$. **(July 2016)**

$Sol:\ Let\ u=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ *be a vector point function of* x,y,z

$$curl\,u=\nabla\times u=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}=\sum i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)$$

$$\text{now div (curl u)}=\nabla.\nabla\times u$$

$$=\left(\sum\frac{\partial}{\partial x}i\right).\sum i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)=\sum\left(\frac{\partial^2 a_3}{\partial x\partial y}-\frac{\partial^2 a_2}{\partial x\partial z}\right)$$

on expanding we get,

$$\frac{\partial^2 a_3}{\partial x\partial y}-\frac{\partial^2 a_2}{\partial x\partial z}+\frac{\partial^2 a_1}{\partial y\partial z}-\frac{\partial^2 a_3}{\partial y\partial x}+\frac{\partial^2 a_2}{\partial z\partial x}-\frac{\partial^2 a_1}{\partial z\partial y}=0$$

Thus $\text{div}(\text{curl}\,u)=0$

4) If $\vec{r} = xi + yj + zk$, then prove that $i)\nabla\times\vec{r} = 0 \quad ii)\nabla^2 r^n = n(n+1)r^{n-2}$ **(July 2016)**

Sol : $i)\nabla\times\vec{r} = 0$

$Given\ \vec{r} = xi + yj + zk$

$$\nabla\times\vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\ \ 0-0\ \ - j(0-0) + k(0-0)$$

$$= 0$$

$ii)\nabla^2 r^n = n(n+1)r^{n-2}$

$$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\frac{\partial}{\partial x}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left\{ nr^{n-1}\frac{x}{r} \right\}$$

$$= \sum \frac{\partial}{\partial x}\ \ nr^{n-2}x\ \ = n\sum\left(r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x \right)$$

$$= n\sum\ \ r^{n-2} + (n-2)r^{n-4}x^2$$

On expanding the summation, we get

$$= n\ \ \ r^{n-2} + (n-2)r^{n-4}x^2\ \ +\ \ r^{n-2} + (n-2)r^{n-4}y^2\ \ +\ \ r^{n-2} + (n-2)r^{n-4}z^2$$

$$= n\ \ 3r^{n-2} + (n-2)r^{n-4}r^2\ \ = nr^{n-2}(3+n-2)$$

$$= n(n+1)r^{n-2}$$

5) Find div $\vec{F}$ **and curl** $\vec{F}$ **where** $\vec{F} = grad\ \ x^3 + y^3 + z^3 - 3xyz$ **(July2016)**

$Sol:\ Let\ \phi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{F} = grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$=\ \ 3x^2 - 3yz\ \ i +\ \ 3y^2 - 3xz\ \ j +\ \ 3z^2 - 3xy\ \ k$$

$$Now\ \ div\vec{F} = \nabla.\vec{F}$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \right).\ \ 3x^2 - 3yz\ \ i +\ \ 3y^2 - 3xz\ \ j +\ \ 3z^2 - 3xy\ \ k$$

$$= \frac{\partial}{\partial x}\ \ 3x^2 - 3yz\ \ i + \frac{\partial}{\partial y}\ \ 3y^2 - 3xz\ \ j + \frac{\partial}{\partial z}\ \ 3z^2 - 3xy\ \ k$$

$$= 6(x+y+z)$$

$$Also\ \ curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

6) A particle moves along the curve whose parametric equations are $x = t^3+1,\ y = t^2, z = 2t+5$ where t is the time. Find the component of its velocity at t = 1 in the direction of I + J + 3K. Find also the component of its acceleration at t = 1 along the normal to I + J + 3K.

(Jan 2016)

Sol: We have $\vec{r} = (t^3+1)\,I + t^2\,J + (2t+5)\,K$ being the position vector of a particle at time t.

$$\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\,I + 2t\,J + 2\,K$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 6t\,I + 2J$$

$$(\vec{v})_{t=1} = 3\,I + 2\,J + 2\,K = \vec{V}\,(say)$$

$$(\vec{a})_{t=1} = 6\,I + 2\,J = \vec{A}\,(say)$$

Now the unit vector in the given direction

$$\vec{D} = I+J+3K \ \text{ is } \ \hat{n} = \frac{I+J+3K}{\sqrt{11}}$$

$\therefore$ The required velocity component in the direction of $\vec{D}$ is given by

$$\vec{V}.\,\hat{n} = (3\,I + 2\,J + 2\,K).\frac{I+J+3K}{\sqrt{11}}$$

$$= \frac{3+2+6}{\sqrt{11}} = \sqrt{11}$$

Also the required acceleration component in the direction of $\vec{D}$ is given by

$$\vec{A}.\,\hat{n} = (6t\,I + 2J).\frac{I+J+3K}{\sqrt{11}}$$

$$= \frac{6+2+3}{\sqrt{11}} = \sqrt{11}$$

Thus the required velocity and acceleration components are $\sqrt{11}$.

7) Verify whether $\vec{A}$=(2x+yz)I+(4y+zx)J-(6z-xy)K is irrotational or not. And find the scalar potential of $\vec{A}$.

(Jan 2016)

$$curl\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & 6z-xy \end{vmatrix}$$

$$= I(-x+x) - J(-y+y) + K(z-z) = 0$$

$\therefore Given\ \vec{F}\ is\ irrotational$

Now we have to find $\phi\, such \nabla\phi = \vec{A}$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (2x+yz)i + (4y+zx)j + (6z-xy)k$$

$$\frac{\partial\phi}{\partial x} = 2x + yz\ , Integrating$$

$$\phi = x^2 + xyz + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = 4y + zx, Integrating$$

$$\phi = 2y^2 + xyz + g(x,z)$$

$$\frac{\partial\phi}{\partial z} = 6z - xy, Integrating$$

$$\phi = 3z^2 + xyz + h(x,y)$$

$$f(y,z) = 2y^2 + 3z^2, g(x,z) = x^2 + 3z^2, h(x,y) = x^2 + 2y^2$$

$$\therefore \phi = x^2 + 2y^2 + 3z^2 + xyz$$

8) If $\vec{A}$ is a vector point function and ϕ is a scalar point function then prove that $\text{div}(\phi\vec{A}) = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi).\vec{A}$ **Jan 2016**

Soln: Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and φ be a scalar point function of x, y , z

Therefore $\varphi\vec{A} = \varphi(a_1 i + a_2 j + a_3 k) = \sum(\varphi a_1)\, i$

$$\nabla.\left(\phi \vec{A}\right) = \frac{\partial}{\partial x}\left(\phi A_1\right) + \frac{\partial}{\partial y}\left(\phi A_2\right) + \frac{\partial}{\partial z}\left(\phi A_3\right) \text{ by the property,}$$

$$= \phi\frac{\partial A_1}{\partial x} + A_1\frac{\partial \phi}{\partial x} + \phi\frac{\partial A_2}{\partial y} + A_2\frac{\partial \phi}{\partial y} + \phi\frac{\partial A_3}{\partial z} + A_3\frac{\partial \phi}{\partial z}$$

$$= \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + A_1\frac{\partial \phi}{\partial x} + A_2\frac{\partial \phi}{\partial y} + A_3\frac{\partial \phi}{\partial z}$$

$$= \phi\left(\nabla.\vec{A}\right) + \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right).\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial y}\hat{k}\right)$$

$$\nabla.\left(\phi \vec{A}\right) = \left(\nabla\phi\right).\vec{A} + \phi\left(\nabla.\vec{A}\right)$$

9) If $\vec{f} = x^2 I + y^2 J + z^2 K$ and $\vec{g} = yzI + zxJ + xyK$, then verify whether $\vec{f}\times\vec{g}$ is solenoidal or not. **(Jan 2016)**

Sol. $\vec{f}\times\vec{g} = \begin{vmatrix} I & J & K \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

$$= I(y^3x - z^3x) - J(x^3y - z^3y) + K(x^3z - y^3z)$$

$$Now\, \nabla.(\vec{f}\times\vec{g}) = \left(\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}J + \frac{\partial}{\partial z}K\right).\left(I(y^3x - z^3x) - J(x^3y - z^3y) + K(x^3z - y^3z)\right)$$

$$= \frac{\partial}{\partial x}(y^3x - z^3x) + \frac{\partial}{\partial y}(x^3y - z^3y) + \frac{\partial}{\partial z}(x^3z - y^3z)$$

$$= y^3 - z^3 + z^3 - x^3 + x^3 - y^3 = 0.$$

Hence $\vec{f}\times\vec{g}$ *is a solenoidal vector*

10) Find the directional derivative of $\phi = x^2 + y^2 + 2z^2$ at P(1, 2, 3) in the direction of line $PQ = 4i - 2j + k$. **(Jan 2016)**

Sol. Given

$$\phi = x^2 + y^2 + 2z^2$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2 + y^2 + 2z^2)\hat{i} + \frac{\partial}{\partial y}(x^2 + y^2 + 2z^2)\hat{j} + \frac{\partial}{\partial z}(x^2 + y^2 + 2z^2)\hat{k}$$

$$=(2x)\hat{i}+(2y)\hat{j}+(4z)\hat{k}$$

$$\nabla\phi\big|_{(1,2,3)}=2\hat{i}+4\hat{j}+12\hat{k}$$

The directional derivative of ϕ at the point (1,2,3) in the direction of vector $4\hat{i}-2\hat{j}+\hat{k}.$ is

$$=(2\hat{i}+4\hat{j}+12\hat{k}).\left(\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{16+4+1}}\right)$$

$$=\frac{1}{\sqrt{21}}\ 8-8+12$$

$$=\frac{12}{\sqrt{21}}$$

$$=i\left\{\frac{\partial}{\partial y}\ 3z^2-3xy\ -\frac{\partial}{\partial z}\ 3y^2-3xz\ \right\}-j\left\{\frac{\partial}{\partial x}\ 3z^2-3xy\ -\frac{\partial}{\partial z}\ 3x^2-3yz\ \right\}$$

$$+k\left\{\frac{\partial}{\partial x}\ 3y^2-3xz\ -\frac{\partial}{\partial y}\ 3x^2-3yz\ \right\}$$

$$=0$$

11) Prove that $\text{curl}(\text{grad}\,\phi)=\vec{0}$ **(Jan 2016, July 2016)**

$$\text{Sol}: grad\,\phi=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$$

$$curl(grad\,\phi)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\sum\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)i=0$$

12) A particle moves on the curve $x=2t^2, y=t^2-4t, z=3t-5,$ where t is the time. Find the components of velocity and acceleration at time t=1 in the direction $\hat{i}+3\hat{j}+2\hat{k}.$ (Jan 2015)

Soln: The position vector at any point (x,y,z) is given $\vec{r}=xi+yj+zk,$ but

$$\vec{r}=2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

Therefore, the velocity and acceleration are

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$$

$$at\, t = 1, \quad \vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

Therefore the component of velocity in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{v}.\hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{16}{\sqrt{14}}$$

Since dot product of two vector is a scalar.

The components of acceleration in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{a}.\hat{n} = (4\hat{i} + 2\hat{j}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{-2}{\sqrt{14}}$$

13**) Find the constants a,b,c such that the vector** **(July 2015)**

$\vec{F} = (x+y+az)\hat{i} + (x+cy+2z)\hat{k} + (bx+2y-z)\hat{j}$ is irrotational.

Soln: Given $\vec{F} = (x+y+az)\hat{i} + (x+cy+2z)\hat{k} + (bx+2y-z)\hat{j}$

Since the vector field is irrotational, therefore $\nabla \times \vec{F} = 0$.

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix}$$

$$= (c+1)\hat{i} - (1-a)\hat{j} + (b-1)\hat{k}$$

$i.e., \ (c+1)\hat{i} - (1-a)\hat{j} + (b-1)\hat{k} = 0$

This is possible only when,

c-1=0, 1-a=0, b-1=0 $\Rightarrow$ a=1, b=1, c=1.

14) Prove that 1. $\nabla r^n = nr^{n-2}\vec{r},$ **(July 2015)**

Soln: We have by the relation x=rcosθ, y=r sinθ.

By definition

$$\nabla r^n = \frac{\partial}{\partial x}\ r^n\ \hat{i} + \frac{\partial}{\partial y}\ r^n\ \hat{j} + \frac{\partial}{\partial z}\ r^n\ \hat{k}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\frac{\partial r}{\partial z}\hat{k}$$

$$But\ \ r^2 = x^2 + y^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \therefore \frac{\partial r}{\partial x} = \frac{x}{r}\ \ lll^{ly}\ \ \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= nr^{n-1}\left(\frac{x}{r}\right)\hat{i} + nr^{n-1}\left(\frac{y}{r}\right)\hat{j} + nr^{n-1}\left(\frac{z}{r}\right)\hat{k}$$

$$= nr^{n-2}x\hat{i} + nr^{n-2}y\hat{j} + nr^{n-2}xz\hat{k}$$

$$= nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= nr^{n-2}\vec{r}$$

15) Find div $\vec{F}$ **and curl** $\vec{F}$ **where** $\vec{F} = grad\ \ x^3 + y^3 + z^3 - 3xyz$ **(July 2015)**

$$Sol:\ Let\,\phi = x^3 + y^3 + z^3 - 3xyz$$

$$\vec{F} = grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$= \ 3x^2 - 3yz\ \ i + \ 3y^2 - 3xz\ \ j + \ 3z^2 - 3xy\ \ k$$

$$Now\ \ div\vec{F} = \nabla.\vec{F}$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).\ \ 3x^2 - 3yz\ \ i + \ 3y^2 - 3xz\ \ j + \ 3z^2 - 3xy\ \ k$$

$$= \frac{\partial}{\partial x}\ 3x^2 - 3yz\ \ i + \frac{\partial}{\partial y}\ 3y^2 - 3xz\ \ j + \frac{\partial}{\partial z}\ 3z^2 - 3xy\ \ k$$

$$= 6(\mathrm{x+y+z})$$

$$Also\ \ curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}\ 3z^2 - 3xy\ - \frac{\partial}{\partial z}\ 3y^2 - 3xz\ \right\} - j\left\{\frac{\partial}{\partial x}\ 3z^2 - 3xy\ - \frac{\partial}{\partial z}\ 3x^2 - 3yz\ \right\}$$

$$+ k\left\{\frac{\partial}{\partial x}\ 3y^2 - 3xz\ - \frac{\partial}{\partial y}\ 3x^2 - 3yz\ \right\}$$

$$= 0$$

16) Prove that $divCurlA = \nabla.\nabla \times A = 0$**.** **(July 2015)**

$Sol:\ Let\ \vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\ be\,a\,vector\ po\text{int}\ function\,of\ x, y, z$

$$curl\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \sum i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)$$

$$now\ divCurlA = \nabla.\nabla \times A$$

$$= \left(\sum \frac{\partial}{\partial x} i\right).\sum i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) = \sum\left(\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z}\right)$$

$on\,\text{exp}\,anding\ we\ get,$

$$\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z} + \frac{\partial^2 a_1}{\partial y \partial z} - \frac{\partial^2 a_3}{\partial y \partial x} + \frac{\partial^2 a_2}{\partial z \partial x} - \frac{\partial^2 a_1}{\partial z \partial y} = 0$$

$$Thus\quad div(CurlA) = 0$$

17) $If\ \ \vec{\text{V}} = \vec{\omega} \times \vec{\text{r}}$ prove that $\text{curl}\vec{\text{V}} = 2\vec{\omega}\,where\,\vec{\omega}\,is\,a$ constant vector. (July 2015)

$Sol:\ Let\,\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k\ be\,the\,cons\,\text{tan}\,t\,vector.$

$we\,have\ \vec{\text{r}} = xi + yj + zk$

$$\vec{\text{V}} = \vec{\omega} \times \vec{\text{r}} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \sum i(\omega_2 z - \omega_3 y)$$

$$Also\ \ curl\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z-\omega_3 y) & (\omega_3 x-\omega_1 \mathrm{z}) & (\omega_1 y-\omega_2\ \mathrm{x}) \end{vmatrix}$$

$$= \sum i\ \ \omega_1-(-\omega_1)\ \ = \sum 2\omega_1 i = 2\ \ \omega_1 i+\omega_2 j+\omega_3 k$$

$$curl\vec{V} = 2\vec{\omega}$$

18) $If\ \ \vec{r}=xi+yj+zk\ and\ |\vec{r}|=r$, Find grad div$\left(\dfrac{\vec{r}}{r}\right)$. (Jan 2015)

$Sol: If\ \ \vec{r}=xi+yj+zk$

$$\frac{\vec{r}}{r} = \frac{x}{r}i+\frac{y}{r}j+\frac{z}{r}k$$

$$div\frac{\vec{r}}{r} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$\nabla.\left(\frac{\vec{r}}{r}\right) = \frac{r-x\dfrac{\partial r}{\partial x}}{r^2}+\frac{r-y\dfrac{\partial r}{\partial y}}{r^2}+\frac{r-z\dfrac{\partial r}{\partial z}}{r^2}$$

$$= \frac{3r-\dfrac{1}{r}\ \ x^2+y^2+z^2}{r^2} = \frac{3r-\dfrac{1}{r}\ \ r^2}{r^2}$$

$$\nabla.\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$$

$$\nabla\ \left(\ \nabla.\left(\frac{\vec{r}}{r}\right)\right) = 2\nabla\left(\frac{1}{r}\right)$$

$$= 2\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right)i+\frac{\partial}{\partial y}\left(\frac{1}{r}\right)j+\frac{\partial}{\partial z}\left(\frac{1}{r}\right)k\right)$$

$$= 2\left(\frac{\partial r}{\partial x}\left(\frac{-1}{r^2}\right)i+\frac{\partial r}{\partial y}\left(\frac{-1}{r^2}\right)j+\frac{\partial r}{\partial z}\left(\frac{-1}{r^2}\right)k\right)$$

$$= \frac{-2}{r^3}\ \ xi+yj+zk$$

$$\nabla\ \left(\ \nabla.\left(\frac{\vec{r}}{r}\right)\right) = \frac{-2}{r^3}\ \vec{r}$$

19) Prove that $\nabla(\varphi\vec{A}) = (\nabla\varphi)\cdot\vec{A}+\varphi(\nabla\cdot\vec{A})$ where φ is a scalar field. **(July 2014)**

Soln: Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and φ be a scalar point function of x, y , z

Therefore $\varphi\vec{A} = \varphi(a_1 i + a_2 j + a_3 k) = \sum(\varphi a_1)\, i$

$$\nabla.\left(\phi\vec{A}\right) = \frac{\partial}{\partial x}\left(\phi A_1\right) + \frac{\partial}{\partial y}\left(\phi A_2\right) + \frac{\partial}{\partial z}\left(\phi A_3\right) \text{ by the property,}$$

$$= \phi\frac{\partial A_1}{\partial x} + A_1\frac{\partial \phi}{\partial x} + \phi\frac{\partial A_2}{\partial y} + A_2\frac{\partial \phi}{\partial y} + \phi\frac{\partial A_3}{\partial z} + A_3\frac{\partial \phi}{\partial z}$$

$$= \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + A_1\frac{\partial \phi}{\partial x} + A_2\frac{\partial \phi}{\partial y} + A_3\frac{\partial \phi}{\partial z}$$

$$= \phi\left(\nabla.\vec{A}\right) + \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right).\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial y}\hat{k}\right)$$

$$\nabla.\left(\phi\vec{A}\right) = \left(\nabla\phi\right).\vec{A} + \phi\left(\nabla.\vec{A}\right)$$

20) Find the constants a,b,c so that the vector function

$\vec{F} = \left(x + 2y + az\right)\vec{i} + \left(bx - 3y - z\right)\vec{j} + \left(4x + y + 2z\right)\vec{k}$ **is rotational .**

(July 2014)

Soln: Since $\vec{F}$ is irrotational

$\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\Rightarrow i\left(c+1\right) - j\left(4-a\right) + k\left(b-2\right) = 0$$

$$\Rightarrow a = 4, b = 2, c = -1$$

21) Show that the vector field $\boldsymbol{F} = (\boldsymbol{x^2 - yz})\boldsymbol{i} + (\boldsymbol{y^2 - zx})\boldsymbol{j} + (\boldsymbol{z^2 - xy})\boldsymbol{k}$ is irrotational and find its scalar potential. **(July 2015)**

Soln: Given $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$

Consider CurlF = ∇ X F = $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$

$=i(-x+x)-j(-y+y)+k(-z+z) = 0$

Since $\text{Curl}(\vec{F}) = 0$

Thus $\vec{F}$ is irrotational

$\frac{\partial \varphi}{\partial x} = (x^2 - yz)$ *integrating* $\varphi = \frac{x^3}{3} - xyz + f_1(y,z)$

Let $\nabla\varphi = F$ i.e $i\frac{\partial \varphi}{\partial x} + j\frac{\partial \varphi}{\partial y} + k\frac{\partial \varphi}{\partial z} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$

$\frac{\partial \varphi}{\partial y} = (y^2 - zx)$ *integrating* $\varphi = \frac{y^3}{3} - xyz + f_1(x,z)$

$\frac{\partial \varphi}{\partial z} = (z^2 - xy)$ *integrating* $\varphi = \frac{z^3}{3} - xyz + f_1(x,y)$

We get $\varphi = x^3 + y^3 + z^3 - 3xyz = c$

22) Find the constant 'a' and 'b' such that $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational. Also find a scalar function ϕ such that $\vec{F} = \nabla\phi$ **(July 2014)**

$$sol: \quad \nabla X\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$= z^2\ b-3\ j + x\ 6-a\ k = 0$$

$$a = 6\ \ b = 3$$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = \ 6xy + z^3\ \ i + \ 3x^2 - z\ \ j + \ 3xz^2 - y\ \ k$$

On solving we get $\phi = 3x^2y + xz^3 + f_1\ \ y,z$ ....(1) $\phi = 3x^2y - yz + f_2\ \ x,z$ ....(2)

$\phi = xz^3 - yz + f_3\ \ x,y$ ....(3)

Comparing (1),(2),(3) we get

$\phi = 3x^2y + xz^3 - yz$

MODULE IV

INTEGRAL CALCULUS

1) Obtain the reduction formula for $\int \sin^m x \cos^n x\, dx$. **(July 2016)**

Sol.: $I_{m,n} = \int \sin^m x \ \cos^n xdx$

$= \int \sin^{m-1} x \ \sin x \cos^n x \ dx = \int u \ v \ dx\,(say)$

we have $\int u\,v\ dx = u\int v\,dx - \iint v\,dx.u'\ dx$

Here $\int u\,dx = \int \sin x \cos^n dx$

Put $\cos x = t \ \ \therefore -\sin xdx = dt$

Hence $\int v\ dx = \int -t^n dt = -\dfrac{t^{n+1}}{n+1} = -\dfrac{\cos^{n+1} x}{n+1}$

Now $I_{m,n} = \sin^{m-1} x \left(\dfrac{-\cos^{n+1} x}{n+1} \right) - \int \dfrac{-\cos^{n+1} x}{n+1}(m-1)\sin^{m-2} x \cos xdx$

i.e., $= -\dfrac{\sin^{m-1} x\cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2} x\cos^{n+2} xdx$

$= -\dfrac{\sin^{m-1} x\cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2} x.\cos^n x.\cos^2 xdx$

$= -\dfrac{\sin^{m-1} x\cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2} x\cos^n x(1-\sin^2 x)dx$

$= -\dfrac{\sin^{m-1} x\cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1}\int \sin^{m-2} x\cos^n xdx - \dfrac{m-1}{n+1}\int \sin^m x\cos^n xdx$

$I_{m,n} = -\dfrac{-\sin^{m-1} x\cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} I_{m-2,n} - \dfrac{m-1}{n+1}\ I_{m,n}$

i.e., $I_{m,n}\left[1 + \dfrac{m-1}{n+1}\right] = \dfrac{1}{n+1}\left[-\sin^{m-1} x\cos^{n+1} x + (m-1)I_{m-2,n}\right]$

$I_{m,n}\left[\dfrac{m-1}{n+1}\right] = \dfrac{1}{n+1}\left[-\sin^{m-1} x\cos^{n+1} x + (m-1)I_{m-2,n}\right]$

$\therefore I_{m,n} = \int \sin^m x\cos^n xdx =$

$\therefore I_{m,n} = -\dfrac{\sin^{m-1} x\cos^{n+1} x}{m+n} + \dfrac{m-1}{m+n} I_{m-2,n}$

2) Solve: $x^2 + y^3 + 6x \quad dx + xy^2 dy = 0$ **(July 2016)**

Sol: $M = x^2 + y^3 + 6x$

$$N = xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad \frac{\partial N}{\partial x} = y^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y^2 \text{ This is near to } N$$

$$Now \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{x} = f(x)$$

$$Hence\ IF = e^{\int f(x)dx} = e^{2\int \frac{1}{x}dx} = e^{\log x^2} = x^2$$

Multiplying the given equation by x^2, *we get*

$$M = x^4 + x^2y^3 + 6x^3 \qquad N = x^3y^2$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 \quad \frac{\partial N}{\partial x} = 3x^2y^2$$

$$\therefore Solution \int M dx + \int N(y) dy = c$$

$$\int (x^4 + x^2y^3 + 6x^3) dx = c$$

$$\frac{x^5}{5} + \frac{x^3y^3}{3} + 2x^3 = c$$

3) Solve: $\frac{dy}{dx} = xy^3 - xy$ **(July 2016)**

Sol: We first note that the given equation is of the form $\frac{dy}{dx} + Py = Qy^n$ with P = x, Q = x

Dividing by y^3, the equation becomes

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}x = x ....(i)$$

$$Let\ us\ put \frac{1}{y^2} = v ....(ii)$$

$$so\ that - \frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}. Then\ equation\ (i)\ becomes$$

$$\left(-\frac{1}{2}\right)\frac{dv}{dx} + vx = x \ \ or \ \ \frac{dv}{dx} - 2vx = -2x ....(iii)$$

This is a linear differential equation in which x is an independent variable and v is the dependent variable with P = -2x and Q = -2x

$$\therefore \int P\,dx = \int -2x\,dx = -x^2$$

$$So\,that\; e^{\int P\,dx} = e^{-x^2}$$

Hence, the general solution of equation (iii) is

$$ve^{\int P\,dx} = \int Qe^{\int P\,dx} + c$$

$$i.e., ve^{-x^2} = \int(-2x)e^{-x^2}\,dx + c = \int e^t\,dt + c,\; where\; t = x^2$$

$$= e^t + c = e^{-x^2} + c$$

Using (ii), this becomes

$$\frac{1}{y^2} = 1 + ce^{x^2}$$

This is the general solution of the given equation.

4) Find the orthogonal trajectories of the family of curves $r^n = a^n \cos n\theta$, where a is a parameter. (July 2016)

Sol: $r^n = a^n \cos n\theta$

$$\Rightarrow n\log r = n\log a + \log(\cos n\theta)$$

Diff. wrt θ *we get,*

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan n\theta$$

$$r\frac{d\theta}{dr} = \tan n\theta$$

$$\therefore \frac{d\theta}{\tan n\theta} = \frac{dr}{r}$$

$$\Rightarrow \int\frac{dr}{r} - \int\cot n\theta\,d\theta = c$$

$$i.e., n\log r - \log(\sin n\theta) = nc$$

$$i.e., \log\left(\frac{r^n}{\sin n\theta}\right) = \log k \quad (say)$$

$$\Rightarrow r^n = k\sin n\theta$$

This is the required orthogonal trajectory.

5) If the temperature of the air is 30^0 C and the substance cools from 100^0C to 70^0C in 15 min find when temperature reaches at 40^0C (Use Newton's law of cooling)

(July 2016)

Sol: According to Newton's law of cooling

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

$$we\ have\ by\ data\ t_1 = 100, \quad t_2 = 30, \quad T = 70\ when\ t = 15$$

$$\therefore T = 30 + 70e^{-kt}$$

$$By\ applying\ the\ initial\ condition, we\ have$$

$$70 = 30 + 70e^{-kt} \ or \ e^{-15k} = \frac{4}{7} \ or \ e^{15k} = \frac{7}{4} = 1.75$$

$$\Rightarrow 15k = \log_e 1.75 \ or \ k = 0.0373$$

$$Hence\ we\ have\ T = 30 + 70e^{-0.0373t}$$

$$We\ have\ to\ find\ t\ when\ T = 40$$

$$\therefore 40 = 30 + 70e^{-0.0373t} \ or \ e^{-0.0373t} = \frac{1}{7}$$

$$e^{0.0373t} = 7 \ or \ 0.0373t = \log_e 7$$

$$\therefore t = \frac{\log_e 7}{0.0373} = 52.2$$

Thus we conclude that it will take 52.2 minutes for a substance to reach a temperature of 40^0C

6) Obtain the reduction formula for $\int \sin^n x\,dx$. Hence evaluate $\int_0^{\pi/2} \sin^n x\,dx$ **(Jan 2016)**

Sol: Let

$$I_n = \int \sin^n x\,dx$$

$$= \int \sin^{n-1}x.\sin x\,dx = \int u\ v\ dx\,(say)$$

We have the rule of integration by parts,

$$\int u\ v\ dx = u\int v\,dx - \iint v\,dx.u'\ dx$$

$$\therefore\ I_n = \sin^{n-1}x(-\cos x) - \int -\cos x\quad n-1\ \sin^{n-2}x.\cos x\ dx$$

$$= -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}x\ dx - (n-1)\ \int \sin^n x\ dx$$

$$i.e.,\ I_n = \sin^{n-1}x.\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$i.e.,\ I_n = 1 + (n-1) = -\sin^{n-1}x.\cos x + (n-1)I_{n-2}$$

$$\therefore\ I_n = \int \sin^n x\ dx = \frac{-\sin^{n-1}x.\cos x}{n} + \frac{n-1}{n}I_{n-2}$$

$$= -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}x.\cos^2 x\ dx$$

$$= -\sin^{n-1}x.\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\ dx$$

Next, let $I_n = \int_0^{\pi/2} \cos^n x dx$

$$\therefore\ \text{from (1)},\ I_n = \left[\frac{\cos^{n-1}x.\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}I_{n-2}$$

But $\cos\ \pi/2\ = 0 = \sin 0.$

Thus $I_n = \dfrac{n-1}{n}I_{n-2}$

We use this recurrence relation to find I_{n-2} by simply replacing n by (n-2)

$$i.e.,\qquad I_{n-2} = \frac{n-3}{n-2}I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4}I_{n-6}$$

Hence, $I_n = \dfrac{n\text{-}1}{n}.\dfrac{n\text{-}3}{n\text{-}2}.\dfrac{n\text{-}5}{n\text{-}4}I_{n\text{-}6}$, by back substitution

Continuing like this, the reduction formula will end up as follows

$$I_n = \frac{n\text{-}1}{n}.\frac{n\text{-}3}{n\text{-}2}.\frac{n\text{-}5}{n\text{-}4}.......\frac{2}{3}I_1\ if\ n\ is\ odd$$

$$= \frac{n\text{-}1}{n}.\frac{n\text{-}3}{n\text{-}2}.\frac{n\text{-}5}{n\text{-}4}.......\frac{1}{2}I_0\ if\ n\ is\ even$$

$$But\ \ I_1 = \int_0^{\pi/2} \sin x\,dx = -\ \cos x\ \Big|_0^{\pi/2} = -(0-1) = 1$$

$$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = - \ x \Big|_0^{\pi/2} = \frac{\pi}{2}$$

Thus we have,

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.......\frac{2}{3}\ if\ n\ is\ odd \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.......\frac{1}{2}.\frac{\pi}{2}\ if\ n\ is\ even \end{cases}$$

This is the required reduction formula.

7) Solve: $4xy + 3y^2 - x \ dx + x \ x + 2y \ dy = 0$ **(Jan 2016)**

 $Let\ M = 4xy + 3y^2 - x\ and\ N = x\ x + 2y\ = x^2 + 2xy$

$\frac{\partial M}{\partial y} = 4x + 6y\ and\ \frac{\partial N}{\partial x} = 2x + 2y.$ The equation is not exact

$$Consider\ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)....close\ to\ N.$$

$$Now\ \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) - \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$$

$Hence\ e^{\int f(x)dx}$ is an integrating factor.

ie., $e^{\int f(x)dx} = e^{\int f \frac{2}{x} dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$$M = 4x^3y + 3x^2y^2 - x^3$$
$$N = x^4 + 2x^3y$$
$$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y$$
$$\frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

Solution of the exact equation is $\int M\,dx + \int N(y)\,dy = c$

$ie., \int \ 4x^3y + 3x^2y^2 - x^3 \ dx + \int 0\,dy = c$

$Thus\ x^4y + x^3y^2 - \frac{x^4}{4} = c$, is the required solution.

8) Find the orthogonal trajectories of the family of curves $r^n = a^n \sin n\theta$,

where a is a parameter. (Jan 2016)

Sol : $r^n = a^n \sin n\theta$

$\Rightarrow n\log r = n\log a + \log(\sin n\theta)$

$Diff.\, wrt\, \theta\, we\, get,$

$\frac{1}{r}\frac{dr}{d\theta} = -\cot n\theta$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\cot n\theta$

$r\frac{d\theta}{dr} = \cot n\theta$

$\therefore \frac{d\theta}{\cot n\theta} = \frac{dr}{r}$

$\Rightarrow \int\frac{dr}{r} - \int tan\, n\theta\, d\theta = c$

$i.e., n\log r - \log(\sec n\theta) = nc$

$i.e.,\ \log\left(\frac{r^n}{\sec n\theta}\right) = \log k\ \ (say)$

$\Rightarrow r^n = k\sec n\theta$

9) Evaluate: $\int_0^\infty \frac{x^6\,dx}{(4+x^2)^{15/2}}$. **(Jan 2016)**

Sol.: Let $x = 2\tan\theta$, so that x = 0 corresponds to $\theta = 0$ and $x \to \infty\ to\ \theta = \pi/2$

$$\int_0^\infty \frac{x^6\,dx}{(4+x^2)^{15/2}} = \int_0^{\pi/2} \frac{64\tan^6\theta}{\sqrt{4^{15}}(1+\tan^2\theta)} 2\sec^2\theta\, d\theta$$

$$= \frac{128}{\sqrt{4^{15}}}\int_0^{\pi/2} \frac{\tan^6\theta}{\sec^{15}\theta}\sec^2\theta\, d\theta$$

$$= \frac{128}{\sqrt{4^{15}}}\int_0^{\pi/2} \frac{\sin^6\theta}{\cos^6\theta}\cos^{13}\theta\, d\theta$$

$$= \frac{128}{\sqrt{4^{15}}} \int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$$

$$= \frac{6144}{\sqrt{4^{15}}(9009)}$$

10) Solve: $x\frac{dy}{dx} + y = x^3 y^6$. **(Jan 2016)**

Sol.: $x\frac{dy}{dx} + y = x^3 y^6$, *dividing by* xy^6 *on both sides, we get*

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2 \text{ ....(1)}$$

$$Put\ y^{-5} = t \Rightarrow -5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$$\therefore (1)\ becomes, -\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$$i.e.,\ \frac{dt}{dx} - \frac{5}{x}t = -5x^2, \text{this is linear in } t,$$

$$Here\ P = -\frac{5}{x},\ Q = -5x^2$$

$$\therefore I.F. = e^{\int P\,dx} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

$$\therefore t(I.F) = \int Q(I.F)dx + c$$

$$y^{-5}\left(\frac{1}{x^5}\right) = \int -5x^2 \frac{1}{x^5}dx + c$$

$$\frac{1}{x^5 y^5} = -5\frac{x^{-2}}{-2} + c$$

$$\therefore \frac{1}{x^5 y^5} = \frac{5}{2x^2} + c$$

11) A body is heated to 110^0C and placed in the air at 10^0C. After one hour its temperature becomes 60^0C. How much additional time is required for it to cool to 30^0C? **(Jan 2016)**

Sol: By the law of cooling the temperature of the body at time t>0 is given by

$$T(t) = T_m + ce^{-kt}$$

When T_m is the temperature of the surroundings and c and k are positive constants to be determined. By the data we have

T(t)=110 at t=0 and T(t)=60 at t = 60 and T_m=10

$$T(t) = T_m + ce^{-kt}$$
$$110 = 10 + ce^0 \Rightarrow \ c = 100$$
$$Also\, 60 = 10 + ce^{-kt} \ \ i.e., 50 = 100e^{-60k}$$
$$e^{-60k} = \frac{1}{2} \quad or \quad e^{60k} = 2 \quad or \quad 60k = \log_e 2$$
$$\therefore \ \ k = 0.01155$$
substituting for T_m, c and k in the expression
$$T - T_m = ce^{-kt}$$
$$T = 10 + 100e^{-0.01155t}$$
This gives the temperature of the body at time $t \geq 0$
$$For\ T = 30$$
$$30 = 10 + 100e^{-0.01155t}$$
$$e^{0.01155t} = \frac{10}{2} = 5$$
$$0.01155t = \log_e 5$$
$$\therefore \ t = \frac{\log_e 5}{0.01155} \quad \Rightarrow \quad t = 139.34$$

i.e., t = approximately 139 minutes.

12) Obtain the reduction formula of the integral $\int \cos^n x dx$. (Jan2015,July 2015)

Sol: $I_n = \int \cos^n x \ dx$

$$= \int \cos^{n-1} x . \cos x \, dx$$

$$\therefore I_n = \cos^{n-1} x.\sin x - \int \sin x \ \ n-1 \ \cos^{n-2} x(-\sin x) \ dx$$
$$= -\cos^{n-1} x.\cos x + (n-1)\int \cos^{n-2} x.\sin^2 x \ dx$$
$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x) \ dx$$
$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x \ dx - (n-1) \int \cos^n x \ dx$$
$$I_n = \cos^{n-1} x.\sin x + (n-1)I_{n-2} - (n-1)\, I_n$$

$$i.e.,\ I_n = 1+(n-1) \ = \cos^{n-1} x.\sin x + (n-1) I_{n-2}$$

$$\therefore\ I_n = \int \cos^n x\ dx = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Next, let $I_n = \int_0^{\pi/2} \cos^n x dx$

$$\therefore\ from (1),\ I_n = \left[\frac{\cos^{n-1} x.\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} \ \backslash$$

$But\ \ \cos\ \pi/2\ = 0 = \sin 0.$

$$Thus\ \ I_n = \frac{n-1}{n} I_{n-2}$$

13) Evaluate $\int_0^{\pi} x \sin^2 x \cos^4 x\, dx$ **(Jan 2015)**

$sol:\ \ Let\ \ I = \int_0^{\pi} x \sin^2 x \cos^4 x\, dx$

$I = \int_0^{\pi} (\pi - x) \sin^2 (\pi - x) \cos^4 (\pi - x) dx$, by a property

$$= \int_0^{\pi} (\pi - x) \sin^2 x \cos^4 x dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x\, dx - \int_0^{\pi} x \sin^2 x \cos^4 x\, dx$$

$$= \pi \int_0^{\pi} \sin^2 x \cos^4 x\, dx - I$$

$$2I = \pi.2 \int_0^{\pi/2} \sin^2 x \cos^4 x\, dx$$

$I = \pi.\dfrac{(1).(3).(1)}{6\times 4\times 2}.\dfrac{\pi}{2}$ by reduction formula.

Thus $I = \pi^2/32$

14) Evaluate $\boldsymbol{\int_0^{2a} x^2 (\sqrt{2ax - x^2})\, dx}$. **(July 2015 ,Jan2014,July 2014)**

Sol: Let $I_1 = \int_0^{2a} x^2 (\sqrt{2ax - x^2})\, dx$

Put x=2asin$^2\theta$ and $dx = 4asin\theta cos\theta d\theta, \theta\ varies\ from\ 0\ to\ \frac{\pi}{2}$

$$= \sqrt{4a^2sin^2\theta(1-sin^2\theta)} \quad = \sqrt{4a^2sin^2\theta cos^2\theta)} = 2asin\theta cos\theta$$

Therefore

$$I_1 = \int_{\theta=0}^{\frac{\pi}{2}} 4a^2sin^4\theta\ 2asin\theta cos\theta 4asin\theta cos\theta$$

$$= 32a^4 \int_0^{\frac{\pi}{2}} sin^6\theta cos^2\theta d\theta$$

Then by reduction formula we have

$$I_1 = \frac{5\pi a^4}{8}$$

15) Find the orthogonal trajectories of the family of curves

$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ 'λ' being the parameter . **(July 2015)**

Sol: $we\, have \frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ ....(1)

Differentiating the (1) equation we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx} = 0$$

$Also\ \ from(1)$ $\quad \frac{x^2}{a^2} - 1 = \frac{-y^2}{b^2+\lambda}$

$$\Rightarrow \frac{x^2-a^2}{a^2} = \frac{-y^2}{b^2+\lambda} \text{.............(3)}$$

$Now, dividing$ (2) by(3) we get

$$\frac{x}{x^2-a^2} = \frac{y}{y^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{x}{x^2-a^2} = \frac{1}{y}\cdot\frac{dy}{dx}$$

$$i.e \quad \frac{x}{a^2} = \frac{-y}{b^2+\lambda}\cdot\frac{dy}{dx} \text{........(2)}$$

Now let us replace $\frac{dy}{dx}$ *by* $\frac{-dx}{dy}$

$$\therefore \frac{x}{x^2-a^2} = \frac{1}{y}\cdot\left(-\frac{dx}{dy}\right)$$

$$or \quad ydy = -\frac{x^2-a^2}{x}dx$$

by separating the variables

$$\Rightarrow \int ydy = -\int xdx + a^2\int\frac{dx}{x} + c$$

$$i.e \ \frac{y^2}{2} = \frac{-x^2}{2} + a^2\log x + c$$

This is the required orthogonal trajectories

16) Solve $\left(1+2xy\cos x^2-2xy\right)dx+\left(\sin x^2-x^2\right)dy=0$ (July 2015)

Sol : *Consider* $\left(1+2xy\cos x^2-2xy\right)dx+\left(\sin x^2-x^2\right)dy=0$

Here $M = 1+2xy\cos x^2-2xy$

$N = \sin x^2 - x^2$

$$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$$

$$\frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{.......given eqn. is exact}$$

The solution of given equation is

$$\int Mdx + \int Ndy = c$$

$$\int\left(1+2xy\cos x^2-2xy\right)dx = c$$

$$x + y\sin x^2 - x^2y = c$$

17) Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$ (Jan 2015)

$$Sol: consider \quad \left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$$

$$Here \quad M=xy^3+y \quad and \quad N=2\left(x^2y^2+x+y^4\right)$$

$$\frac{\partial M}{\partial y}=3xy^2+1 \qquad \frac{\partial N}{\partial x}=2\left(2xy^2+1\right)$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} ....given\ eqn. is\ not\ exact$$

$$hence\ \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y}$$

$$I.F=e^{\int\frac{1}{y}dy}=y$$

Now multiply the given equation with I.F. we get exact diff equation

$$i.e., \left(xy^4+y^2\right)dx+2\left(x^2y^3+xy+y^5\right)dy=0$$

The silution of given equation is

$$\int Mdx+\int N(y)dy=c$$

$$\int\left(xy^4+y^2\right)dx+\int 2y^5dy=c$$

$$\frac{x^2y^4}{2}+xy^2+\frac{y^6}{3}=c$$

18) Evaluate $\int_0^{\pi/2}\sin^7\theta\cos^6\theta d\theta$. **(June 2013)**

Sol: $I_{m,n}=\int_0^{\pi/2}\sin^7\theta\cos^6\theta d\theta$

$$I_{m,n}=\frac{m-1}{m+n}.\frac{m-3}{m+n-2}.......\frac{2}{3+n}.\frac{1}{n+1}$$

if m is odd and n is even or odd

$$I_{7,6}=\frac{6}{13}\frac{4}{11}\frac{2}{9}\frac{1}{7}=\frac{16}{3003}$$

19) Derive the Reduction formula for $\int_0^{\pi/2} \cos^n x\, dx,$ (July 2014)

Where n is a positive integer.

Sol: $Let\ \ I_n = \int \cos^n x\ dx$

$$= \int \cos^{n-1} x.\cos x\, dx$$

$$\therefore\ I_n = \cos^{n-1} x.\sin x - \int \sin x \quad n-1\ \ \cos^{n-2} x(-\sin x)\ dx$$

$$= -\cos^{n-1} x.\cos x + (n-1)\int \cos^{n-2} x.\sin^2 x\ dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\ dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \cos^{n-2} x\ dx - (n-1)\int \cos^n x\ dx$$

$$I_n = \cos^{n-1} x.\sin x + (n-1)I_{n-2} - (n-1)\,I_n$$

$$i.e.,\ \ I_n = \ 1+(n-1)\ = \cos^{n-1} x.\sin x + (n-1)I_{n-2}$$

$$\therefore\ \ I_n = \int \cos^n x\ dx = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Next, let $I_n = \int_0^{\pi/2} \cos^n x dx$

$$\therefore\ \ from(1),\ I_n = \left[\frac{\cos^{n-1} x.\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

$But\ \ \cos\ \ \pi/2\ \ = 0 = \sin 0.$

$$Thus\ \ I_n = \frac{n-1}{n} I_{n-2}$$

20) $Solve\ \frac{dy}{dx} + x sin 2y = x^3 \cos^2 y$ **(July 2014)**

Sol: $\frac{dy}{dx} + x sin 2y = x^3 \cos^2 y$

Dividing the given equation throughout by $\cos^2$y

$$\sec^2 y\frac{dy}{dx}+2x\tan y = x^3........(1)$$

Put $\tan y = z$ then $\sec^2 y\frac{dy}{dx}=\frac{dz}{dx}$

(1) becomes $\frac{dz}{dx}+2xz = x^3.......(2)....LDE$

$P = 2x, Q = x^3$

$I.F. = e^{\int Pdx} = e^{\int 2xdx} = e^{x^2}$

General Solution,

$z.e^{x^2} = \int x^3 e^{x^2} dx + C........\text{put } x^2 = t$

$=\frac{1}{2}e^t(t-1)+C=\frac{1}{2}e^{x^2}(x^2-1)+C$

$\therefore \tan y = \frac{1}{2}(x^2-1)+Ce^{-x^2}$

21) Solve $x+2y^3\ \frac{dy}{dx} = y$ **.** **(July 2015)**

$So\ln:\ \ Let\frac{dx}{dy}=\frac{x+2y^3}{y}=\frac{x}{y}+2y^2$

$\frac{dx}{dy}-\frac{x}{y}=2y^2...........LDE$

$P=-\frac{1}{y}, Q = 2y^2$

$I.F. = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

General Solution,

$x.\frac{1}{y} = \int 2y^2\frac{1}{y}dy + C = \frac{y^2}{2}+C$

$x = y^3 + cy$

22) Find the orthogonal trajectory of the cardiods $r = a\ 1-\cos\theta$ **, using the differential equation method .** **(Jan 2015)**

Sol: Differentiating the given equation w.r.t θ,we get $\frac{dr}{d\theta}=a\sin\theta$.Substituting for a in the given equation,we get

$r=\left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{dr}{d\theta}$.........DE of given equation

$Changing\ \frac{dr}{d\theta}\ to -r^2\frac{d\theta}{dr} \quad r=\left(\frac{1-\cos\theta}{\sin\theta}\right)\left(-r^2\frac{d\theta}{dr}\right)$

$\frac{dr}{r}+\ \cos ec\theta-\cot\theta\ \ d\theta=0$........DE of orthogonal trajectories

solving this equation,we get

$\log r+\log\ \cos ec\theta-\cot\theta\ -\log\sin\theta=\log c$

$r\frac{\cos ec\theta-\cot\theta}{\sin\theta}=c\Rightarrow r\frac{1-\cos\theta}{\sin^2\theta}=c$

$r=c\ \ 1+\cos\theta$, this is requried orthogonal trajectories.

23) Solve $\frac{dy}{dx}+xsin2y=x^3cos^2$y. **(Jan 2015)**

Sol: Dividing the equation by $\cos^2 y$, we get

$$(\sec^2 y)\frac{dy}{dx}+\frac{2x\sin y\cos y}{\cos^2 y}=x^3$$

$$or \qquad (\sec^2 y)\frac{dy}{dx}+2x\tan y=x^3$$

Let us put $\tan y=t$. Then, $\sec^2 y\frac{dy}{dx}+\frac{dt}{dx}$, and equation (i) becomes

$$\frac{dt}{dx}+2xt=x^3$$

This is a linear differential equation in which x is an independent variable and t is the dependent variable, with $P=2x,\ Q=x^3$. Therefore,

$$\int P\,dx=x^2, \quad and \quad I.F.=e^{\int P\,dx}=e^{x^2}$$

Hence, the general solution of equation (ii) is

$$te^{x^2}=\int x^3e^{x^2}dx+c=\frac{1}{2}\int ue^u du+c, \quad where\ u=x^2$$

$$=\frac{1}{2}(u-1)e^u+c=\frac{1}{2}(x^2-1)e^{x^2}+c$$

Recalling that t = tan y, this becomes

$$\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

24) Find the orthogonal trajectory of the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is the parameter.

(Jan 2015)

Sol: Differentiating the given equation we get,

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda} \cdot \frac{dy}{dx} = 0$$

$$or \qquad \lambda = -\frac{b^2 x + a^2 y(dy/dx)}{x + y(dy/dx)}$$

This gives $a^2 + \lambda = a^2 - \frac{b^2 x + a^2 y(dy/dx)}{x + y(dy/dx)} = \frac{(a^2 - b^2)x}{x + y(dy/dx)}$,

$$b^2 + \lambda = b^2 - \frac{b^2 x + a^2 y(dy/dx)}{x + y(dy/dx)} = \frac{(b^2 - a^2)y(dy/dx)}{x + y(dy/dx)}$$

Substituting these into the given equation, we obtain

$$\frac{x\{x + y(dy/dx)\}}{(a^2 - b^2)} + \frac{y\{x + y(dy/dx)\}}{(b^2 - a^2)(dy/dx)} = 1,$$

$$x^2 + xy\frac{dy}{dx} - \frac{xy}{(dy/dx)} - y^2 = a^2 - b^2,$$

$$(x^2 - y^2) + xy\left\{\frac{dy}{dx} - \frac{1}{(dy/dx)}\right\} = a^2 - b^2. \qquad (i)$$

This is the differential equation for the given family.

Changing (dy/dx) to $-(dx/dy)$ in this equation, we obtain the equation

$$(x^2 - y^2) + xy\left\{-\frac{dx}{dy} + \frac{1}{(dx/dy)}\right\} = a^2 - b^2,$$

$$(x^2 - y^2) + xy\left\{-\frac{1}{(dy/dx)} + \frac{dy}{dx}\right\} = a^2 - b^2. \qquad (ii)$$

This is the differential equation for the orthogonal trajectories. We observe that this equation is identical with equation (i). Thus, the given family and the family of its orthogonal trajectories have one and the same differential equation. This means that the given family is self-orthogonal.

MODULE V
LINEAR ALGEBRA-I

1) Find the rank of the matrix

$$A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

(July 2016)

$$Soln:\ A=\begin{bmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix} \sim A=\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

$$R_2^1 = R_2 - 2R_1, R_3^1 = R_3 - 3R_1, R_4 - 6R_1$$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

$$R_4^1 = 4R_4 - 9R_3$$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 0 & -33 & -22\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 0 & 3 & 2\end{bmatrix}$$

$$Thus\ \rho(A) = 4$$

2) *Find the* largest eigen value and the corresponding eigen vector of the matrix

$$A=\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix} \text{ by power method use } 1,0,0 \text{ as initial vector take five iterations}$$

(July 2016)

$$Soln:\ Let\ X_0=\begin{bmatrix}1\\0\\0\end{bmatrix} be\ initial\ eigen\ vector$$

$$AX_0=\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0.5\end{bmatrix} = 2X^1$$

$$AX^1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = 2.5X^2$$

1. $$AX^2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.9285 \end{bmatrix} = 2.8X^3$$

2. $$AX^3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9285 \end{bmatrix} = 2.928\begin{bmatrix} 1 \\ 0 \\ 0.9757 \end{bmatrix} = 2.926X^4$$

3. $$AX^4 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9757 \end{bmatrix} = 2.9575\begin{bmatrix} 1 \\ 0 \\ 0.9918 \end{bmatrix} = 2.9918X^5$$

3) $\text{Re}\,duce\,the\,matrix\,to\,the\,diagonal\;\;form\;A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

(July 2016)

$So\ln:\; A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

$The\,characteristic\,equation\,is\,|A - \lambda I| = \begin{vmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{vmatrix} = 0$

$solving\;we\;get\,\lambda = 2, -5$

$for\;\lambda = 2, the\,eigen\,vector\,is\;\; A - \lambda I \quad X \;=\; 0$

$$\begin{bmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -19-2 & 7 \\ -42 & 16-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$we\;get\;X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$for\;\lambda = -5, the\,eigen\,vector\,is\;\; A - \lambda I \quad X \;=\; 0$

$$\begin{bmatrix} -19-\lambda & 7 \\ -42 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -19+5 & 7 \\ -42 & 16+5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$we\ get\ X_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$The\ modal\ matrix\ P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$The\ diagonal\ matrix D = P^{-1}AP$

$$D = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$$

4) *Solve the system by Gauss – seidel method* upto 3 iterations to solve with (0,0,0) as initial values $10x + y + z = 12, x + 10y + z = 12, x + y + 10z = 12.$ (J*uly* 2016)

Soln :

First iteration : $x = 0, y = 0, z = 0$

$x_1 = \frac{1}{10}\ 12 - y - z$;

$y_1 = \frac{1}{10}\ 12 - x - z$;

$z_1 = \frac{1}{10}\ 12 - x - y$

$x_1 = 1.2, y_1 = 1.08, z_1 = 0.972$

Second iteration :

$x_2 = \frac{1}{10}\ 12 - 1.08 - 0.972$;

4. $y_2 = \frac{1}{10}\ 12 - 0.9948 - 0.972$;

$z_2 = \frac{1}{10}\ 12 - 0.9948 - 1.0033$

$x_2 = 0.9948, y_2 = 1.0033, z_2 = 1.0001$

Third iteration :

$x_3 = \frac{1}{10}\ 12 - 1.0033 - 1.0001$;

$y_3 = \frac{1}{10}\ 12 - 0.9996 - 1.0001$;

$z_3 = \frac{1}{10}\ 12 - 0.9996 - 1.0000$

$x_3 = 0.9996, y_3 = 1.0000, z_3 = 1.000$

fourth iteration :

$x_3 = \frac{1}{10}\ 12 - 1.000 - 1.0000\ ;$

$y_3 = \frac{1}{10}\ 12 - 1 - 1.0000\ ;$

$z_3 = \frac{1}{10}\ 12 - 1 - 1$

$x_4 = 1.0000, y_4 = 1.0000, z_4 = 1.0000$

Thus $x = 1, y = 1, z = 1$

5) Determine the the largest eigen value and the corresponding eigen vector of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} u \sin g\ Power\ method$$

Take $1, 0, 0\ ^T$ *as the* int *ial eigen vector and perform four iterations July* 2016

$$So \ln : A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

1. $Let\ X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} be\ any\ arbitrary\ initial\ eigen\ vector$

$$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = 2X^1$$

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.5X^2$$

$$AX^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = 2.8X^3$$

$$AX^3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = 2.8X^4$$

6) *Show that the transformation*

2.
$y_1 = 2x_1 + x_2 + x_3$

$y_2 = x_1 + x_2 + 2x_3$

$y_1 = x_1 - 2x_3$ *is regular. Write down the inverse transformation.* *July* 2016

Soln: In matrix notation the given transformation is Y = AX

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$|A| = -1 \neq 0$, *The matrix A is non-singular and hence the transformation is regular*

$$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

3. The Inverse transformation is given by $X = A^{-1}Y$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x = 2y_1 - 23y_2 - y_3,\ y = -4y_1 + 5y_2 + 3y_3\ \ z = y_1 - y_2 - y_3$

7) Reduce the quadratic form:

4. $3x^2 + 5y^2 + 3z^2 - 2yz + 2xz - 2yx$ *into canonical form.* *July* 2016

$$Soln: The\ matrix P = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0,$$

solving we get $\lambda = 2, 3, 6$

$for\,\lambda = 2\, eigen\, vector\, is$

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, the\, normalised\, vector\;\; form\, of\;\; X_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

$for\,\lambda = 3\, eigen\, vector\, is$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, the\, normalised\, vector\;\; form\, of\;\; X_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$or\,\lambda = 6\, eigen\, vector\, is$

6. $$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$

$the\, normalised\, vector\;\; form\, of\;\; X_3 = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

The orthogonal modal matrix for P is

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-\sqrt{2}}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix},$$

$the\, quadratic\;\; form\, to\, the\, required\, canonical\;\; form\, is$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 2y_1^2 + 3y_2^2 + 6y_3^2$$

8) Solve the following system of equations by Gauss-Jordan method: (Jan 2016)

$x+y+z=8;\ -x-y+2z=-4;\ 3x+5y-7z=14$

$$Soln: A=\begin{bmatrix}1 & 1 & 1 & 8\\ -1 & -1 & 2 & -4\\ 3 & 5 & -7 & 14\end{bmatrix}$$

$R_2^1=R_2+R_1, R_3^1=R_3-3R_1$

$$\begin{bmatrix}1 & 1 & 1 & 8\\ 0 & 0 & 3 & 4\\ 0 & 2 & -10 & -10\end{bmatrix}$$

$$R_2 \leftrightarrow R_3\begin{bmatrix}1 & 1 & 1 & 8\\ 0 & 2 & -10 & -10\\ 0 & 0 & 3 & 4\end{bmatrix}$$

$R_1^1=R_1+R_2$

$$\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & -1 & 5 & 5\\ 0 & 0 & 3 & 4\end{bmatrix}$$

$R_2^1=R_2^1-5R_3$

$$A=\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & -1 & 0 & -\frac{5}{3}\\ 0 & 0 & 1 & \frac{4}{3}\end{bmatrix}$$

$x=5, y=5/3, z=4/3$

9) *Verify the transformation* $y_1=19x_1-9x_2+2x_3, y_2=-4x_1+2x_2-x_3; y_3=-2x_1+x_2$

(Jan 2016)

Sol: Given Y=AX

$$Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}, A=\begin{bmatrix}19 & -9 & 2\\ -4 & 2 & -1\\ -2 & 1 & 0\end{bmatrix}, X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$|A|=1\neq 0 \rightarrow$ *the matrx is regular the inverse tranformtion is given by* $X=A^{-1}Y$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1 & 2 & 5\\ 2 & 4 & 11\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

$x_1 = y_1 + 2y_2 + 5y_3$

$x_2 = 2y_1 + 4y_2 + 11y_3$

$x_1 = -1y_2 + 2y_3$

10) Reduce the matrix to the diagonal form $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ **(Jan 2016)**

$Sol: A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

$The\ characteristic\ equation\ is\ |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$solving\ we\ get\ \lambda = \pm 2$

$for\ \lambda = 2, the\ eigen\ vector\ is\ \ A - \lambda I \quad X = 0$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-2 & 1 \\ 3 & -1-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$we\ get\ X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$for\ \lambda = -2, the\ eigen\ vector\ is\ \ A - \lambda I \quad X = 0$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 & 1 \\ 3 & -1+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$we\ get\ X_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$The\ \mathrm{mod}\ al\ matrix\ P = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

$The\ diagonal\ matrix D = P^{-1}AP$

$$D = \begin{bmatrix} 0.75 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

11) $Solve\ the\ system\ by\ Gauss-seidel\ method:$

$20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25.$

Perform three iterations. *Jan* 2016

$Sol: First\ iteration: x = 0, y = 0, z = 0$

$x_1 = 0.85, y_1 = -1.0275, z_1 = 1.01098$

Second iteration :

$$x_2 = \frac{1}{20}\ 17 + 1.0275 + 2\ \ 1.01098 \quad ;$$

$$y_2 = \frac{1}{20}\ -18 - 3\ \ 1.0025\ \ + 1.01098\ \ ;$$

$$z_2 = \frac{1}{20}\ 25 - 2\ \ 1.0025\ \ + 3\ \ -0.9998$$

$$x_2 = 1.0025, y_2 = -0.9998, z_2 = -0.9998$$

Third iteration :

$$x_3 = \frac{1}{20}\ 17 + 0.9998 + 2\ \ 0.9998 \quad ;$$

$$y_3 = \frac{1}{20}\ -18 - 3\ \ 0.9997\ \ + 0.9998\ \ 1.0025;$$

$$z_3 = \frac{1}{20}\left(25 - 2\left(1.0025\right) - 3\left(-1.000\right)\right)$$

$$x_3 = 0.9997, y_3 = -1.000, z_3 = 1.0000$$

12) Determine the the largest eigen value and the corresponding eigen vector of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ using Power method}$$

Take $\ 1,0,0\ ^T$ *as the initial eigen vector and perform four iterations* *Jan* 2016

$$\text{Sol:} A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\text{Let } X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ be any arbitrary initial eigen vector}$$

$$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = 2X^1$$

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.5X^2$$

$$AX^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = 2.8X^3$$

$$AX^2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = 2.8X^3$$

13) Reduce the quadratic form:

$8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz$ into canonical form. *Jan* 2016

$$So\ln: The\ matrix P = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0, \text{solving we get } \lambda = 0, 3, 15$$

for $\lambda = 0$ *eigen vector is*

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \text{the normalised vector form of } X_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

for $\lambda = 3$ *eigen vector is*

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \textit{the normalised vector form of } X_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$or \lambda = 15$ *eigen vector is*

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \textit{the normalised vector form of } X_2 = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

The orthogonal modal matrix for P is

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}, \textit{the quadratic form to the required canonical form is}$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 0y_1^2 + 3y_2^2 + 15y_3^2 = 3y_2^2 + 15y_3^2$$

14) Solve the following system of equations by Gauss's elimination method: (July 2015)

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + x_4 = -6$$

Sol : Consider augmented matrix $A:B$ *by* $R_1 \leftrightarrow R_4$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - 5R_1$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix}$$

$R_4 \to 5R_4 + 4R_3$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & 0 & -117 & : & 234 \end{bmatrix}$$

$Hence\,we\,have\; x_1 + x_2 + x_3 + 4x_4 = -6$

$2x_2 - x_4 = 6$

$5x_3 - 3x_4 = 1$

$-117x_4 = 234$

$\therefore x_4 = -2, x_3 = -1, x_2 = 2 \; and \; x_1 = 1 \, is\, the\, reqd.\, soln.$

15) Using Rayleigh's power method to find the largest Eigen value and the corresponding Eigen vector of the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad X^{(0)} = [1,0,0]$$

(July 2015)

$$Sol: \; AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix} = \begin{bmatrix} 7.3332 \\ -3.3332 \\ 3.3332 \end{bmatrix} = 7.3332\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 7.818 \\ -3.818 \\ 3.818 \end{bmatrix} = 7.818\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 7.952 \\ -3.952 \\ 3.952 \end{bmatrix} = 7.952\begin{bmatrix} 1 \\ -0.4969 \\ 0.4969 \end{bmatrix}$$

The largest Eigen value is $\lambda = 7.952$ and the corresponding Eigen vector is

$$1 \quad -0.4969 \quad 0.4969\,'$$

16) Find the matrix P which diagonalizes the matrix $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$ **(Jan 2015)**

Sol:

The characteristic equation of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & -1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$-(1+\lambda)\left[\ 2+\lambda \quad 3+\lambda\ \right] = 0$$

$$\lambda = -1, -2, -3$$

Let us now form the system of equations

$$(-1-\lambda)x + y + 2z = 0$$

$$0x - (2+\lambda)y - z = 0$$

$$0x + 0y - (3+\lambda)z = 0$$

$case(i) : Let \lambda = -1$ and the corresponding equations are

$$y + 2z = 0$$

$$-y + z = 0$$

$$-2z = 0$$

one variable can be chosen arbitrarily

$Let x = k_1, y = 0, z = 0$

Thus

$x_1 = (k_1 0, 0)^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

$case(ii): Let \lambda = -2$ and the corresponding equations are

x+y+2z=0

$-z = 0 \Rightarrow z = 0$

Two variables can be chosen arbitrarily

$Let y = k_1$, then $x = k_2$

Thus

$$x_2 = (k_2, k_1, 0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{is the eigen vector corresponding to } \lambda = -2$$

$case(iii): Let \lambda = -3$ and the corresponding equations are

2x+y+2z=0

$y - z = 0 \Rightarrow y = z$

one variable can be chosen arbitrarily

$Let y = k_1$, then $z = k_1$and $x = -3k_1$

Thus

$$x_2 = (-3k_1, k_1, k_1) = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{is the eigen vector corresponding to } \lambda = -2$$

$$P = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

17) Show that the transformation $y_1=x_1+2x_2+5x_3$, $y_2=2x_1+4x_2+11x_3$, $y_3=- x_{2+}2x_3$ is regular, write down the inverse transformation.

(Jan 2015)

Sol: Take Y=AX, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 11 \\ 0 & -1 & 2 \end{bmatrix}$

Then |A|=1≠0……. It is regular

.

$$A^{-1} = \frac{Adj\ A}{|A|} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Then X=A^{-1}X

$$x_1 = 19y_1 - 9y_2 + 2y_3$$
$$x_2 = -4y_1 + 2y_2 - y_3$$
$$x_3 = -2y_1 + y_2$$

Is the Inverse Transformation.

18) Reduce the quadratic form $x^2+5y^2+z^2+2yz+6xz+2xy$ to the canonical form and specify the matrix of transformation.

(Jan2015,July 2014)

Sol: Here A= $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Take characteristic equation as $|A-\lambda I|=0$

Then the Eigen values are $\lambda=-2,3,6$ and the corresponding Eigen vectors:

$$\lambda = -2, X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\lambda = 6, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

then

$$P^{-1} = \begin{bmatrix} -0.8662 & 0 & 0.8662 \\ 0.43301 & -1.11803 & 0.43301 \\ 0.43301 & 1.11803 & 0.43301 \end{bmatrix}$$

Then $D=P^{-1}AP=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = -2y_1^2 + 3y_2^2 + 6y_3^2$

19) Find the rank of the matrix $\begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$. **(July 2014)**

Sol:

Using row transformation $\begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1, R_5 \to R_5 - R_1,$

$$\sim \begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \ldots\ldots\ldots \frac{R_3}{2}, \frac{R_4}{3}, \frac{R_5}{4}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2, R_5 \to R_5 - R_2,$

$$\sim \begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -91R_2 + R_1,$

$$\sim \begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

20) Investigate the value of λ and μ **so that the equations**
$2x+3y+5z=9, 7x+3y-2z=8, 2x+3y+\lambda z=\mu$
have i)unique solution
ii) no solution
iii) infinite number of solutions. (Jan 2015)

Soln: $A:B = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$

$R_2 \to 2R_2 - 7R_1$

$R_3 \to R_3 - R_1$

$$A:B = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

$i)$unique solution :$\lambda \neq 5$

$ii)$no solution :$\lambda = 5, \mu \neq 9$

$iii)$infinite number of solutions:$\lambda = 5, \mu = 9$

21) Using elementary row transformation find the rank of the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$**.**

(June2014)

Soln:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} ....R_1 \Leftrightarrow R_4$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ RankA=2

22) For what values of λ and μ**, the following simultaneous equations have i) a unique solution i**

ii) no solution iii) an infinite number of solutions?

$x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + \lambda z = \mu$ **(July 2014)**

Sol:

$$A:B = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

$i)$unique solution :$\lambda \neq 3$

$ii)$no solution :$\lambda = 3, \mu \neq 10$

$iii)$infinite number of solutions:$\lambda = 3, \mu = 10$

23) Applying Gauss elimination method solve

$2x+3y-z=5$, $4x+4y-3z=3$, $2x-3y+2z=2$

(July 2014, Jan 2015)

Sol:

$$A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 4 & 4 & -3 & : & 3 \\ 2 & -3 & 2 & : & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & -2 & -1 & : & -7 \\ 0 & -6 & 3 & : & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & 2 & 1 & : & 7 \\ 0 & -2 & 1 & : & -1 \end{bmatrix}$$

$R_1 \to 2R_1 - 3R_2, R_3 \to R_3 + R_2$

$$A:B = \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 2 & : & 6 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3, R_2 \to R_2 - R_3$

$$A:B = \begin{bmatrix} 4 & 0 & 0 & : & 4 \\ 0 & 2 & 0 & : & 4 \\ 0 & 0 & 2 & : & 6 \end{bmatrix}$$

Hence $4x = 4, 2y = 4, 2z = 6$

$\therefore x = 1, y = 2, z = 3$

24) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

(Jan 2015)

Sol: Given A= $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

The number of non-zero rows=4.

$\therefore \rho(A)=4$.

25) Solve the system of equations by Gauss siedel method:

2x+5y+7z=52, 2x+y-z=0, x+y+z=9. (June2014)

Sol: The augmented matrix is given by

$$A:B = \begin{bmatrix} 1 & 1 & 1: & 9 \\ 2 & 1 & -1: & 0 \\ 2 & 5 & 7: & 52 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$A:B \sim \begin{bmatrix} 1 & 1 & 1: & 9 \\ 0 & -1 & -3: & -18 \\ 0 & 3 & 5: & 34 \end{bmatrix}$$

Apply $R_1 \to R_1 + R_2$, $R_3 \to R_3 + 3R_2$

$$A:B \sim \begin{bmatrix} 1 & 0 & -2: & -9 \\ 0 & -1 & -3: & -18 \\ 0 & 0 & -4: & -20 \end{bmatrix}$$

Apply $R_2 \to -1/4(R_3)$

$$A:B \sim \begin{bmatrix} 1 & 0 & -2: & -9 \\ 0 & -1 & -3: & -18 \\ 0 & 0 & 1: & 5 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_3$, $R_1 \to R_1 + 2R_3$

$$A:B \sim \begin{bmatrix} 1 & 0 & 0: & 1 \\ 0 & -1 & 0: & -3 \\ 0 & 0 & 1: & 5 \end{bmatrix}$$

The equations associated with above augmented matrix are,
x=1, y=3 and z=5 is the required solution.

26) Show that the transformation given below $y_1 = 2x_1 + x_2 + x_3, y_2 = x_1 + x_2 + 2x_3, y_3 = x_1 - 2x_3$, is regular and find the inverse transformation. (July2014)

Sol: The given transformation in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Let A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$|A| = 2(-2-0) - 1(-2-2) + 1(0-1) = -1 \neq 0$$

$\Rightarrow$ the transformation is regular

We compute $A^{-1} = \dfrac{1}{|A|} adjA$

$$adjA = \begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Inverse transformation is given by $X = A^{-1}Y$

27) Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form using characteristics equation.

(July 2014)

Sol: The characteristic equation of the given matrix is

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^3 + \lambda^2 + 5\lambda - 5 = 0$

$\lambda = 1,\ 2.24,\ -2.24$

Let us now form the system of equations

$(1-\lambda)x + 2y - 2z = 0$

$x + (2-\lambda)y + z = 0$

$-x - y - \lambda z = 0$

case(i): Let $\lambda = 1$ and the corresponding equations are

$x - y + z = 0$

$x + y + z = 0$

$$\frac{x}{-1-1} = \frac{-y}{1-1} = \frac{z}{1+1}$$

$X_1 = (-1, 0, 1)$is the eigen vector corresponding to$\lambda = 1$

case(ii): Let $\lambda = 2.24$ and the corresponding equations are

$-3.24x + 2y - 2z = 0$

$x + 0.24y + z = 0$

$-x - y - 2.24z = 0$

$$\frac{x}{1.24} = \frac{-y}{-1} = \frac{z}{1}$$

$X_2 = (1.24, 1, 1)$is the eigen vector corresponding to$\lambda = 2.24$

case(iii): Let $\lambda = -2.24$ and the corresponding equations are

$1.24x + 2y - 2z = 0$

$x + 4.24y + z = 0$

$-x - y + 2.24z = 0$

$$\frac{x}{0.24} = \frac{-y}{-1} = \frac{z}{-1}$$

$X_3 = (0.24, 1, -1)$is the eigen vector corresponding to$\lambda = -2.24$

$$P = \begin{bmatrix} -1 & 1.24 & 0.24 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\therefore D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2.24 & 0 \\ 0 & 0 & -2.24 \end{bmatrix}$$

28) Find the eigen values and eigen vector corresponding to the largest eigen value of the matrix $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$. (Jan 2015)

Soln: The characteristic equation of the given matrix is

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{vmatrix}=0$$

$$\lambda^3-7\lambda^2-36=0$$

$$\lambda=6,-2,3$$

Let us now form the system of equations

$$(1-\lambda)x+y+3z=0$$
$$x+(5-\lambda)y+z=0$$
$$3x+y+(1-\lambda)z=0$$

For the largest eigen value $\lambda=-1$ and the corresponding equations are

$$5x+y+3z=0$$
$$x-y+z=0$$
$$3x+y+5z=0$$

on simplication we get

$x=3, y=1, z=-2$

$\therefore$ corresponding eigen vector is $X=\begin{bmatrix}3\\ 1\\ -2\end{bmatrix}$

29) If $P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$ **is a modal matrix of the matrix** $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ **and inverses of P is**

$P^{-1}=\begin{bmatrix}-0.5 & 0 & 0.5\\ 0.33 & -0.33 & 0.33\\ 0.16 & 0.33 & 0.16\end{bmatrix}$**, then transform A in to diagonal form and hence find** A^4 **.**

(July 2014)

Sol:

$$D = P^{-1}AP = \begin{bmatrix} -0.5 & 0 & 0.5 \\ 0.33 & -0.33 & 0.33 \\ 0.16 & 0.33 & 0.16 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}\begin{bmatrix} -0.5 & 0 & 0.5 \\ 0.33 & -0.33 & 0.33 \\ 0.16 & 0.33 & 0.16 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

30) Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ **in to canonical form by an appropriate orthogonal transformation which transforms** $x_1\ x_2\ x_3$ **in terms of new variables** $y_1\ y_2\ y_3$. **(Jan 2015)**

Sol: The coefficient matrix of the given quadratic form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

$$\lambda = 1, 2, 4$$

Let us now form the system of equations

$$\left|\begin{matrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{matrix}\right| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case(i): Let $\lambda = 1$ and the corresponding equations are

$$X_1 = \ k_1 \quad 0 \quad 0 \ ^T$$

We find that $\|X_1\| = \sqrt{\mathrm{k}_1^2 + 0 + 0} = k_1$

The normalized form of $X_1 = \left[\quad 0 \quad 0\right]^T$

case(ii): Let $\lambda = 2$ and the corresponding equations are

$$X_2 = \ 0 \quad k_2 \quad k_2 \ ^T$$

We find that $\|X_2\| = \sqrt{0 + \mathrm{k}_2^2 + \mathrm{k}_2^2} = \sqrt{2}k_2$

The normalized form of $X_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$

case(iii): Let $\lambda = 4$ and the corresponding equations are

$$X_3 = \ 0 \quad k_3 \quad -k_3 \ ^T$$

We find that $\|X_3\| = \sqrt{0 + \mathrm{k}_3^2 + \mathrm{k}_3^2} = \sqrt{2}k_3$

The normalized form of $X_3 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T$

$$Q = \ X_1 X_2 X_3 \ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

The canonical form

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = y_1^2 + 2y_2^2 + 4y_3^2$$

31) Find the nature of the quadratic forms for which corresponding eigen values of the corresponding matrices are given as (Jan 2015)

Matrix	Eigen values
A	2,3,4
B	-3,-4,-5
C	0,3,6
D	0,-3,-4
E	-2,3,4

Sol:

Matrix	Eigen values	Nature of quadratic form
A	2,3,4	+ve definite
B	-3,-4,-5	-ve definite
C	0,3,6	+ve semi definite
D	0,-3,-4	-ve semi definite
E	-2,3,4	Indefinite

32) Find the matrix P which diagonalizes the matrix $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$ **(July 2014)**

Sol: The characteristic equation of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & -1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$-(1+\lambda)\begin{bmatrix} 2+\lambda & 3+\lambda \end{bmatrix} = 0$$

$$\lambda = -1, -2, -3$$

Let us now form the system of equations

$$(-1-\lambda)x + y + 2z = 0$$

$$0x - (2+\lambda)y - z = 0$$

$$0x + 0y - (3+\lambda)z = 0$$

$case(i) : Let \lambda = -1$ and the corresponding equations are

$$y + 2z = 0$$

$$-y + z = 0$$

$$-2z = 0$$

one variable can be chosen arbitrarily

$Letx = k_1, y = 0, z = 0$

Thus

$$x_1 = (k_1 0, 0)^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{is the eigen vector corresponding to} \lambda = -1$$

$case(ii): Let \lambda = -2$ and the corresponding equations are

x+y+2z=0

$-z = 0 \Rightarrow z = 0$

Two variables can be chosen arbitrarily

$Let y = k_1$, then $x = k_2$

Thus

$$x_2 = (k_2, k_1, 0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{is the eigen vector corresponding to } \lambda = -2$$

$case(iii): Let \lambda = -3$ and the corresponding equations are

2x+y+2z=0

$y - z = 0 \Rightarrow y = z$

one variable can be chosen arbitrarily

$Let y = k_1$, then $z = k_1$ and $x = -3k_1$

Thus

$$x_2 = (-3k_1, k_1, k_1) = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \text{is the eigen vector corresponding to } \lambda = -2$$

$$P = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

32) Reduce the quadratic form $x^2+5y^2+z^2+2yz+6xz+2xy$ to the canonical form and specify the matrix of transformation.

(Jan2015,July 2014)

Sol: Here A= $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Take characteristic equation as |A-λI|=0

Then the Eigen values are λ=-2,3,6 and the corresponding Eigen vectors:

$$\lambda = -2, X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3, \quad X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\lambda = 6, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

then

$$P^{-1} = \begin{bmatrix} -0.8662 & 0 & 0.8662 \\ 0.43301 & -1.11803 & 0.43301 \\ 0.43301 & 1.11803 & 0.43301 \end{bmatrix}$$

Then D=P^{-1}AP= $\begin{bmatrix} -2\,0\,0 \\ 0\,3\,0 \\ 0\,0\,6 \end{bmatrix} = -2y_1^2 + 3y_2^2 + 6y_3^2$