SYLLABUS

Engineering Mathematics-I

Subject Code: 15MAT11 IA Marks: 20
Hours/Week: 04 Exam. Hours: 03
Total Hours: 50 Exam. Marks: 80

Course Objectives

To enable students to apply knowledge of Mathematics in various engineering fields by making hem to learn the following:

- nth derivatives of product of two functions and polar curves.
- Partial derivatives.
- Vectors calculus.
- Reduction formulae of integration to solve First order differential equations
- Solution of system of equations and quadratic forms.

Module –1

Differential Calculus -1:

Determination of nth order derivatives of Standard functions - Problems. Leibnitz's theorem (without proof) - problems.

Polar Curves - angle between the radius vector and tangent, angle between two curves, Pedal equation for polar curves. Derivative of arc length - Cartesian, Parametric and Polar forms (without proof) - problems. Curvature and Radius of Curvature – Cartesian, Parametric, Polar and Pedal forms(without proof) and problems. **10hrs**

Module –2

Differential Calculus -2

Taylor's and Maclaurin's theorems for function of o ne variable(statement only)- problems. Evaluation of Indeterminate forms.

Partial derivatives – Definition and simple problems, Euler's theorem(without proof) – problems, total derivatives, partial differentiation of composite functions-problems, Jacobians-definition and problems .
10hrs

Module –3

Vector Calculus:

Derivative of vector valued functions, Velocity, Acceleration and related problems, Scalar and Vector point functions.Definition Gradient, Divergence, Curl- problems . Solenoidal and Irrotational vector fields. Vector identities - div (F A), curl (F A),curl (grad F), div (curl A).

10hrs

Module- 4

Integral Calculus:

Reduction formulae $\int \sin^n x\, dx$ $\int \cos^n x\, dx$ $\int \sin^n x \cos^m x\, dx$,, (m and n are positive integers), evaluation of these integrals with standard limits (0 to $\pi/2$) and problems.

Differential Equations:

Solution of first order and first degree differential equations – Exact, reducible to exact and Bernoulli's differential equations. **Applications-** orthogonal trajectories in Cartesian and polar forms. Simple problems on Newton's law of cooling. **10hrs**

Module –5

Linear Algebra Rank of a matrix by elementary transformations, solution of system of linear equations - Gauss- elimination method, Gauss- Jordan method and Gauss-Seidel method. Rayleigh's power method to find the largest Eigen value and the corresponding Eigen vector. Linear transformation, diagonalisation of a square matrix, Quadratic forms, reduction to Canonical form **10hrs**

COURSE OUTCOMES

On completion of this course students are able to

- Use partial derivatives to calculate rates of change of multivariate functions
- Analyse position, velocity and acceleration in two or three dimensions using the calculus of vector valued functions
- Recognize and solve first order ordinary differential equations, Newton's law of cooling
- Use matrices techniques for solving systems of linear equations in the different areas of linear algebra.

Engineering Mathematics – I

MODULE I

DIFFERENTIAL CALCULUS-I

CONTENTS:

SUCCESSIVE DIFFERENTIATION

In this lesson, the idea of differential coefficient of a function and its successive derivatives will be discussed. Also, the computation of n^{th} derivatives of some standard functions is presented through typical worked examples.

1. **Introduction:-** Differential calculus (DC) deals with problem of calculating rates of change. When we have a formula for the distance that a moving body covers as a function of time, DC gives us the formulas for calculating the body's **velocity** and **acceleration** at any instant.

- **Definition of derivative of a function y = f(x):-**

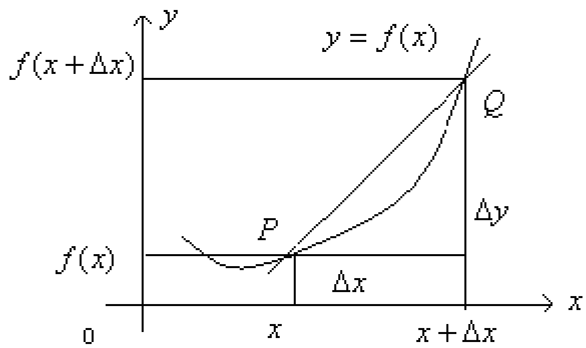


Fig.1. Slope of the line PQ is $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

The derivative of a function y = f(x) is the function $f'(x)$ whose value at each x is defined as

$$\frac{dy}{dx} = f'(x) \text{ = Slope of the line PQ (See Fig.}\mathbf{1}\text{)}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{-------- (1)}$$

$$= \lim_{\Delta x \to 0} \text{(Average rate change)}$$

= Instantaneous rate of change of *f* at *x* provided the limit exists.

The instantaneous velocity and acceleration of a body (moving along a line) at any instant *x* is the derivative of its position co-ordinate *y = f(x)* w.r.t x, i.e.,

$$\text{Velocity} = \frac{dy}{dx} = f'(x) \quad \text{--------- (2)}$$

And the corresponding acceleration is given by

$$\text{Acceleration} = \frac{d^2y}{dx^2} = f''(x) \quad \text{---------- (3)}$$

Successive Differentiation:-

The process of differentiating a given function again and again is called as **Successive differentiation** and the results of such differentiation are called **successive derivatives.**

- The higher order differential coefficients will occur more frequently in spreading a function all fields of scientific and engineering applications.
- Notations:

  i. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots$, n[th] order derivative: $\frac{d^ny}{dx^n}$

  ii $f'(x)$, $f''(x)$, $f'''(x)$, ……, n[th] order derivative: $f^n(x)$

  iii Dy, D^2y, D^3y, ………..., n[th] order derivative: D^ny

  iv y', y'', y''', ……, n[th] order derivative: $y^{(n)}$

  v. $y_1, y_2, y_3 \ldots$, n[th] order derivative: y_n

- **<u>Successive differentiation – A flow diagram</u>**

Input function: $y = f(x)$ → [Operation d/dx] → Output function $y' = \frac{df}{dx} = f'(x)$ (first order derivative)

Input function $y' = f'(x)$ → [Operation d/dx] → Output function $y'' = \frac{d^2f}{dx^2} = f''(x)$ (second order derivative)

Input function $y'' = f''(x)$ → [Operation d/dx] → Output function $y''' = \frac{d^3f}{dx^3} = f'''(x)$ (third order derivative)

Input function $y^{n-1} = f^{n-1}(x)$ → [Operation d/dx] → Output function $y^n = \frac{d^nf}{dx^n} = f^n(x)$ (nth order derivative)

Calculation of n[th] derivatives of some standard functions

- Below, we present a table of n[th] order derivatives of some standard functions for ready reference.

Sl. No	y = f(x)	$y_n = \frac{d^n y}{dx^n} = D^n y$
1	e^{mx}	$m^n e^{mx}$
2	a^{mx}	$m^n (\log a)^n a^{mx}$
3	$(ax+b)^m$	i. $m(m-1)(m-2)\ldots(m-n+1)a^n (ax+b)^{m-n}$ for all m. ii. 0 if $m<n$ iii. $(n!)a^n$ if $m=n$ iv. $\frac{m!}{(m-n)!}x^{m-n}$ if $m<n$
4	$\frac{1}{(ax+b)}$	$\frac{(-1)^n n!}{(ax+b)^{n+1}}a^n$
5.	$\frac{1}{(ax+b)^m}$	$\frac{(-1)^n (m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n$
6.	$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)!}{(ax+b)^n}a^n$
7.	$\sin(ax+b)$	$a^n \sin(ax+b+n\pi/2)$
8.	$\cos(ax+b)$	$a^n \cos(ax+b+n\pi/2)$
9.	$e^{ax}\sin(bx+c)$	$r^n e^{ax}\sin(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}(b/a)$
10.	$e^{ax}\cos(bx+c)$	$r^n e^{ax}\cos(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}(b/a)$

- We proceed to illustrate the proof of some of the above results, as only the above functions are able to produce a **sequential change** from one derivative to the other. Hence, in general we cannot obtain readymade formula for nth derivative of functions other than the above.

1. Consider e^{mx}. Let $y = e^{mx}$. Differentiating w.r.t x, we get

$y_1 = me^{mx}$. Again differentiating w.r.t x, we get

$y_2 = m(me^{mx}) = m^2 e^{mx}$

Similarly, we get

$y_3 = m^3 e^{mx}$

$y_4 = m^4 e^{mx}$

…………….

And hence we get

$$y_n = m^n e^{mx} \quad \therefore \frac{d^n}{dx^n}\left[e^{mx}\right] = m^n e^{mx}.$$

2. $(ax+b)^m$

let $y=(ax+b)^m$ Differentiating w.r.t x,

$y_1=m\,(ax+b)^{m-1}a$. Again differentiating w.r.t x, we get

$y_2=m\,(m-1)\,(ax+b)^{m-2}a^2$

Similarly, we get

$y_3=m\,(m-1)\,(m-2)\,(ax+b)^{m-3}a^3$

…………………………………

And hence we get

$y_n=m\,(m-1)\,(m-2)\ldots\ldots\,(m-n+1)\,(ax+b)^{m-n}a^n$ for all m.

Case (i) If $m=n$ (m-positive integer),then the above expression becomes

$$y_n=n\,(n-1)\,(n-2)\ldots\ldots3.2.1\,(ax+b)^{n-n}a^n$$

i.e. $y_n=(n!)a^n$

Case (ii) If m<n,(i.e. if n>m) which means if we further differentiate the above expression, the

right hand site yields zero. Thus $D^n\left[(ax+b)^m\right]=0 \quad if\ (m<n)$

Case (iii) If m>n, then $y_n=m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}a^n$ becomes

$$=\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)(m-n)!}{(m-n)!}(ax+b)^{m-n}a^n$$

. i.e $$y_n=\frac{m!}{(m-n)!}(ax+b)^{m-n}a^n$$

3. $\dfrac{1}{(ax+b)^m}$

Let $y=\dfrac{1}{(ax+b)^m}=(ax+b)^{-m}$

Differentiating w.r.t x

$y_1=-m(ax+b)^{-m-1}a=(-1)m(ax+b)^{-(m+1)}a$

$y_2=(-1)m\left[-(m+1)\right](ax+b)^{-(m+1)-1}a=(-1)^2m(m+1)(ax+b)^{-(m+2)}a^2$

Similarly, we get $y_3=(-1)^3m(m+1)(m+2)(ax+b)^{-(m+3)}a^3$

$y_4=(-1)^4m(m+1)(m+2)(m+3)(ax+b)^{-(m+4)}a^4$

……………………………

$y_n=(-1)^nm(m+1)(m+2)\ldots\ldots(m+n-1)(ax+b)^{-(m+n)}a^n$

This may be rewritten as

$$y_n=\frac{(-1)^n(m+n-1)(m+n-2)\ldots\ldots(m+1)m(m-1)!}{(m-1)!}(ax+b)^{-(m+n)}a^n$$

or $$y_n=\frac{(-1)^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n$$

4. $\frac{1}{(ax+b)}$

Putting $m=1$, in the result

$$D^n\left[\frac{1}{(ax+b)^m}\right]=\frac{(-1)^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n$$

we get $D^n\left[\frac{1}{(ax+b)}\right]=\frac{(-1)^n(1+n-1)!}{(1-1)!(ax+b)^{1+n}}a^n$

or $D^n\left[\frac{1}{(ax+b)}\right]=\frac{(-1)^n n!}{(ax+b)^{1+n}}a^n$

Find the nth derivative of the following examples

1. (a) $\log(9x^2-1)$ (b) $\log\left[(4x+3)e^{5x+7}\right]$ (c) $\log_{10}\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}$

Sol: (a) Let $y=\log(9x^2-1)=\log\left[(3x+1)(3x-1)\right]$

$$y=\log(3x+1)+\log(3x-1) \qquad (\because \log(AB)=\log A+\log B)$$

$$\therefore y_n=\frac{d^n}{dx^n}\left\{\log(3x+1)\right\}+\frac{d^n}{dx^n}\left\{\log(3x-1)\right\}$$

$$\text{i.e } y_n=\frac{(-1)^{n-1}(n-1)!}{(3x+1)^n}(3)^n+\frac{(-1)^{n-1}(n-1)!}{(3x-1)^n}(3)^n$$

(b) Let $y=\log\left[(4x+3)e^{5x+7}\right]=\log(4x+3)+\log e^{5x+7}$

$$=\log(4x+3)+(5x+7)\log_e e \qquad (\because \log A^B=B\log A)$$

$$\therefore y=\log(4x+3)+(5x+7) \qquad (\because \log_e e=1)$$

$$\therefore y_n=\frac{(-1)^{n-1}(n-1)!}{(4x+3)^n}(4)^n+0$$

$$\left(\begin{aligned}&\because D(5x+6)=5\\ &D^2(5x+6)=0\\ &D^n(5x+1)=0\ (n>1)\end{aligned}\right)$$

(c) Let $y=\log_{10}\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}$

$$=\frac{1}{\log_e 10}\left\{\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}\right\} \qquad \because \log_{10}X=\frac{\log_e X}{\log_e 10}$$

$$= \frac{1}{\log_e 10}\left\{\frac{1}{2}\log\left\{\frac{(3x+5)^2(2-3x)}{(x+1)^6}\right\}\right\} \qquad \because \log A^B = B\log A$$

$\because \log\left(\frac{A}{B}\right) = \log A - \log B$

$$= \frac{1}{2\log_e 10}\left\{\log(3x+5)^2 + \log(2-3x) - \log(x+1)^6\right\}$$

$$\therefore \quad y = \frac{1}{2\log_e 10}\left\{2\log(3x+5) + \log(2-3x) - 6\log(x+1)\right\}$$

Hence,

$$y_n = \frac{1}{2\log_e 10}\left\{2.\frac{(-1)^{n-1}(n-1)!}{(3x+5)^n}(3)^n + \frac{(-1)^{n-1}(n-1)!}{(2-3x)^n}(-3)^n - 6.\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}(1)^n\right\}$$

2. (a) $e^{2x+4} + 6^{2x+4}$ (b) $\cosh 4x + \cosh^2 4x$

(c) $e^{-x}\sinh 3x\cosh 2x$ (d) $\frac{1}{(4x+5)} + \frac{1}{(5x+4)^4} + (6x+8)^5$

Sol: (a) Let $y = e^{2x+4} + 6^{2x+4}$

$$= e^{2x}e^4 + 6^{2x}6^4$$

$$\therefore \quad y = e^4(e^{2x}) + 1296(6^{2x})$$

hence $$y_n = e^4\frac{dn}{dx^n}(e^{2x}) + 1296\frac{dn}{dx^n}(6^{2x})$$

$$= e^4\left\{2^n e^{2x}\right\} + 1296\left\{2^n(\log 6)^n 6^{2x}\right\}$$

(b) Let $y = \cosh 4x + \cosh^2 4x$

$$= \left(\frac{e^{4x}+e^{-4x}}{2}\right) + \left(\frac{e^{4x}+e^{-4x}}{2}\right)^2$$

$$= \frac{1}{2}\left(e^{4x} + e^{-4x}\right) + \frac{1}{4}\left((e^{4x})^2 + (e^{-4x})^2 + 2(e^{4x})(e^{-4x})\right)$$

$$y = \frac{1}{2}\left(e^{4x} + e^{-4x}\right) + \frac{1}{4}\left(e^{8x} + e^{-8x} + 2\right)$$

hence, $$y_n = \frac{1}{2}\left[4^n e^{4x} + (-4)^n e^{-4x}\right] + \frac{1}{4}\left[8^n e^{8n} + (-8)^n e^{-8n} + 0\right]$$

(c) Let $y = e^{-x}\sinh 3x\cosh 2x$

$$= e^{-x}\left\{\frac{e^{3x}-e^{-3x}}{2}\right\}\left\{\frac{e^{2x}+e^{-2x}}{2}\right\}$$

$$= \frac{e^{-x}}{4}\left((e^{3x} - e^{-3x})(e^{2x} + e^{-2x})\right)$$

$$= \frac{e^{-x}}{4}\left[e^{5x} - e^{-x} + e^{x} - e^{-5x}\right]$$

$$= \frac{1}{4}\left[e^{4x} - e^{-2x} + 1 - e^{-6x}\right]$$

$$y = \frac{1}{4}\left[1 + e^{4x} - e^{-2x} - e^{-6x}\right]$$

Hence, $$y_n = \frac{1}{4}\left[0 + (4)^n e^{4x} - (-2)^n e^{-2x} - (-6)^n e^{-6x}\right]$$

(d) Let $y = \frac{1}{(4x+5)} + \frac{1}{(5x+4)^4} + (6x+8)^5$

Hence, $$y_n = \frac{dn}{dx^n}\left\{\frac{1}{(4x+5)}\right\} + \frac{dn}{dx^n}\left\{\frac{1}{(5x+4)^4}\right\} + \frac{dn}{dx^n}\left[(6x+8)^5\right]$$

$$= \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (4+n-1)!}{(4-1)!(5x+4)^{4+n}}(5)^n + 0$$

i.e $$y_n = \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (3+n)!}{3!(5x+4)^{n+4}}(5)^n$$

Evaluate

1. (i) $\frac{1}{x^2 - 6x + 8}$ (ii) $\frac{1}{1 - x - x^2 + x^3}$ (iii) $\frac{x^2}{2x^2 + 7x + 6}$

(iv) $\left(\frac{x+2}{x+1}\right) + \frac{1}{4x^2 + 12x + 9}$ (v) $\tan^{-1}\left(x/a\right)$ (vi) $\tan^{-1} x$ (vii) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

Sol: (i) Let $y = \frac{1}{x^2 - 6x + 8}$. The function can be rewritten as $y = \frac{1}{(x-4)(x-2)}$

This is proper fraction containing two distinct linear factors in the denominator.

So, it can be split into partial fractions as

$y = \frac{1}{(x-4)(x-2)} = \frac{A}{(x-4)} + \frac{B}{(x-2)}$ Where the constant A and B are found

as given below.

$$\frac{1}{(x-4)(x-2)} = \frac{A(x-2) + B(x-4)}{(x-4)(x-2)}$$

$\therefore \quad 1 = A(x-2) + B(x-4)$ -------------(*)

Putting x = 2 in (*), we get the value of B as $B = -1/2$

Similarly putting x = 4 in (*), we get the value of A as $A = 1/2$

$$\therefore\ y = \frac{1}{(x-4)(x-2)} = \frac{(1/2)}{x-4} + \frac{(-1/2)}{x-2} \quad \text{Hence}$$

$$y_n = \frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-4}\right) - \frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-2}\right)$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x-4)^{n+1}}(1)^n\right] - \frac{1}{2}\left[\frac{(-1)^n n!}{(x-2)^{n+1}}(1)^n\right]$$

$$= \frac{1}{2}(-1)^n n!\left[\frac{1}{(x-4)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right]$$

(ii) Let $y = \dfrac{1}{1-x-x^2+x^3} = \dfrac{1}{(1-x)-x^2(1-x)} = \dfrac{1}{(1-x)(1-x^2)}$

ie $y = \dfrac{1}{(1-x)(1-x)(1+x)} = \dfrac{1}{(1-x)^2(1+x)}$

Though *y* is a proper fraction, it contains a repeated linear factor $(1-x)^2$ in its

denominator. Hence, we write the function as

$y = \dfrac{A}{(1-x)} + \dfrac{B}{(1-x)^2} + \dfrac{C}{1+x}$ in terms of partial fractions. The constants *A*, *B*, *C*

are found as follows:

$$y = \frac{1}{(1-x)^2(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$$

ie $1 = A(1-x)(1+x) + B(1+x) + C(1-x)^2$ -------------(**)

Putting x = 1 in (**), we get *B* as $B = 1/2$

Putting x = -1 in (**), we get *C* as $C = 1/4$

Putting x = 0 in (**), we get $1 = A + B + C$

$\therefore\ A = 1 - B - C = 1 - 1/2 - 1/4 = 1/4$

$\therefore\ A = 1/4$

Hence, $y = \dfrac{(1/4)}{(1-x)} + \dfrac{(1/2)}{(1-x)^2} + \dfrac{(1/4)}{(1+x)}$

$$\therefore\ y_n = \frac{1}{4}\left[\frac{(-1)^n n!}{(1-x)^{n+1}}(1)^n\right] + \frac{1}{2}\left[\frac{(-1)^n (2+n-1)!}{(2-1)!(1-x)^{2+n}}(1)^n\right] + \frac{1}{4}\left[\frac{(-1)^n n!}{(1+x)^{n+1}}(1)^n\right]$$

$$= \frac{1}{4}(-1)^n n!\left[\frac{1}{(1-x)^{n+1}} + \frac{1}{(1+x)^{n+1}}\right] + \frac{1}{2}\left[\frac{(-1)^n (n+1)!}{(1-x)n+2}\right]$$

(iii) Let $y = \dfrac{x^2}{2x^2+7x+6}$ **(VTU July-05)**

This is an improper function. We make it proper fraction by actual division and later spilt that into partial fractions.

i.e $x^2 \div (2x^2+7x+6) = \dfrac{1}{2} + \dfrac{(-\frac{7}{2}x-3)}{2x^2-7x+6}$

$\therefore\ y = \dfrac{1}{2} + \dfrac{-\frac{7}{2}x-3}{(2x+3)(x+2)}$ Resolving this proper fraction into partial fractions,

we get

$y = \dfrac{1}{2} + \left[\dfrac{A}{(2x+3)} + \dfrac{B}{(x+2)}\right]$. Following the above examples for finding A & B, we get

$$y = \frac{1}{2} + \left[\frac{\frac{9}{2}}{2x+3} + \frac{(-4)}{x+2}\right]$$

Hence, $y_n = 0 + \dfrac{9}{2}\left[\dfrac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right] - 4\left[\dfrac{(-1)^n n!}{(x+2)^{n+1}}(1)^n\right]$

i.e $y_n = (-1)^n n!\left[\dfrac{\frac{9}{2}(2)^n}{(2x+3)^{n+1}} - \dfrac{4}{(x+2)^{n+1}}\right]$

(iv) Let $y = \underbrace{\dfrac{(x+2)}{(x+1)}}_{\text{(i)}} + \underbrace{\dfrac{x}{4x^2+12x+9}}_{\text{(ii)}}$

Here (i) is improper & (ii) is proper function. So, by actual division (i) becomes

$\left(\dfrac{x+2}{x+1}\right) = 1 + \left(\dfrac{1}{x+1}\right)$. Hence, y is given by

$$y = 1 + \left(\frac{1}{x+1}\right) + \frac{1}{(2x+3)^2} \qquad [\because\ (2x+3)^2 = 4x^2+12x+9]$$

Resolving the last proper fraction into partial fractions, we get

$\dfrac{x}{(2x+3)^2} = \dfrac{A}{(2x+3)} + \dfrac{B}{(2x+3)^2}$. Solving we get

$A = \frac{1}{2}$ and $B = -\frac{3}{2}$

$$\therefore \; y = 1 + \left(\frac{1}{1+x}\right) + \left[\frac{\frac{1}{2}}{(2x+3)} + \frac{-\frac{3}{2}}{(2x+3)^2}\right]$$

$$\therefore \; y_n = 0 + \left[\frac{(-1)^n n!}{(1+x)^n}(1)^n\right] + \frac{1}{2}\left[\frac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right] - \frac{3}{2}\left[\frac{(-1)^n (n+1)!}{(2x+3)n+2}(2)^n\right]$$

(v) $\tan^{-1}(x/a)$

Let $y = \tan^{-1}(x/a)$

$$\therefore \; y_1 = \frac{1}{1+(x/a)^2}\left(\frac{1}{a}\right) = \frac{a}{x^2+a^2}$$

$$y_n = D^n y = D^{n-1}(y_1) = D^{n-1}\left(\frac{a}{x^2+a^2}\right)$$

Consider $\dfrac{a}{x^2+a^2} = \dfrac{a}{(x+ai)(x-ai)}$

$= \dfrac{A}{(x+ai)} + \dfrac{B}{(x-ai)}$, on resolving into partial fractions.

$= \dfrac{(-1/2i)}{(x+ai)} + \dfrac{(1/2i)}{(x-ai)}$, on solving for A & B.

$$\therefore \; D^{n-1}\left(\frac{a}{x^2+a^2}\right) = D^{n-1}\left(\frac{-\frac{1}{2i}}{x+ai}\right) + D^{n-1}\left(\frac{\frac{1}{2i}}{x-ai}\right)$$

$$= \left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right] + \left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right] \text{-----------}(*)$$

We take transformation $x = r\cos\theta \quad a = r\sin\theta$ where $r = \sqrt{x^2+a^2}$, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

$$x + ai = r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$x - ai = r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

$$\frac{1}{(x-ai)^n} = \frac{1}{r^n e^{-in\theta}} = \frac{e^{in\theta}}{r^n}, \quad \frac{1}{(x+ai)^n} = \frac{e^{-in\theta}}{r^n}$$

now(*) is $y_n = \dfrac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left(e^{in\theta} - e^{-in\theta}\right)$

$$y_n = \frac{(-1)^{n-1}}{2\,i\,r^n}(2\,i\sin n\theta) \Rightarrow \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

(vi) Let $y = \tan^{-1} x$ .Putting a = 1 in Ex.(v) we get

y_n which is same as above with $r = \sqrt{x^2+1}$ $\theta = \tan^{-1}(1/x)$

$\theta = \cot^{-1}(x)$ *or* $x = \cot\theta$

$\therefore r = \sqrt{\cot^2\theta + 1} = \operatorname{cosec}\theta \Rightarrow \frac{1}{r^n} = \frac{1}{\operatorname{cosec}^n\theta} = \sin^n\theta$

$D^n(\tan^{-1} x) = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ *where* $\theta = \cot^{-1} x$

(vii) Let $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

put $x = \tan\theta$ $\theta = \tan^{-1} x$

$$\therefore \quad y = \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan(\pi/4 + \theta)\right] \quad \because \tan(\pi/4 + \theta) = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \pi/4 + \theta = \frac{\pi}{4} + \tan^{-1}(x)$$

$$y = \frac{\pi}{4} + \tan^{-1}(x)$$

$$y_n = 0 + D^n(\tan^{-1} x)$$

$$= \left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right] + \left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right]$$

nth derivative of trigonometric functions:

1. $\sin(ax+b)$.

Let $y = \sin(ax+b)$. Differentiating w.r.t x,

$y_1 = \cos(ax+b).a$ As $\sin(x + \frac{\pi}{2}) = \cos x$

We can write $y_1 = a\sin(ax + b + \pi/2)$.

again differentiating w.r.t x, $y_2 = a\cos(ax + b + \pi/2).a$

Again using $\sin(x + \frac{\pi}{2}) = \cos x$,we get y_2 as

$y_2 = a\sin(ax + b + \pi/2 + \pi/2).a$

i.e. $y_2 = a^2\sin(ax + b + 2\pi/2)$.

Similarly, we get

$y_3 = a^3\sin(ax + b + 3\pi/2)$.

$y_4 = a^4\sin(ax + b + 4\pi/2)$.

$y_n = a^n\sin(ax + b + n\pi/2)$.

2. $e^{ax}\sin(bx+c)$

$Let\ y=e^{ax}\sin(bx+c)$.....(1)

Differentiating using product rule ,we get

$y_1=e^{ax}\cos(bx+c)b+\sin(bx+c)ae^{ax}$

i.e. $y_1=e^{ax}[a\sin(bx+c)+b\cos(bx+c)]$. For computation of higher order derivatives

it is convenient to express the constants 'a' and 'b' in terms of the constants r and

θ defined by $a=r\cos\theta$ & $b=r\sin\theta$,so that

$r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}(b/a)$.thus,

y_1 can be rewritten as

$y_1=e^{ax}[(r\cos\theta)\sin(bx+c)+(r\sin\theta)\cos(bx+c)]$

or $y_1=e^{ax}[r\{\sin(bx+c)\cos\theta+\cos(bx+c)\sin\theta\}]$

i.e. $y_1=re^{ax}[\sin(bx+c+\theta)]$.........(2)

Comparing expressions (1) and (2), we write y_2 as

$y_2=r^2e^{ax}\sin(bx+c+2\theta)$

$y_3=r^3e^{ax}\sin(bx+c+3\theta)$

Continuing in this way, we get

$y_4=r^4e^{ax}\sin(bx+c+4\theta)$

$y_5=r^5e^{ax}\sin(bx+c+5\theta)$

…………………………….

$y_n=r^ne^{ax}\sin(bx+c+n\theta)$

$\therefore D^n[e^{ax}\sin(bx+c)]=r^ne^{ax}\sin(bx+c+n\theta)$,where

$r=\sqrt{a^2+b^2}$ & $\theta=\tan^{-1}(b/a)$

Solve the following:

1. (i) $\sin^2 x+\cos^3 x$ (ii) $\sin^3\cos^3 x$ (iii) $\cos x\cos 2x\cos 3x$
   (iv) $\sin x\sin 2x\sin 3x$ (v) $e^{3x}\cos 2x$ (vi) $e^{2x}(\sin^2 x+\cos^3 x)$

The following formulae are useful in solving some of the above problems.

(i) $\sin^2 x=\dfrac{1-\cos 2x}{2}$ $(ii)\cos^2 x=\dfrac{1+\cos 2x}{2}$

(iii) $\sin 3x=3\sin x-4\sin^3 x$ $(iv)\cos 3x=4\cos^3 x-3\cos x$

(v) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(vi) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(vii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(viii) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Sol: (i) Let $y = \sin^2 x + \cos^3 x = \left(\frac{1-\cos 2x}{2}\right) + \frac{1}{4}(\cos 3x + 3\cos x)$

$$\therefore y_n = \frac{1}{2}\left[-2^n \cos\left(2x + \frac{n\pi}{2}\right)\right] + \frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right]$$

(ii)Let $y = \sin^3 x \cos^3 x = \left(\frac{\sin 2x}{2}\right)^3 = \frac{\sin^3 2x}{8} = \frac{1}{8}\left[\frac{-\sin 6x + 3\sin 2x}{4}\right]$

$$= \frac{1}{32}[3\sin 2x - \sin 6x]$$

$$y_n = \frac{1}{32}\left[3.2^n \sin\left(2x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right)\right]$$

(iii))Let y = $\cos 3x \cos x \cos 2x$

$$= \frac{1}{2}(\cos 4x + \cos 2x)\cos 2x = \frac{1}{2}[\cos 4x \cos 2x + \cos^2 2x]$$

$$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 2x) + \frac{1-\cos 4x}{2}\right]$$

$$= \frac{1}{4}\cos 6x + \frac{\cos 2x}{4} + \frac{1}{4}(1 - \cos 4x)$$

$$\therefore y_n = \frac{1}{4}6^n \cos\left(6x + \frac{n\pi}{2}\right) + \frac{2^n \cos\left(2x + \frac{n\pi}{2}\right)}{4} - \frac{4^n \cos\left(4x + \frac{n\pi}{2}\right)}{4}$$

(iv))Let y = $\sin 3x \sin x\, sn2x$

$$= \frac{1}{2}[\sin(2x) - \sin 4x]\sin 2x$$

$$= \frac{1}{2}[\sin^2 2x - \sin 4x \sin 2x]$$

$$= \frac{1}{2}\left[\frac{1-\cos 4x}{2} - \frac{1}{2}(\sin 2x - \sin 6x)\right]$$

$$= \left[\left(\frac{1-\cos 4x}{4}\right) - \frac{1}{4}(\sin 2x - \sin 6x)\right]$$

$$y_n = \frac{1}{4}\left[4^n \cos\left(4x + \frac{n\pi}{2}\right) - 2^n \sin\left(2x + \frac{n\pi}{2}\right) + 6^n \sin\left(6x + \frac{n\pi}{2}\right)\right]$$

(v) Let $y = e^{3x}\cos 2x$

$\therefore y_n = re^{3x}\cos(2x + n\theta)$

where $r = \sqrt{3^2 + 2^2} = \sqrt{13}$ & $\theta = \tan^{-1}\left(\frac{2}{3}\right)$

(vi) Let $y = e^{2x}\left(\sin^2 x + \cos^3 x\right)$

We know that $\left(\sin^2 x + \cos^3 x\right) = \dfrac{1-\cos 2x}{2} + \dfrac{1}{4}\left(\cos 3x + 3\cos x\right)$

$$\therefore y = e^{2x}\left(\sin^2 x + \cos^3 x\right) = e^{2x}\left[\frac{1-\cos 2x}{2}\right] + \frac{e^{2x}}{4}\left(\cos 3x + 3\cos x\right)$$

$$\therefore y = \frac{1}{2}\left(e^{2x} - e^{2x}\cos 2x\right) + \frac{1}{4}\left(e^{2x}\cos 3x + 3e^{2x}\cos x\right)$$

Hence,

$$y_n = \frac{1}{2}\left[2^n e^{2x} - r_1^n e^{2x}\cos(2x + n\theta_1)\right] + \frac{1}{4}\left[r_2^n e^{2x}\cos(3x + n\theta_2) + 3r_3^n e^{2x}\cos(x + n\theta_3)\right]$$

where $r_1 = \sqrt{2^2+2^2} = \sqrt{8}$; $r_2 = \sqrt{2^2+3^2} = \sqrt{13}$; $r_3 = \sqrt{2^2+1^2} = \sqrt{5}$

$\theta_1 = \tan^{-1}\left(\dfrac{2}{2}\right)$; $\theta_2 = \tan^{-1}\left(\dfrac{3}{2}\right)$; $\theta_3 = \tan^{-1}\left(\dfrac{1}{2}\right)$;

Leibnitz's Theorem

Leibnitz's theorem is useful in the calculation of n^{th} derivatives of product of two functions.

Statement of the theorem:

If u and v are functions of x, then

$$D^n(uv) = D^n uv + {}^nC_1 D^{n-1}uDv + {}^nC_2 D^{n-2}uD^2v + \ldots + {}^nC_r D^{n-r}uD^rv + \ldots uD^nv,$$

where $D = \dfrac{d}{dx}$, ${}^nC_1 = n$, ${}^nC_2 = \dfrac{n(n-1)}{2}$,........, ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

Examples

1. If $x = \sin t, y = \sin pt$ *prove that*

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (p^2 - n^2)y_n = 0$$

Solution: Note that the function $y = f(x)$ is given in the parametric form with a parameter *t*.

So, we consider

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p\cos pt}{\cos t} \qquad (p - \text{constant})$$

or $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{p^2\cos^2 pt}{\cos^2 t} = \dfrac{p^2(1-\sin^2 pt)}{1-\sin^2 t} = \dfrac{p^2(1-y^2)}{1-x^2}$

or $(1-x^2)y_1^2 = p^2(1-y^2)$

So that $(1-x^2)y_1^2 - p^2(1-y^2)$ Differentiating w.r.t. x,

$$(1-x^2)(2y_1y_2) + y_1^2(-2x) - p^2(-2yy_1) = 0$$

$(1-x^2)y_2 - xy_1 + p^2 y = 0$ --------------- (1) $[\div 2y_1$, throughout]

Equation (1) has second order derivative y_2 in it. We differentiate (1), n times, term wise,

using Leibnitz's theorem as follows.

$D^n[(1-x^2)y_2 - xy_1 - p^2 y] = 0$

i.e $D^n\{(1-x^2)y_2\} - D^n\{xy_1\} + D^n\{p^2 y\} = 0$ ---------- (2)

(a) (b) (c)

Consider the term (a):

$D^n[(1-x^2)y_2]$. Taking $u = y_2$ and $v = (1-x^2)$ and applying Leibnitz's theorem we get

$D^n[uv] = D^n uv + {}^nC_1 D^{n-1}uDv + {}^nC_2 D^{n-2}D^2 v + {}^nC_3 D^{n-3}uD^3 v + \ldots$

i.e

$D^n[y_2(1-x^2)] = D^n(y_2).(1-x^2) + {}^nC_1 D^{n-1}(y_2).D(1-x^2) + {}^nC_2 D^{n-2}(y_2)D^2(1-x^2) + {}^nC_3 D^{n-3}(y_2)D^3(1-x^2) +$

$= y_{(n)+2}(1-x^2) + ny_{(n-1)+2}.(-2x) + \frac{n(n-1)}{2!}y_{(n-2)+2}.(-2) + \frac{n(n-1)(n-2)}{3!}.y_{(n-3)+2}.(0) + \ldots$

$D^n[(1-x^2)y_2] = (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$ ----------- (3)

Consider the term (b):

$D^n[xy_1]$. Taking $u = y_1$ and $v = x$ and applying Leibnitz's theorem, we get

$D^n[y_1(x)] = D^n(y_1).(x) + {}^nC_1 D^{n-1}y_1.D(x) + {}^nC_2 D^{n-2}(y_1).D^2(x) + \ldots.$

$= y_{(n)+1}.x + ny_{(n-1)+1} + \frac{n(n-1)}{2!}y_{(n-2)+2}(0) + \ldots.$

$D^n[xy_1] = xy_{n+1} + ny_n$ ---------- (4)

Consider the term (c):

$D^n(p^2 y) = p^2 D^n(y) = p^2 y_n$ --------- (5)

Substituting these values (3), (4) and (5) in Eq (2) we get

$\{(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n\} - \{xy_{n+1} + ny_n\} + \{p^2 y_n\} = 0$

ie $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n + ny_n - ny_n + p^2 y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (p^2 - n^2)y_n = 0$ as desired.

2. If $\sin^{-1} y = 2\log(x+1)$ or $y = \sin[2\log(x+1)]$ or $y = \sin[\log(x+1)^2]$ or $y = \sin\log(x^2+2x+1)$, show that $(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

(VTU Jan-03)

Sol: Out of the above four versions, we consider the function as

$\sin^{-1}(y) = 2\log(x+1)$

Differentiating w.r.t x, we get

$$\frac{1}{\sqrt{1-y^2}}(y_1)=\left(\frac{2}{x+1}\right) \text{ ie } (x+1)y_1 = 2\sqrt{1-y^2}$$

Squaring on both sides

$$(x+1)^2 y_1^2 = 4(1-y^2)$$

Again differentiating w.r.t x,

$$(x+1)^2 (2y_1y_2) + y_1^2 (2(x+1)) = 4(-2yy_1)$$

or $$(x+1)^2 y_2 + (x+1)y_1 = -4y \quad (\div 2y_1)$$

or $$(x+1)^2 y_2 + (x+1)y_1 + 4y = 0 \qquad \text{-----------*}$$

Differentiating * w.r.t x, n-times, using Leibnitz's theorem,

$$\left\{D^n y_2 (x+1)^2 + nD^{n-1}(y_2)2(x+1) + \frac{n(n-1)}{2\,!}D^{n-2}(y_2)(2)\right\} + \left\{D^n (g_1)(x+1) + nD^{n-1}y_1(1)\right\} + 4D^n y = 0$$

On simplification, we get

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

3. If $x=\tan(\log y)$, then find the value of

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1}$$ **(VTU July-04)**

Sol: Consider $x=\tan(\log y)$

i.e. $\tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$

Differentiating w.r.t x,

$$y_1 = e^{\tan^{-1}x}.\frac{1}{1+x^2} = \frac{y}{1+x^2}$$

$$\therefore (1+x^2)y_1 = y \qquad ie\,(1+x^2)y_1 - y = 0 \qquad \text{-----------*}$$

We differentiate * n-times using Leibnitz's theorem,

We get

$$D^n\left[(1+x^2)y_1\right] - D^n(y) = 0$$

ie.

$$\left[D^n(y_1)(1+x^2) + {}^nC_1D^{n-1}(y_1)D(1+x^2) + {}^nC_2D^{n-2}(y_1)D^2(1+x^2)+\ldots\right] - \left[D^n y\right] = 0$$

ie $$\left\{y_{n+1}(1+x^2) + ny_n(2x) + \frac{n(n-1)}{2!}y_{n-1}(2) + 0 + \ldots\right\} - y_n = 0$$

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$$

4. If $y^{1/m} + y^{-1/m} = 2x$, or $y = \left[x+\sqrt{x^2-1}\right]^m$ or $y = \left[x-\sqrt{x^2-1}\right]^m$

Show that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$ **(VTU Feb-02)**

Sol: Consider $y^{1/m} + y^{-1/m} = 2x \quad \Rightarrow \quad y^{1/m} + \dfrac{1}{y^{1/m}} = 2x$

$$\Rightarrow \left(y^{1/m}\right)^2 - 2x\left(y^{1/m}\right) + 1 = 0$$ Which is quadratic equation in $y^{1/m}$

$$\therefore\ y^{1/m} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = \left(x \pm \sqrt{x^2 - 1}\right) \Rightarrow y^{1/m} = \left(x \pm \sqrt{x^2 - 1}\right)$$

$$\therefore\ y = \left(x \pm \sqrt{x^2 - 1}\right)^m$$

so, we can consider $y = \left[x + \sqrt{x^2 - 1}\right]^m$ or $y = \left[x - \sqrt{x^2 - 1}\right]^m$

Let us take $y = \left[x + \sqrt{x^2 - 1}\right]^m$

$$\therefore\ y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left(1 + \frac{1}{2\sqrt{x^2 - 1}}(2x)\right)$$

$$y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left(\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}\right)$$

or

$\left(\sqrt{x^2 - 1}\right)y_1 = my$. On squaring

$\left(x^2 - 1\right)y_1^2 = m^2 y^2$.

Again differentiating w.r.t x,

$$\left(x^2 - 1\right)2y_1y_2 + y_1^2(2x) = m^2(2yy_1)$$

or

$$\left(x^2 - 1\right)y_2 + xy_1 = m^2 y \quad (\div 2y_1)$$

or

$$\left(x^2 - 1\right)y_2 + xy_1 - m^2 y = 0 \qquad \text{------------(*)}$$

Differentiating (*) n- times using Leibnitz's theorem and simplifying, we get

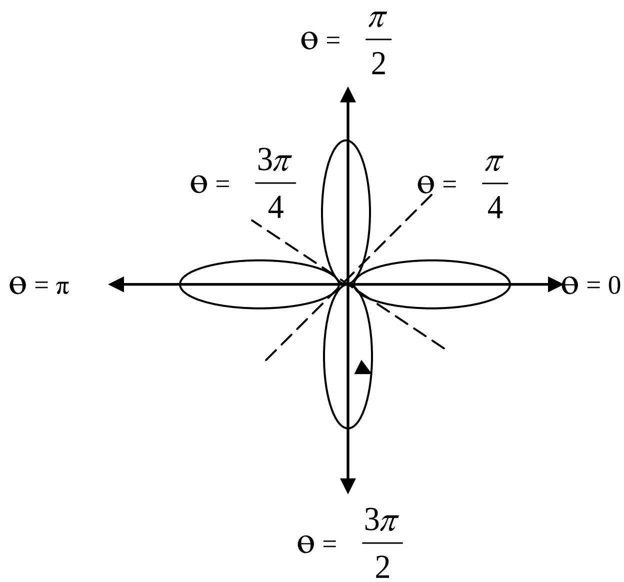
$$\left(x^2 - 1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2 - m^2\right)y_n = 0$$

POLAR CURVES

Angle between Polar Curves:

Introduction:- We are familiar with Cartesian coordinate system for specifying a point in the xy – plane. Another useful system for similar purpose is Polar coordinate system, and the curves specified by these coordinates are referred to as polar curves.

- **A polar curve by name "three-leaved rose" is displayed below:**



- Any point P can be located on a plane with co-ordinates (r, θ) called **polar co-ordinates** of P where r = **radius vector** OP,(with pole 'O') θ = projection of $\vec{OP}$ on the **initial axis** OA.(See Fig.)
- The equation $r = f(\theta)$ is known as a **polar curve**.
- Polar coordinates (r, θ) can be related with Cartesian coordinates (x, y) through the relations
- **Fig.1. Polar coordinate system**
  $x = r\cos\theta$ & $y = r\sin\theta$.

<u>Theorem 1</u>: Angle between the radius vector and the tangent:

i.e., With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$

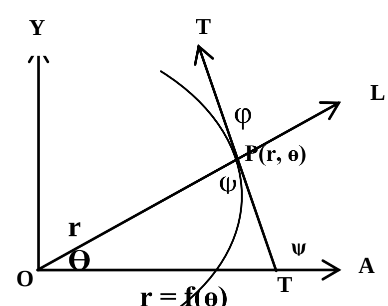


Fig.2. Angle between radius vector and the tangent

- **Proof:-** Let "ϕ" be the angle between the radius vector OPL and the tangent TPT^1 at the point `P` on the polar curve $r = f(\theta)$. (See fig.2)
  From Fig.2,

$$\psi = \theta + \phi$$

$$\tan\psi = \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

i.e. $$\frac{dy}{dx} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \text{................(1)}$$

On the other hand, we have $x = r\cos\theta$; $y = r\sin\theta$ differentiating these, w.r.t θ,

$$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right) \ \& \ \frac{dy}{d\theta} = r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)}{r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right)} \ \textit{dividing the Nr \& Dr by} \ \frac{dr}{d\theta}\cos\theta$$

$$\frac{dy}{dx} = \frac{r(d\theta/dr) + \tan\theta}{-(r\,d\theta/dr)\tan\theta + 1}$$

i.e. $$\frac{dy}{dx} = \frac{\tan\theta + (r\,d\theta/dr)}{1 - \tan\theta(r\,d\theta/dr)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Comparing equations (1) and (2)

we get $\tan\phi = r\,d\theta/dr$

- **Note that** $\cot\phi = \left(\frac{1}{r}\frac{dr}{d\theta}\right)$
- **<u>A Note on Angle of intersection of two polar curves</u>**:-

  If ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ then the angle intersection of the curves is given by $|\phi_1 - \phi_2|$

<u>Theorem 2</u>: The length "p" of perpendicular from pole to the tangent in a polar curve

i.e.(i) $p = r\sin\phi$ **or** **(ii)** $\frac{1}{p^2} = \frac{1}{r^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

Proof:- In the Fig.3, note that ON = p, the length of the perpendicular from the pole to the tangent at p on $r = f(\theta)$.from the right angled triangle OPN,

$$\sin\phi = \frac{ON}{OP} \Rightarrow ON = (OP)\sin\phi$$

i.e. $p = r\sin\phi \ldots\ldots\ldots\ldots(i)$

Consider $\frac{1}{p} = \frac{1}{r\sin\phi} = \frac{1}{r}\operatorname{cosec}\phi$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1 + \cot 2\phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$$

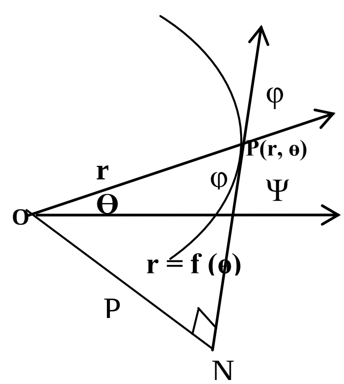


Fig.3 Length of the perpendicular from the pole to the tangent

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{ ....}(ii)$$

Note:-If $u = \frac{1}{r}$, we get $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

In this session, we solve few problems on angle of intersection of polar curves and pedal equations.

Examples:-

Find the acute angle between the following polar curves

1. $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$ **(VTU-July-2003)**
2. $r = (\sin\theta + \cos\theta)$ and $r = 2\sin\theta$ **(VTU-July-2004)**
3. $r = 16\sec^2(\theta/2)$ and $r = 25\operatorname{cosec}^2(\theta/2)$
4. $r = a\log\theta$ and $r = a/\log\theta$ **(VTU-July-2005)**
5. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Sol:

1. Consider

$r = a(1+\cos\theta)$

Diff w.r.t θ

$\frac{dr}{d\theta} = -a\sin\theta$

$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$

$\tan\phi_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$

$= -\cot\theta/2$

Consider

$r = b(1-\cos\theta)$

Diff w.r.t θ

$\frac{dr}{d\theta} = b\sin\theta$

$r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$

$\tan\phi_1 = -\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$

$= \tan\theta/2$

i.e $\tan\phi_1 = \tan(\pi/2 + \theta/2) \Rightarrow \phi_1 = (\pi/2 + \theta/2)$

$\tan\phi_1 = \tan\theta/2 \Rightarrow \phi_1 = \phi_2$

Angle between the curves

$|\phi_1 - \phi_2| = |(\pi/2 + \theta/2) - \theta/2| = \pi/2$

Hence, the given curves intersect orthogonally.

2. Consider

$r = (\sin\theta + \cos\theta)$

Diff w.r.t θ

Consider

$r = 2\sin\theta$

Diff w.r.t θ

$\frac{dr}{d\theta} = \cos\theta - \sin\theta$

$r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$

$\tan\phi_1 = \frac{\tan\theta + 1}{1 - \tan\theta}$ (÷ Nr & Dr $\cos\theta$)

i.e $\tan\phi_1 = \frac{\tan\theta + 1}{1 - \tan\theta} = \tan\left(\pi/4 + \theta\right)$

$\Rightarrow \phi_1 = \pi/4 + \theta$

$\frac{dr}{d\theta} = 2\cos\theta$

$r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta}$

$\tan\phi_2 = \tan\theta$

$\Rightarrow \phi_2 = \theta$

$\therefore$ Angle between the curves $= |\phi_1 - \phi_2| = \left|\left(\pi/4 + \theta\right) - \theta\right| = \pi/4$

3. Consider

$r = 16\sec^2\left(\theta/2\right)$

Diff w.r.t θ

$\frac{dr}{d\theta} = 32\sec^2\left(\theta/2\right)\tan\left(\theta/2\right).1/2$

$= 16\sec^2\left(\theta/2\right)\tan\left(\theta/2\right)$

$r\frac{d\theta}{dr} = \frac{16\sec^2\left(\theta/2\right)}{16\sec^2\left(\theta/2\right)\tan\left(\theta/2\right)}$

$\tan\phi_1 = \cot\theta/2 = \tan\left(\pi/2 - \theta/2\right)$

$\Rightarrow \phi_1 = \left(\pi/2 - \theta/2\right)$

Consider

$r = 25\operatorname{cosec}^2\left(\theta/2\right)$

Diff w.r.t θ

$\frac{dr}{d\theta} = -50\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right).1/2$

$= -25\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right)$

$r\frac{d\theta}{dr} = \frac{25\operatorname{cosec}^2\left(\theta/2\right)}{-25\operatorname{cosec}^2\left(\theta/2\right)\cot\left(\theta/2\right)}$

$\tan\phi_2 = -\tan\theta/2 = \tan\left(-\theta/2\right)$

$\Rightarrow \phi_2 = -\theta/2$

Angle of intersection of the curves $= |\phi_1 - \phi_2| = \left|\left(\pi/2 - \theta/2\right) - \left(-\theta/2\right)\right|$

$= \pi/2$

4. Consider

$r = a\log\theta$

Diff w.r.t θ

$\frac{dr}{d\theta} = a/\theta$

$r\frac{d\theta}{dr} = a\log\theta\left(\theta/a\right)$

$\tan\phi_1 = \theta\log\theta.........(i)$

Consider

$r = a/\log\theta$

Diff w.r.t θ

$\frac{dr}{d\theta} = -a/\left(\log\theta\right)^2 . 1/\theta$

$r\frac{d\theta}{dr} = -\left(a/\log\theta\right)\left(\frac{\left(\log\theta\right)^2\theta}{a}\right)$

$\tan\phi_2 = -\theta\log\theta.........(ii)$

We know that

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

$$= \frac{\theta\log\theta - (-\theta\log\theta)}{1 + (\theta\log\theta)(-\theta\log\theta)}$$

i.e $\tan(\phi_1 - \phi_2) = \dfrac{2\theta\log\theta}{1 - (\theta\log\theta)^2}$ ....(iii)

From the data: $a\log\theta = r = a/\log\theta \Rightarrow (\log\theta)^2 = 1 \quad or\ \log\theta = \pm 1$

As θ is acute, we take by $\theta = 1 \Rightarrow \theta = e$ $\|NOTE\|$

Substituting $\theta = e$ in (iii), we get

$$\tan(\phi_1 - \phi_2) = \frac{2e\log e}{1 - (e\log e)^2} = \left(\frac{2e}{1 - e^2}\right) \qquad (\because \log_e^e = 1)$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1 - e^2}\right)$$

5. Consider

$$r = \frac{a\theta}{1+\theta} \text{ as}$$

$$\frac{1}{r} = \frac{1+\theta}{a\theta} = \frac{1}{a}\left(1/\theta + 1\right)$$

Diff w.r.t θ

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{a}\left(1/\theta^2\right)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{r}{a\theta^2}$$

$$r\frac{d\theta}{dr} = \frac{a\theta^2}{r}$$

$$\tan\phi_1 = \frac{a\theta^2}{a\theta/(1+\theta)}$$

$$\therefore \tan\phi_1 = \theta(1+\theta)$$

Consider

$$r = \frac{a\theta}{1+\theta^2}$$

$$\therefore (1+\theta^2) = a/r$$

Diff w.r.t θ

$$2\theta = -a/r^2 \frac{dr}{d\theta}$$

$$\frac{-2r\theta}{a} = \frac{1}{r}\frac{dr}{d\theta}$$

i.e $r\dfrac{d\theta}{dr} = \dfrac{-a}{2r\theta}$

$$\tan\phi_2 = -\frac{a}{2\theta}\left(\frac{1+\theta^2}{a}\right)$$

$$\tan\phi_2 = -\frac{1}{2\theta}(1+\theta^2)$$

Now, we have

$$\frac{a\theta}{1+\theta} = r = \frac{a}{1+\theta^2} \Rightarrow a\theta(1+\theta^2) = a(1+\theta)$$

$$\text{or } \theta + \theta^3 = 1 + \theta \Rightarrow \theta^3 = 1 \text{ or } \theta = 1$$

$\therefore \tan\phi_1 = 2$ & $\tan\phi_2 = (-1)$

Consider $\tan(\phi_1 - \phi_2) = \left| \dfrac{\tan\phi_1 - \tan\phi_2}{1 + (\tan\phi_1)(\tan\phi_2)} \right|$

$$= \left| \frac{2 - (-1)}{1 + (2)(-1)} \right| = |-3| = 3$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}(3)$$

<u>Pedal equations (p-r equations</u>):- Any equation containing only **p** & **r** is known as pedal equation of a polar curve.

<u>Working rules to find pedal equations</u>:-

(i) Eliminate r and ϕ from the Eqs.: (i) $r = f(\theta)$ & $p = r\sin\phi$

(ii) Eliminate only θ from the Eqs.: (i) $r = f(\theta)$ & $\therefore \dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

- Find the pedal equations for the polar curves:-
  1. $\dfrac{2a}{r} = 1 - \cos\theta$
  2. $r = e^{\theta\cot\alpha}$
  3. $r^m = a^m \sin m\theta + b^m \cos m\theta$ **(VTU-Jan-2005)**
  4. $l/r = 1 + e\cos\theta$

Sol:

1. Consider $\dfrac{2a}{r} = 1 - \cos\theta$ ……….(i)

Diff. w.r.t θ

$$2a\left(-1/r^2\right)\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-r\sin\theta}{2a}$$

$$r\frac{d\theta}{dr} = -\frac{2a}{r}\frac{1}{\sin\theta}$$

$$\tan\phi = -\frac{(1-\cos\theta)}{\sin\theta} = -\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = -\tan(\theta/2)$$

$$\tan\phi = \tan(-\theta/2) \Rightarrow \phi = -\theta/2$$

Using the value of ϕ is $p = r\sin\phi$, we get

$$p = r\sin(-\theta/2) = -r\sin\theta/2 \text{............}(ii)$$

Eliminating "θ" between (i) and (ii)

$$p^2 = r^2 \sin^2 \theta/2 = r^2\left(\frac{1-\cos\theta}{2}\right) = \frac{r^2}{2}\left(\frac{2a}{r}\right) \quad \text{[See eg: - (i)]}$$

$$p^2 = ar.$$

This eqn. is only in terms of p and *r* and hence it is the pedal equation of the polar curve.

2. Consider $r = e^{\theta\cot\alpha}$

Diff. w.r.t θ

$$\frac{dr}{d\theta} = e^{\theta\cot\alpha}(\cot\alpha) = r\cot\alpha \quad (\because r = e^{\theta\cot\alpha})$$

We use the equation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}(r\cot\alpha)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}(r^2\cot^2\alpha) = \frac{1}{r^2}(1+\cot^2\alpha) = \frac{1}{r^2}\operatorname{cosec}^2\alpha$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\alpha$$

$$p^2 = r^2/\operatorname{cosec}^2\alpha \quad \text{or} \quad r^2 = p^2\operatorname{cosec}^2\alpha \quad \text{is the required pedal equation}$$

3.Consider $r^m = a^m \sin m\theta + b^m \cos m\theta$

Diff. w.r.t θ

$$mr^{m-1}\frac{dr}{d\theta} = a^m(m\cos m\theta) + b^m(-m\sin m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta} = a^m\cos m\theta - b^m\sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^m\cos m\theta - b^m\sin m\theta}{a^m\sin m\theta + b^m\cos m\theta}$$

$$\cot\phi = \frac{a^m\cos m\theta - b^m\sin m\theta}{a^m\sin m\theta + b^m\cos m\theta}$$

Consider $p = r\sin\phi, \frac{1}{p} = \frac{1}{r}\operatorname{cosec}\phi$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left[1+\left(\frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[\frac{(a^m \sin m\theta + b^m \cos m\theta)^2 + (a^m \cos m\theta - b^m \sin m\theta)^2}{(a^m \sin m\theta + b^m \cos m\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2m} + b^{2m}}{r^{2m}}\right]$$

$$\Rightarrow p^2 = \frac{r^{2(m+1)}}{a^{2m} + b^{2m}}$$ is the required *p*-*r* equation

4. Consider $l/r = (1+ e\cos\theta)$

Diff w.r.t θ

$$l\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right) = -e\sin\theta \Rightarrow l/r\left(1/r\,\frac{dr}{d\theta}\right) = e\sin\theta$$

$$l/r\,(\cot\phi) = e\sin\theta$$

$$\therefore \cot\phi = (r/l)\, e\sin\theta$$

We have $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$ (see eg: 3 above)

Now $\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{l^2 + e^2 r^2 \sin^2\theta}{l^2}\right]$

$$= \frac{1}{r^2}\left(1 + e^2 r^2/l^2 \sin^2\theta\right)$$

$$1 + e\cos\theta = l/r \Rightarrow \qquad e\cos\theta = \frac{l-r}{r}$$

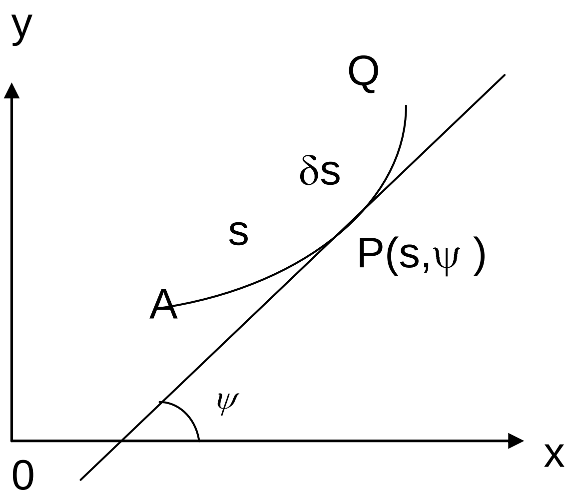
$$\cos\theta = \left(\frac{l-r}{re}\right) \Rightarrow \sin^2\theta = 1 - \cos^2\theta \Rightarrow = 1 - \left(\frac{l-r}{re}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{l^2 + e^2 r^2\left\{1 - \left(\frac{l-r}{re}\right)^2\right\}}{l^2}\right]$$

On simplification $\frac{1}{p^2} = \left(\frac{e^2 - 1}{e^2}\right) + \frac{2}{lr}$

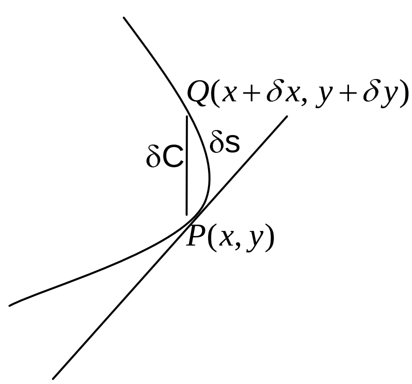
DERIVATIVES OF ARC LENGTH:

Consider a curve C in the XY plane. Let A be a fixed point on it. Let P and Q be two neighboring positions of a variable point on the curve C. If 's' is the distance of P from A measured along the curve then 's' is called the arc length of P. Let the tangent to C at P make an angle ψ with X-axis. Then (s,ψ) are called the intrinsic co-ordinates of the point P. Let the arc length AQ be $s + \delta s$. Then the distance between P and Q measured along the curve C is δs. If the actual distance between P and Q is δC. Then $\delta s=\delta C$ in the limit $Q \to P$ along C.



$$i.e. \quad \underset{Q\to P}{Lt} \frac{\delta s}{\delta C} = 1$$

Cartesian Form:



Let $y = f(x)$ be the Cartesian equation of the curve C and let $P(x, y)$ *and* $Q(x+\delta x, y+\delta y)$ be any two neighboring points on it as in fig.

Let the arc length $PQ = \delta s$ and the chord length $PQ = \delta C$. Using distance between two points formula we have $PQ^2 = (\delta C)^2 = (\delta x)^2 + (\delta y)^2$

$$\therefore \left(\frac{\delta C}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \quad or \quad \frac{\delta C}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

$$\Rightarrow \frac{\delta s}{\delta x} = \frac{\delta s}{\delta C} \cdot \frac{\delta C}{\delta x} = \frac{\delta s}{\delta C} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

We note that $\delta x \to 0$ as $Q \to P$ along C, also that when $Q \to P, \frac{\delta s}{\delta C} = 1$

$\therefore$ When $Q \to P$ i.e. when $\delta x \to 0$, from (1) we get

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \to (1)$$

Similarly we may also write

$$\frac{\delta s}{\delta y} = \frac{\delta s}{\delta C} \cdot \frac{\delta C}{\delta y} = \frac{\delta s}{\delta C} \sqrt{1 + \left(\frac{\delta x}{\delta y}\right)^2}$$

and hence when $Q \to P$ this leads to

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \quad \to (2)$$

Parametric Form: Suppose $x = x(t)$ *and* $y = y(t)$ is the parametric form of the curve C. Then from (1)

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} = \frac{1}{dx/dt} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\therefore \frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \to (3)$$

<u>**Note**</u>: Since ψ is the angle between the tangent at P and the X-axis,

we have $\frac{dy}{dx} = \tan\psi$

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + y'^{2}} = \sqrt{1 + \tan^2\psi} = sec\psi$$

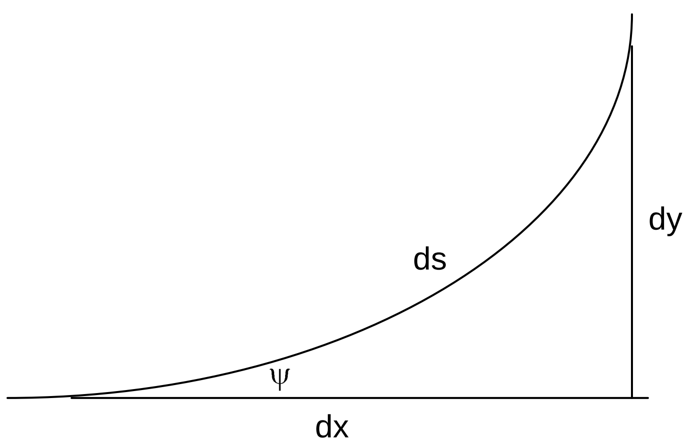
Similarly

$$\frac{ds}{dy} = \sqrt{1 + \frac{1}{y'^{2}}} = \sqrt{1 + \frac{1}{\tan^2\psi}} = \sqrt{1 + \cot^2\psi} = co\sec\psi$$

i.e. $\cos\psi = \frac{dx}{ds}$ *and* $\sin\psi = \frac{dy}{ds}$

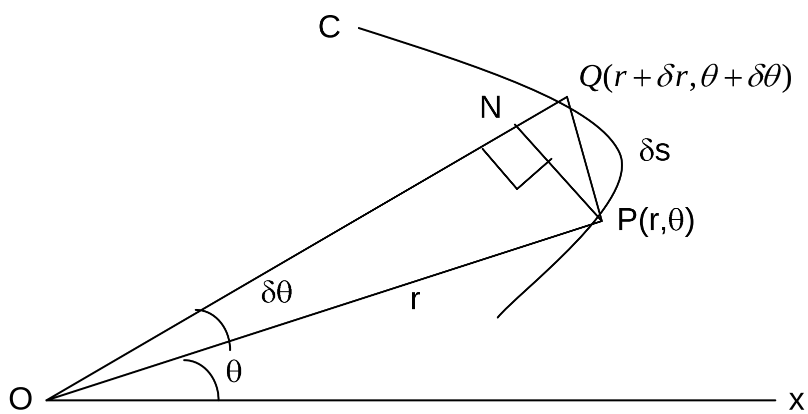
$$\therefore \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \Rightarrow ds^2 = dx^2 + dy^2$$

We can use the following figure to observe the above geometrical connections among dx, dy, ds *and* ψ.



Polar Curves:

Suppose $r = f(\theta)$ is the polar equation of the curve C and $P(r,\theta)$ *and* $Q(r+\delta r, \theta+\delta\theta)$ be two neighboring points on it as in figure:



Consider PN $\perp$ OQ.

In the right-angled triangle OPN, We have $sin\ \delta\theta = \frac{PN}{OP} = \frac{PN}{r} \Rightarrow PN = r\ sin\ \delta\theta = r\delta\theta$

since $\text{Sin}\delta\theta = \delta\theta$ when $\delta\theta$ *is very small*.

From the figure we see that, $cos\ \delta\theta = \frac{ON}{OP} = \frac{ON}{r} \Rightarrow ON = r\ cos\ \delta\theta = r(1) = r$

$\because \cos\delta\theta = 1\ when\ \ \delta\theta \to 0$

$\therefore NQ = OQ - ON = (r+\delta r) - r = \delta r$

$From\ \square PNQ,\ PQ^2 = PN^2 + NQ^2\ i.e,\ (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2$

$$\Rightarrow \frac{\delta C}{\delta\theta} = \sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2} \quad \therefore \frac{\delta S}{\delta\theta} = \frac{\delta S}{\delta C}\square\frac{\delta C}{\delta\theta} = \frac{\delta S}{\delta C}\sqrt{r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2}$$

We note that $when\ Q \to P\ along\ the\ curve,\ \delta\theta \to 0\ also\ \frac{\delta S}{\delta C} = 1$

$$\therefore when\ Q \to P,\ \frac{dS}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \to (4)$$

$$Similarly,\quad (\delta C)^2 = (r\delta\theta)^2 + (\delta r)^2 \Rightarrow \frac{\delta C}{\delta r} = \sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$and \quad \frac{\delta S}{\delta r} = \frac{\delta S}{\delta C} \cdot \frac{\delta C}{\delta r} = \frac{\delta S}{\delta C}\sqrt{1 + r^2\left(\frac{\delta\theta}{\delta r}\right)^2}$$

$$\therefore when\ Q \to P,\ we\ get\ \frac{dS}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \to (5)$$

Note:

We know that $\tan\phi = r\dfrac{d\theta}{dr}$

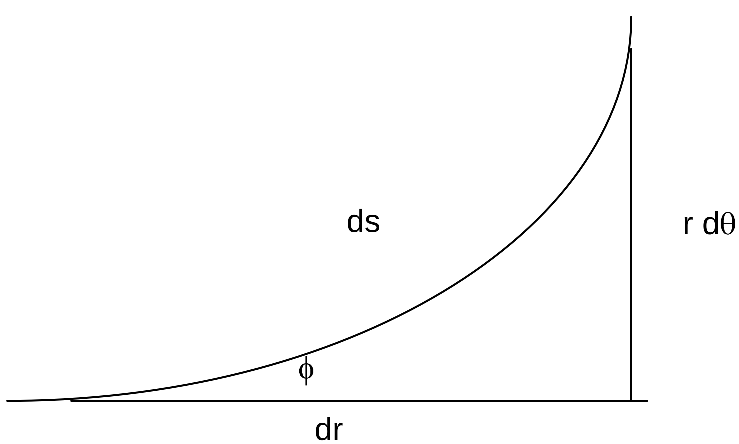
$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r^2 cot^2\phi} = r\sqrt{1 + cot^2\phi} = rco\sec\phi$$

Similarly

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = \sqrt{1 + tan^2\phi} = sec\phi$$

$$\therefore \frac{dr}{ds} = cos\phi \quad and \quad \frac{d\theta}{ds} = \frac{1}{r} sin\phi$$

The following figure shows the geometrical connections among ds, dr, dθ and ϕ



Thus we have :

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}, \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \quad and \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Example 1: $\frac{\mathrm{ds}}{\mathrm{dx}}$ and $\frac{\mathrm{ds}}{\mathrm{dy}}$ for the curve $\mathrm{x}^{2/3} + y^{2/3} = a^{2/3}$

$$\mathrm{x}^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}\mathrm{x}^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad \Rightarrow y' = \frac{\mathrm{x}^{\frac{-1}{3}}}{\mathrm{y}^{\frac{-1}{3}}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Hence $\frac{\mathrm{ds}}{\mathrm{dx}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}}$

$$= \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{a^{2/3}}{x^{2/3}}} = \left(\frac{a}{x}\right)^{1/3}$$

Similarly

$$\frac{\mathrm{ds}}{\mathrm{dy}} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{x^{2/3}}{y^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{y^{2/3}}}$$

$$= \sqrt{\frac{a^{2/3}}{y^{2/3}}} = \left(\frac{a}{y}\right)^{1/3}$$

Example 2: *Find* $\frac{\mathrm{ds}}{\mathrm{dx}}$ for the curve $\mathrm{y} = \mathrm{a}\ \log\left(\frac{\mathrm{a}^2}{\mathrm{a}^2\text{-}\mathrm{x}^2}\right)$

$$y = a\log a^2 - a\log\ \ a^2 - x^2 \ \ \Rightarrow \frac{dy}{dx} = -a\left(\frac{-2x}{a^2 - x^2}\right) = \frac{2ax}{a^2 - x^2}$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4a^2x^2}{a^2 - x^2}}$$

$$= \sqrt{\frac{a^2 - x^2 \ \ + 4a^2x^2}{a^2 - x^2 \ \ ^2}} = \sqrt{\frac{a^2 + x^2 \ \ ^2}{a^2 - x^2 \ \ ^2}}$$

$$= \frac{a^2 + x^2}{a^2 - x^2}$$

Example 3: If $\mathrm{x} = \mathrm{ae^t sint}$, $\mathrm{y} = \mathrm{ae^t cost}$, find $\frac{ds}{dt}$

$$\mathrm{x} = \mathrm{ae^t sint} \Rightarrow \frac{dx}{dt} = \mathrm{ae^t sint} + \mathrm{ae^t cost}$$

$$y = ae^t\cos t \Rightarrow \frac{dy}{dt} = ae^t\cos t - ae^t\sin t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2e^{2t}\ (cos\, t + sin\, t)^2 + a^2e^{2t}\ (cos\, t - sin\, t)^2}$$

$$= ae^t\sqrt{2\ (cos^2t + sin^2t)} \quad = a\sqrt{2}\ e^t \qquad \because\ (a+b)^2 + (a-b)^2 = 2\ (a^2 + b^2)$$

Example 4: If $x = a\left[\cos t + \log\tan\frac{t}{2}\right], y = a\sin t$, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = a\left[-sin\, t + \frac{sec^2\ t/2}{2\cdot tan\ t/2}\right] = a\left[-sin\, t + \frac{1}{2sin\frac{t}{2}cos\frac{t}{2}}\right]$$

$$= a\left[-sin\, t + \frac{1}{sin\, t}\right] = a\,\frac{1 - sin^2t}{sin\, t} \quad = \frac{a\, cos^2t}{sin\, t} = a\, cost\cdot cot\, t$$

$$\frac{dy}{dt} = a\, cos\, t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{a^2cos^2t\, cot^2t + a^2cos^2t}$$

$$= \sqrt{a^2cos^2t\ (cot^2t + 1)}$$

$$= \sqrt{a^2cos^2t\cdot co\sec^2 t} = \sqrt{a^2cot^2t}$$

$$= a\, cot\, t$$

Example 5: If $x = a\cos^3 t$, $y = \sin^3 t$, find $\frac{ds}{dt}$

$$\frac{dx}{dt} = -3\, a\, cos^2t\, sin\, t,\ \ \frac{dy}{dt} = 3\, a\, sin^2t\, cos\, t$$

$$\therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad = \sqrt{9a^2cos^4t\, sin^2t + 9a^2sin^4t\, cos^2t}$$

$$= \sqrt{9a^2cos^2t\, sin^2t\ (cos^2t + sin^2t)} \ = 3\, a\, sin\, t\, cos\, t$$

Example 6: If $r^2 = a^2 \cos 2\theta$, Show that $r \frac{ds}{d\theta}$ is constant

$$r^2 = a^2 cos2\theta \Rightarrow 2r\frac{dr}{d\theta} = -2a^2 sin2\theta \quad \Rightarrow \frac{dr}{d\theta} = \frac{-a^2}{r} sin2\theta$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + \frac{a^4}{r^2} sin^2 2\theta} = \frac{1}{r}\sqrt{r^4 + a^4 sin^2 2\theta}$$

$$\therefore r\frac{ds}{d\theta} = \sqrt{r^4 + a^4 sin^2 2\theta} = \sqrt{a^4 cos^2 2\theta + a^4 sin^2 2\theta}$$

$$= a^2\sqrt{cos^2\theta + sin^2 2\theta} = a^2 = \text{constant} \qquad \therefore r\frac{ds}{d\theta} \text{ is constant for } r^2 = a^2 \cos 2\theta$$

Example 7: For the curve $\theta = \cos^{-1}\left(\frac{r}{k}\right) - \frac{\sqrt{k^2 - r^2}}{r}$, Show that $r\frac{ds}{dr}$ is constant.

$$\frac{d\theta}{dr} = \frac{-1}{\sqrt{1-\frac{r^2}{k^2}}} \cdot \frac{1}{k} - \frac{r\left(\frac{-2r}{2\sqrt{k^2-r^2}}\right) - \sqrt{k^2-r^2}(1)}{r^2} = \frac{-1}{\sqrt{k^2-r^2}} + \frac{r^2 + k^2 - r^2}{r^2\sqrt{k^2-r^2}}$$

$$= \frac{-1}{\sqrt{k^2-r^2}} + \frac{k^2}{r^2\sqrt{k^2-r^2}} = \frac{-r^2+k^2}{r^2\sqrt{k^2-r^2}} \qquad = \frac{\sqrt{k^2-r^2}}{r}$$

$$\therefore \frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)2}$$

$$= \sqrt{1 + r^2\frac{k^2-r^2}{r^2}} = \frac{\sqrt{r^2+k^2-r^2}}{r} = \frac{k}{r}$$

Hence $r\frac{ds}{dr} = k$ (constant)

Example 8: For a polar curve $r = f\ \theta$ show that $\frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}, \frac{ds}{d\theta} = \frac{r^2}{p}$

We know that $cos\,\phi = \frac{dr}{ds}$ *and* $\frac{d\theta}{ds} = \frac{1}{r} sin\,\phi$

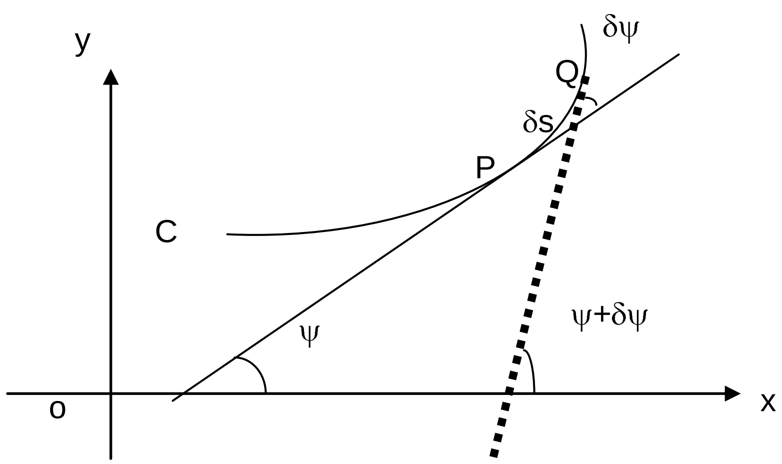
$$\therefore \frac{dr}{ds} = cos\varphi = \sqrt{1 - sin^2\phi} = \sqrt{1 - \frac{p^2}{r^2}} = \frac{\sqrt{r^2 - p^2}}{r} \qquad \because p = rsin\varphi$$

$$\therefore \frac{\mathrm{ds}}{\mathrm{dr}} = \frac{r}{\sqrt{r^2 - p^2}}$$

$$Also \quad \frac{\mathrm{ds}}{\mathrm{d}\theta} = \frac{r}{\sin\phi} = \frac{r}{p/r} = \frac{r^2}{p}$$

CURVATURE:

Consider a curve C in XY-plane and let P, Q be any two neighboring points on it. Let arc AP=s and arc PQ=δs. Let the tangents drawn to the curve at P, Q respectively make angles ψ and ψ+δψ with X-axis i.e., the angle between the tangents at P and Q is δψ. While moving from P to Q through a distance'δs', the tangent has turned through the angle 'δψ'. This is called the bending of the arc PQ. Geometrically, a change in ψ represents the bending of the curve C and the ratio $\frac{\delta\psi}{\delta s}$ represents the ratio of bending of C between the point P & Q and the arc length between them.



$\therefore$Rate of bending of Curve at P is $\frac{d\psi}{ds} = \underset{Q\to P}{Lt} \frac{\delta\psi}{\delta s}$

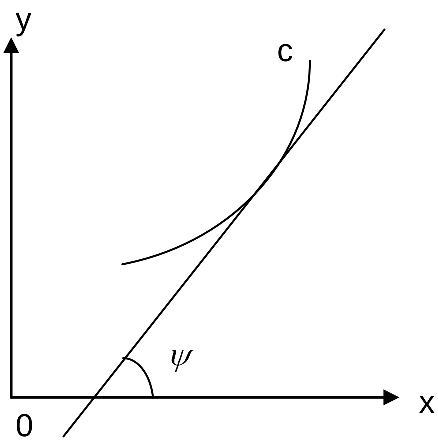
This rate of bending is called the curvature of the curve C at the point P and is denoted by κ (kappa). Thus $\kappa = \frac{\mathrm{d}\psi}{\mathrm{ds}}$We note that the curvature of a straight line is zero since there exist no bending i.e. κ=0, and that the curvature of a circle is a constant and it is not equal to zero since a circle bends uniformly at every point on it

If $\kappa \neq 0$, then $\frac{1}{\kappa}$ is called the radius of curvature and is denoted by ρ (rho - Greek letter).

$$\therefore \rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

<u>Radius of curvature in Cartesian form</u> :

Suppose y = f(x) is the Cartesian equation of the curve considered in figure.



we have $y' = \frac{dy}{dx} = tan\,\psi \Rightarrow y'' = \frac{d^2y}{dx^2} = sec^2\,\psi \cdot \frac{d\psi}{dx} = 1 + tan^2\psi \; \frac{d\psi}{ds}\cdot\frac{ds}{dx}$

But we know that $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\therefore \frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\cdot\frac{d\psi}{ds}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{ds}{d\psi} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\therefore \rho = \frac{ds}{d\psi} = \frac{\left[1+\;y'^{\,2}\right]^{3/2}}{y''}$$

This is the expression for radius of curvature in Cartesian form.

NOTE: We note that when y'=∞, we find ρ using the formula $\rho = \dfrac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\left(\dfrac{d^2x}{dy^2}\right)}$

Example 9: Find the radius of curvature of the curve $x^3+y^3 = 2a^3$ at the point (a, a).

$$x^3+y^3=2a^3 \Rightarrow 3x^2+3y^2\cdot y'=0 \quad \Rightarrow y'=-\frac{x^2}{y^2} \;\; hence\ at\ \ (a,a)\ ,\ y'=-1$$

$$\therefore y''=-\left[\frac{y^2(2x)-x^2(2y)y'}{y^4}\right],\ hence\ at\ \ (a,a)\ ,\ y''=-\left[\frac{2a^3+2a^3}{a^4}\right]=-\frac{4}{a}$$

$$\therefore\ \rho=\frac{\left[1+(y')^2\right]^{3/2}}{y''}=\frac{\left[1+(-1)^2\right]^{3/2}}{-4/a}\ \ i.e.,\ |\rho|=\frac{a}{4}\cdot 2\sqrt{2}\quad=\quad\frac{a}{\sqrt{2}}$$

Example 10: Find the radius of curvature for $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at the point where it meets the line y=x.

On the line y = x, $\sqrt{x}+\sqrt{x}=\sqrt{a}\quad i.e\ 2\sqrt{x}=\sqrt{a}\quad or\quad x=\dfrac{a}{4}$

i.e., We need to find ρ at $\left(\dfrac{a}{4},\dfrac{a}{4}\right)$

$$\sqrt{x}+\sqrt{y}=\sqrt{a}\Rightarrow\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0\ \ i.e\ \ y'=-\sqrt{\frac{y}{x}},\ hence\ at\left(\frac{a}{4},\frac{a}{4}\right),y'=-1$$

$$Also,\ y''=-\left[\frac{\sqrt{x}\dfrac{1}{2\sqrt{y}}\cdot y'-\sqrt{y}\dfrac{1}{2\sqrt{x}}}{x}\right]$$

$$\therefore at\left(\frac{a}{4},\frac{a}{4}\right),y''=-\left[\frac{\sqrt{\frac{a}{4}}\dfrac{1}{2\sqrt{\frac{a}{4}}}\cdot(-1)-\sqrt{\frac{a}{4}}\dfrac{1}{2\sqrt{\frac{a}{4}}}}{\frac{a}{4}}\right]=-\frac{(-\frac{1}{2}-\frac{1}{2})}{\frac{a}{4}}=-\frac{(-1)}{\frac{a}{4}}=\frac{4}{a}$$

$$\therefore \rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+(-1)^2\right]^{3/2}}{4/a} = \frac{a}{4}2\sqrt{2} = \frac{a}{\sqrt{2}}$$

Example 11: Show that the radius of curvature for the curve y = 4 Sin x - Sin 2x

at x = $\pi/2$ is $5\sqrt{5}/4$

y = 4 sin x - sin 2x $\Rightarrow y' = 4\cos x - 2\cos 2x$

$\therefore$ when x $= \pi/2$, $y' = 4\cos\pi/2 - 2\cos\pi = 0 - 2(-1) = 2$

Also, $y'' = -4\sin x + 4\sin 2x$ and when x $= \pi/2$, $y'' = -4\sin\pi/2 + 4\sin\pi = -4$

$$\therefore \rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+2^2\right]^{3/2}}{-4} \Rightarrow |\rho| = \frac{5\sqrt{5}}{4}$$

Example 12: Find the radius of curvature for $xy^2 = a^3 - x^3$ at (a, 0).

$$xy^2 = a^3 - x^3 \Rightarrow y^2 + 2xy\,y' = -3x^2$$

$$\therefore y' = \frac{-3x^2 - y^2}{2xy} \text{ and at } (a,0),\ \ y' = \infty$$

In such cases we write $\frac{dx}{dy} = \frac{2xy}{-3x^2 - y^2}$ and $at(a,0), \frac{dx}{dy} = 0$

Also $\frac{dx}{dy} = \frac{-2xy}{3x^2+y^2} \Rightarrow \frac{d^2x}{dy^2} = \left[\frac{(3x^2+y^2)\left(2\frac{dx}{dy}y + 2x\right) - 2xy\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2+y^2)^2}\right]$

$$\therefore At\ (a,0),\ \frac{d^2x}{dy^2} = \left[\frac{(3a^2+0)(0+2a) - 0}{(3a^2+0)^2}\right] = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2} = \frac{\left[1+o^2\right]^{3/2}}{-2/3a} \ or \quad |\rho| = \frac{3a}{2}$$

An expression for the radius of curvature in the case of a parametric curve $x = x(t),\ y = y(t)$

$$\rho = \frac{\{(\dot{x})^2 + (\dot{y})^2\}^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

1. Find the radius of curvature of the curve

$x = a\log(\text{sect} + \text{tant}),\ \ \text{y} = \text{asect}$

$\Rightarrow x = a\log(\text{sect} + \text{tant})$

$$\frac{dx}{dt} = \frac{a}{\sec t + \tan t}\sec t\tan t + \sec^2 t = \frac{a\sec t(\text{seet} + \text{tant})}{(\text{seet} + \text{tant})}$$

$$\therefore \frac{dx}{dt} = a\sec t$$

Also $y = a\sec t$ *gives*

$$\frac{dy}{dt} = a\sec t\tan t$$

$$Now\ , y_1 = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{a\sec t\tan t}{a\sec t}$$

$$y_1 = \tan t$$

Differentiating w.r.t x we get

$$y_2 = \sec^2 t\frac{dt}{dx}$$

$$\therefore y_2 = \frac{\sec t}{a}$$

$$we\, have\ \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{a(1 + \tan^2 t)^{3/2}}{\sec t}$$

$$\rho = a\sec^2 t$$

2. *Show that the radius of curvature at any point θ on the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos(\theta/2)$*

>> $x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1+\cos\theta) \quad ; \quad \frac{dy}{d\theta} = a\sin\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$\therefore \quad y_1 = \tan(\theta/2)$

Differentiating w.r.t. x we get,

$$y_2 = \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{d\theta}{dx}$$

$$= \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)} = \frac{\sec^2(\theta/2)}{4a\cos^2(\theta/2)}$$

$\therefore \quad y_2 = \frac{1}{4a}\sec^4(\theta/2)$

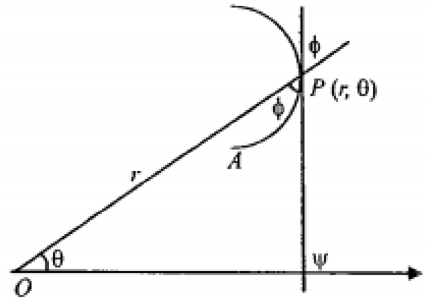
We have $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{[1+\tan^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)}$$

$$= \frac{[\sec^2(\theta/2)]^{3/2}\cdot 4a}{\sec^4(\theta/2)} = \frac{4a\sec^3(\theta/2)}{\sec^4(\theta/2)}$$

Thus $\rho = 4a\cos(\theta/2)$

An expression for the radius of curvature in the case of a polar curve $r = f(\theta)$



Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at $P(r, \theta)$.

Let ψ be the angle made by the tangent at P with the initial line.

Let A be a fixed point on the curve and let $\overset{\frown}{AP} = s$.

We have $\psi = \theta + \phi$

$$\therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} \quad ie., \ \frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\text{or} \quad \rho = \frac{\left(\frac{ds}{d\theta}\right)}{1 + \frac{d\phi}{d\theta}} \qquad \ldots (1)$$

We know that $\tan\phi = r\,\frac{d\theta}{dr} = r \Big/ \left(\frac{dr}{d\theta}\right)$

ie., $\tan\phi = \frac{r}{r_1}$ where $r_1 = \frac{dr}{d\theta}$

Differentiating w.r.t θ we get,

$$\sec^2\phi \, \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

$$\text{or} \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2 \sec^2\phi} = \frac{r_1^2 - r r_2}{r_1^2(1 + \tan^2\phi)}$$

$$ie., \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - r r_2}{r_1^2[1 + (r^2/r_1^2)]} = \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

Hence $1+\dfrac{d\phi}{d\theta} = 1+\dfrac{r_1^2 - r r_2}{r^2+r_1^2} = \dfrac{r^2+r_1^2+r_1^2-r r_2}{r^2+r_1^2}$

ie., $$1+\frac{d\phi}{d\theta} = \frac{r^2+2r_1^2-r r_2}{r^2+r_1^2} \qquad \ldots (2)$$

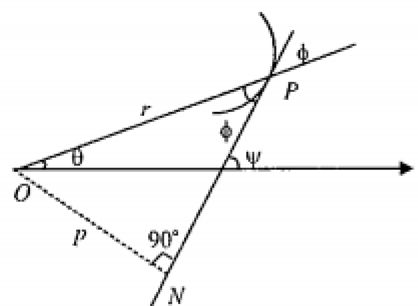
Also, we know that $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2+r_1^2} \qquad \ldots (3)$$

Using (2) and (3) in (1) we get

$$\rho = \sqrt{r^2+r_1^2}\cdot\frac{(r^2+r_1^2)}{r^2+2r_1^2-r r_2}$$

Thus in the polar form , $$\boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r r_2}}$$

An expression for the radius of curvature in the case of a pedal curve



Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at P. Let ψ be the angle made by the tangent at P with the initial line. Draw $ON = p$, a perpendicular from the pole to the tangent.

We have from the $\Delta\ ONP$, $\sin\phi = \dfrac{p}{r}$

ie., $$p = r\sin\phi \qquad \ldots (1)$$

Differentiating (1) w.r.t r we get,

$$\frac{dp}{dr} = r\cos\phi\,\frac{d\phi}{dr} + 1\cdot\sin\phi$$

But we know that, $\sin\phi = r\dfrac{d\theta}{ds}$ and $\cos\phi = \dfrac{dr}{ds}$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\phi}{dr}\frac{dr}{ds} + r\frac{d\theta}{ds} = r\left[\frac{d\phi}{ds}+\frac{d\theta}{ds}\right] = r\frac{d}{ds}(\phi+\theta)$$

But $\phi+\theta = \psi$

$$\therefore \quad \frac{dp}{dr} = r\frac{d\psi}{ds} \quad \text{or} \quad \frac{ds}{d\psi} = r\frac{dr}{dp}$$

Thus $$\boxed{\rho = r\frac{dr}{dp}}$$

1. *Show that the radius of curvature of the curve* $r^n = a^n \cos n\theta$ *varies inversely as* r^{n-1}

>> $r^n = a^n \cos n\theta$

$\Rightarrow \quad n \log r = n \log a + \log(\cos n\theta)$

Differentiating w.r.t. θ we have,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n\sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$\therefore \quad r_1 = -r\tan n\theta$

Hence $r_2 = \dfrac{d^2 r}{d\theta^2} = -r_1 \tan n\theta - n\, r \sec^2 n\theta$

We have $\quad \rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$$\therefore \quad \rho = \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r(-r_1\tan n\theta - nr\sec^2 n\theta)}$$

$$= \frac{(r^2)^{3/2}(\sec^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2(1 + \tan^2 n\theta + n\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{\sec^2 n\theta(1+n)} = \frac{r\sec n\theta}{(1+n)}$$

Thus $\quad \rho = \dfrac{r}{1+n}\sec n\theta$

But $\quad a^n / r^n = \sec n\theta$ by data.

$$\therefore \quad \rho = \frac{r}{1+n}\cdot\frac{a^n}{r^n} = \left[\frac{a^n}{1+n}\right]\frac{1}{r^{n-1}}$$

ie., $\quad \rho = \text{const}\cdot\dfrac{1}{r^{n-1}}$

Thus $\quad \rho \propto 1/r^{n-1}$

2. *Find the radius of curvature of the curve* $r = a\sin n\theta$ *at the pole.*

>> $r = a\sin n\theta$

$\therefore \quad r_1 = an\cos n\theta,\ r_2 = -an^2\sin n\theta$

At the pole we have $\theta = 0$. When $\theta = 0$: $r = 0,\ r_1 = an,\ r_2 = 0$

We have $\rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$

$\therefore \quad \rho = \dfrac{(a^2 n^2)^{3/2}}{2a^2 n^2} = \dfrac{a^3 n^3}{2a^2 n^2} = \dfrac{an}{2}$

Thus $\rho = an/2$ at the pole.

3. *Find the radius of curvature of the curve* $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}(a/r)$ *at any point on it.*

>> Differentiating the given equation w.r.t. r we have,

$$\frac{d\theta}{dr} = \frac{1}{a}\cdot\frac{2r}{2\sqrt{r^2 - a^2}} - \left\{\frac{-1}{\sqrt{1-(a/r)^2}}\cdot\frac{-a}{r^2}\right\}$$

$$= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{r}{\sqrt{r^2 - a^2}}\cdot\frac{a}{r^2}$$

$$= \frac{1}{\sqrt{r^2 - a^2}}\left(\frac{r}{a} - \frac{a}{r}\right) = \frac{r^2 - a^2}{\sqrt{r^2 - a^2}\cdot ar}$$

ie., $$\frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{ar} \quad \ldots (1)$$

We prefer to find the pedal equation of the given curve and then apply the formula for ρ in the pedal form.

From (1) $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{r^2 - a^2}}$ *ie.,* $\cot\phi = \dfrac{a}{\sqrt{r^2 - a^2}}$

Consider $p = r\sin\phi$

$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \quad$ *ie.,* $\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$

$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{a^2}{r^2 - a^2}\right]$

ie., $\quad \frac{1}{p^2} = \frac{1}{r^2}\left[\frac{r^2}{r^2 - a^2}\right] \quad$ *ie.,* $\frac{1}{p} = \frac{1}{\sqrt{r^2 - a^2}}$

$\therefore \quad p = \sqrt{r^2 - a^2}$ is the pedal equation of the curve.

Differentiating w.r.t. p we get,

$$1 = \frac{2r}{2\sqrt{r^2 - a^2}}\frac{dr}{dp} \quad \text{ie., } \sqrt{r^2 - a^2} = r\frac{dr}{dp} = \rho$$

Thus $\quad \rho = \sqrt{r^2 - a^2}$

MODULE II

DIFFERENTIAL CALCULUS-II

CONTENTS:

Taylor's Mean Value Theorem:

(Generalized Mean Value Theorem):

(English Mathematician Brook Taylor 1685-1731)

Statement:

Suppose a function $f(x)$ satisfies the following two conditions:

(i) $f(x)$ and it's first (n-1) derivatives are continuous in a closed interval $[a,b]$

(ii) $f^{(n-1)}(x)$ is differentiable in the open interval (a,b)

Then there exists at least one point c in the open interval (a,b) such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \frac{(b-a)^3}{\lfloor 3} f'''(a) + \ldots..$$

$$\ldots.. + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} f^{(n)}(c) \to (1)$$

Taking $b = a + h$ and for $0 < \theta < 1$, the above expression (1) can be rewritten as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \ldots. + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{h^n}{\lfloor n} f^{(n)}(a+\theta h) \to (2)$$

Taking b=x in (1) we may write

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \frac{(x-a)^3}{\lfloor 3} f'''(a) + \ldots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + R_n \to (3)$$

$$Where \quad R_n = \frac{(x-a)^n}{\lfloor n} f^{(n)}(c) \to \text{Remainder term after } n \text{ terms}$$

When $n \to \infty$, we can show that $|R_n| \to 0$, thus we can write the Taylor's series as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \ldots.$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) \to (4)$$

Using (4) we can write a Taylor's series expansion for the given function f(x) in powers of (x-a) or about the point 'a'.

Maclaurin's series:
(Scottish Mathematician Colin Maclaurin's 1698-1746)

When a=0, expression (4) reduces to a Maclaurin's expansion given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \ldots.$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{\lfloor n} f^{(n)}(0) \rightarrow (5)$$

Example 1: Obtain a Taylor's expansion for $f(x) = sin\, x$ in the ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to the fourth degree term.

The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x) = f(\frac{\pi}{4}) + (x - \frac{\pi}{4}) f'(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{\lfloor 2} f''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3} f'''(\frac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^4}{\lfloor 4} f^{(4)}(\frac{\pi}{4}) \ldots. \rightarrow (1)$$

$$f(x) = sin\, x \Rightarrow f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \;; \qquad f'(x) = cos\, x \Rightarrow f'\left(\frac{\pi}{4}\right) = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x) = -sin\, x \Rightarrow f''\left(\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -cos\, x \Rightarrow f'''\left(\frac{\pi}{4}\right) = -cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x) = sin\, x \Rightarrow f^{(4)}\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x) = \frac{1}{\sqrt{2}} + (x - \frac{\pi}{4})(\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^2}{\lfloor 2}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^4}{\lfloor 4}(\frac{1}{\sqrt{2}}) \ldots.$$

$$f(x) = \frac{1}{\sqrt{2}}\left[1 + (x - \frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{\lfloor 2} + \frac{(x-\frac{\pi}{4})^3}{\lfloor 3} - \frac{(x-\frac{\pi}{4})^4}{\lfloor 4} + \ldots\right]$$

Example 2………………………….: Obtain a Taylor's expansion for $f(x) = \log_e x$ up to the term containing $x-1^{\,4}$ and hence find $\log_e(1.1)$.

The Taylor's series for $f(x)$ about the point 1 is

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{\lfloor 2} f''(1) + \frac{(x-1)^3}{\lfloor 3} f'''(1) + \frac{(x-1)^4}{\lfloor 4} f^{(4)}(1) \ldots. \to (1)$$

Here $f(x) = \log_e x \Rightarrow f(1) = \log 1 = 0$; $\quad f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$; $\quad f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = 2$

$f^{(4)}(x) = -\frac{6}{x^4} \Rightarrow f^{(4)}(1) = -6$ etc.,

Using all these values in (1) we get

$$f(x) = \log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{\lfloor 2}(-1) + \frac{(x-1)^3}{\lfloor 3}(2) + \frac{(x-1)^4}{\lfloor 4}(-6) \ldots.$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \ldots.$$

Taking x=1.1 in the above expansion we get

$$\Rightarrow \log_e(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} \ldots. = 0.0953$$

Example 18: Using Taylor's theorem Show that

$$\log_e(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad for\ 0 < \theta < 1, x > 0$$

Taking n=3 in the statement of Taylor's theorem, we can write

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{\lfloor 2} f''(a) + \frac{x^3}{\lfloor 3} f'''(a+\theta x) \to (1)$$

Consider $f(x) = \log_e x \quad \Rightarrow f'(x) = \frac{1}{x}$; $f''(x) = -\frac{1}{x^2}$ *and* $f'''(x) = \frac{2}{x^3}$

Using these in (1), we can write,

$$\log(a+x) = \log a + x\left(\frac{1}{a}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{a^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{(a+\theta x)^3}\right) \to (2)$$

For a=1 in (2) we write,

$$\log(1+x) = \log 1 + x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3}$$

Since $x > 0$ *and* $\theta > 0$, $(1+\theta x)^3 > 1$ *and therefore* $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Example 19: Obtain a Maclaurin's series for $f(x) = \sin x$ up to the term containing x^5.

The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots. \rightarrow (1)$$

Here $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$ $\quad f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$ $\quad f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = \sin 0 = 0$ $\quad f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = \cos 0 = 1$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = \sin x$ as

$$f(x) = \sin x = 0 + x\,(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots.$$

$$\Rightarrow \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots.$$

Indeterminate Forms:

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0$ *and* 1^∞ which do not represent any value. Such expressions are called Indeterminate Forms.

We can evaluate such limits that lead to indeterminate forms by using L'Hospital's Rule (French Mathematician 1661-1704).

L'Hospital's Rule:

If f(x) and g(x) are two functions such that

(i) $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$

(ii) $f'(x)$ *and* $g'(x)$ *exist and* $g'(a) \neq 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

The above rule can be extended, i.e, if

$f'(a) = 0$ *and* $g'(a) = 0$ *then* $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = ...$

Note:

1. We apply L'Hospital's Rule only to evaluate the limits that in $\frac{0}{0}, \frac{\infty}{\infty}$ forms. Here we differentiate the numerator and denominator separately to write $\frac{f'(x)}{g'(x)}$ and apply the limit to see whether it is a finite value. If it is still in $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form we continue to differentiate the numerator and denominator and write further $\frac{f''(x)}{g''(x)}$ and apply the limit to see whether it is a finite value. We can continue the above procedure till we get a definite value of the limit.

2. To evaluate the indeterminate forms of the form $0 \times \infty, \infty - \infty,$ we rewrite the functions involved or take L.C.M. to arrange the expression in either $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ and then apply L'Hospital's Rule.
3. To evaluate the limits of the form $0^0, \infty^0$ *and* 1^∞ i.e, where function to the power of function exists, call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

4. We can use the values of the standard limits like

$\lim_{x \to 0} \frac{\sin x}{x} = 1;\ \lim_{x \to 0} \frac{\tan x}{x} = 1; \lim_{x \to 0} \frac{x}{\sin x} = 1; \lim_{x \to 0} \frac{x}{\tan x} = 1; \lim_{x \to 0} \cos x = 1;$ *etc*

Evaluate the following limits:

Example 1: Evaluate $\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}$

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x - 1}{3\tan^2 x\sec^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6\tan x\sec^4 x+6\tan^3 x\sec^2 x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\cos x}{6\sec^6 x+24\tan^2 x\sec^4 x+18\tan^2 x\sec^4 x+12\tan^4 x\sec^2 x}=-\frac{1}{6}$$

Method 2:

$$\lim_{x\to 0}\frac{\sin x - x}{\tan^3 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{\sin x - x}{x^3}}{\left(\frac{\tan x}{x}\right)^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x - x}{x^3}\left(\frac{0}{0}\right)\quad \because \lim_{x\to 0}\frac{\tan x}{x}=1$$

$$=\lim_{x\to 0}\frac{\cos x - 1}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x}{6x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$$

Example 2: Evaluate $\lim_{x\to 0}\frac{a^x - b^x}{x}$

$$\lim_{x\to 0}\frac{a^x - b^x}{x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1}$$

$$=\log a-\log b=\log\frac{a}{b}$$

Example 3: Evaluate $\lim_{x\to 0}\frac{x\sin x}{e^x - 1\;^{2}}$

$$\lim_{x\to 0}\frac{x\sin x}{e^x-1\;^{2}}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x + x\cos x}{2\;\;e^x-1\;\;e^x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x+\cos x - x\sin x}{2\left[e^x.e^x+(e^x-1)e^x\right]}=\frac{1+1-0}{2[1+0]}=\frac{2}{2}=1$$

Example 4: Evaluate $\lim\limits_{x\to 0}\dfrac{x\,e^x-\log(1+x)}{x^2}$

$$\lim_{x\to 0}\frac{x\,e^x-\log(1+x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+x\,e^x-\dfrac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+e^x+x\,e^x+\dfrac{1}{1+x^{\;2}}}{2}=\frac{1+1+0+1}{2}=\frac{3}{2}$$

Example 5: Evaluate $\lim\limits_{x\to 0}\dfrac{\cosh x-\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sinh x+\sin x}{\sin x+x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cosh x+\cos x}{\cos x+\cos x-x\sin x}=\frac{1+1}{1+1-0}=\frac{2}{2}=1$$

Example 6: Evaluate $\lim\limits_{x\to 0}\dfrac{\cos x-\log(1+x)-1+x}{\sin^2 x}$

$$\lim_{x\to 0}\frac{\cos x-\log(1+x)-1+x}{\sin^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\sin x-\dfrac{1}{1+x}+1}{\sin 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-\cos x+\dfrac{1}{(1+x)^2}}{2\cos 2x}=\frac{-1+1}{2}=0$$

Example 7: Evaluate $\lim\limits_{x\to 1}\dfrac{x^x-x}{x-1-\log x}$

$$\lim_{x\to 1}\frac{x^x-x}{x-1-\log x}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)-1}{1-\dfrac{1}{x}}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x^x(1+\log x)^2+x^{x-1}}{\dfrac{1}{x^2}}=\frac{1+1}{1}=2$$

$$since\;\; y=x^x\Rightarrow\log y=x\log x\Rightarrow\frac{1}{y}y'$$

$$=1+\log x\Rightarrow y'=y(1+\log x)$$

$$then\;\frac{d}{dx}(x^x)=x^x(1+\log x)$$

Example 8: Evaluate $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sec^2 x-2\tan x}{1+\cos 4x}$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x-2\tan x}{1+\cos 4x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{4}}\frac{2\sec^2 x\tan x-2\sec^2 x}{-4\sin 4x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{4}}\frac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x}$$

$$= \frac{2\left(\sqrt{2}\right)^4 + 4\left(\sqrt{2}\right)^2 (1)^2 - 4\left(\sqrt{2}\right)^2}{16} = \frac{8}{16} = \frac{1}{2}$$

Example 9: Evaluate $\lim_{x \to a} \frac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)}$

$$\lim_{x \to a} \frac{\log(\sin x.\cos ec\ a)}{\log(\cos a.\sec x)} \left(\frac{0}{0}\right) = \lim_{x \to a} \frac{\left[\frac{\cos x \cos ec\ a}{\sin x.\cos ec\ a}\right]}{\left[\frac{\sec x \tan x.\cos a}{\cos a.\sec x}\right]} = \lim_{x \to a} \frac{\cot x}{\tan x} = \cot^2 a$$

Example 10: Evaluate $\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$

$$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{\sin x + x\cos x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{\cos x + \cos x - x\sin x} = \frac{1+1+2}{1+1-0} = 2$$

Example 11: Evaluate $\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2}$

$$\lim_{x \to 0} \frac{x\cos x - \log(1+x)}{x^2} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x - \sin x - x\cos x + \frac{1}{(1+x)^2}}{2} = \frac{-0-0-0+1}{2} = \frac{1}{2}$$

Example 12: Evaluate $\lim_{x \to 0} \frac{\log(1-x^2)}{\log \cos x}$

$$\lim_{x \to 0} \frac{\log(1-x^2)}{\log \cos x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\left[\frac{-2x}{1-x^2}\right]}{\left(\frac{-\sin x}{\cos x}\right)} = \lim_{x \to 0} \frac{2x\cos x}{(1-x^2)\sin x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2\cos x - 2x\sin x}{(1-x^2)\cos x - 2x\sin x} = \frac{2-0}{1-0} = 2$$

Example 13: Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sec^2 x - \cos x}{3\sin^2 x \cos x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2\sec^2 x \tan x + \sin x}{6\sin x \cos^2 x - 3\sin^3 x} \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6\cos^3 x-12\sin^2 x\cos x-9\sin^2 x\cos x}=\frac{0+2+1}{6-0-0}=\frac{3}{6}=\frac{1}{2}$$

Method 2:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\frac{\tan x-\sin x}{x^3}}{\left(\frac{\sin x}{x}\right)^3}=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\left(\frac{0}{0}\right)\quad\because\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}=\frac{0+2+1}{6}=\frac{3}{6}=\frac{1}{2}$$

Example 14: Evaluate $\lim_{x\to 0}\frac{\tan x-x}{x^2\tan x}$

$$\lim_{x\to 0}\frac{\tan x-x}{x^2\tan x}=\lim_{x\to 0}\frac{\frac{\tan x-x}{x^3}}{\left(\frac{\tan x}{x}\right)}=\lim_{x\to 0}\frac{\tan x-x}{x^3}\left(\frac{0}{0}\right)\qquad\because\lim_{x\to 0}\frac{\tan x}{x}=1$$

$$=\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{0+2}{6}=\frac{1}{3}$$

Example 15: Evaluate $\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}$

$$\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\log(1+bx)}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{ae^{ax}+ae^{-ax}}{b/(1+bx)}$$

$$=\frac{a+a}{b}=\frac{2a}{b}$$

Example 16: Evaluate $\lim_{x\to 0}\frac{a^x-1-x\log a}{x^2}$

$$\lim_{x\to 0}\frac{a^x-1-x\log a}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a^x\log a-\log a}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{a^x(\log a)^2}{2}=\frac{1}{2}(\log a)^2$$

Example 17: Evaluate $\lim_{x\to 0}\frac{e^x-\log(e+ex)}{x^2}$

$$\lim_{x\to 0}\frac{e^x-\log(e+ex)}{x^2}=\lim_{x\to 0}\frac{e^x-\log e(1+x)}{x^2}=\lim_{x\to 0}\frac{e^x-\log e-\log(1+x)}{x^2}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{e^x-\dfrac{1}{1+x}}{2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x+\dfrac{1}{(1+x)^2}}{2}=\frac{1+1}{2}=1$$

Limits of the form $\left(\frac{\infty}{\infty}\right)$:

Example 18: Evaluate $\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}$

$$\lim_{x\to 0}\frac{\log(\sin 2x)}{\log(\sin x)}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{(2\cos 2x/\sin 2x)}{(\cos x/\sin x)}=\lim_{x\to 0}\frac{2\cot 2x}{\cot x}=\lim_{x\to 0}\frac{2\tan x}{\tan 2x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sec^2 x}{2\sec^2 2x}=\frac{2}{2}=1$$

Example 19: Evaluate $\lim_{x\to 0}\frac{\log x}{\cos ecx}$

$$\lim_{x\to 0}\frac{\log x}{\cos ecx}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 0}\frac{1\backslash x}{-\cos ec\ x.\cot x}=\lim_{x\to 0}\frac{-\sin^2 x}{x\cos x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-2\sin 2x}{\cos x-x\sin x}=\frac{0}{1-0}=0$$

Example 20: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}$

$$\lim_{x\to\frac{\pi}{2}}\frac{\log\cos x}{\tan x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=-\lim_{x\to\frac{\pi}{2}}\frac{-\tan x}{\sec^2 x}=\lim_{x\to\frac{\pi}{2}}\frac{-\sin x\cos x}{1}=\frac{-0}{1}=0$$

Example 21: Evaluate $\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}$

$$\lim_{x\to 1}\frac{\log(1-x)}{\cot\pi x}\left(\frac{\infty}{\infty}\right)=\lim_{x\to 1}\frac{-1/(1-x)}{-\pi\cos ec^2\pi x}=\lim_{x\to 1}\frac{\sin^2\pi x}{\pi(1-x)}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{2\pi\sin\pi x\cos\pi x}{-\pi}=\frac{0}{-\pi}=0$$

Example 22: Evaluate $\lim_{x\to 0}\log_{\tan 2x}\tan 3x$

$$\lim_{x\to 0}\log_{\tan 2x}\tan 3x=\lim_{x\to 0}\left(\frac{\log\tan 3x}{\log\tan 2x}\right)\left(\frac{\infty}{\infty}\right)\qquad \because \log_b a=\frac{\log_e a}{\log_e b}$$

$$= \lim_{x\to 0}\left(\frac{3\sec^2 3x/\tan 3x}{2\sec^2 2x/\tan 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right) = \lim_{x\to 0}\left(\frac{3/\sin 3x.\cos 3x}{2/\sin 2x.\cos 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{6/\sin 6x}{4/\sin 4x}\right) = \lim_{x\to 0}\left(\frac{6\sin 4x}{4\sin 6x}\right)\left(\frac{0}{0}\right) = \lim_{x\to 0}\left(\frac{24\cos 4x}{24\cos 6x}\right) = \frac{24}{24} = 1$$

Example 23: Evaluate $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x - e^a)}$

$$\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}\left(\frac{\infty}{\infty}\right) = \lim_{x\to a}\frac{1/(x-a)}{e^x/(e^x-e^a)} = \lim_{x\to a}\frac{(e^x-e^a)}{e^x(x-a)}\left(\frac{0}{0}\right) = \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} = \frac{e^a}{e^a} = 1$$

Limits of the form $0\times\infty$ **:** To evaluate the limits of the form $0\times\infty$ **,** we rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 24: Evaluate $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x$

$$\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x \quad 0\times\infty \ \ form \ = \lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\left(\frac{1}{x}\right)}\left(\frac{0}{0}\right) = \lim_{x\to\infty}\frac{a^{\frac{1}{x}}(\log a)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty} a^{\frac{1}{x}}(\log a) = a^0\log a = \log a$$

Example 25: Evaluate $\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x$

$$\lim_{x\to\frac{\pi}{2}}(1-\sin x)\tan x \quad 0\times\infty \ \ form \ = \lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)}{\cot x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\cos ec^2 x} = \frac{0}{1} = 0$$

Example 26: Evaluate $\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x$

$$\lim_{x\to 1}\sec\frac{\pi}{2x}.\log x \quad \infty\times 0 \ \ form \ = \lim_{x\to 1}\frac{\log x}{\cos\frac{\pi}{2x}}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1/x}{-\frac{\pi}{2}\left(\sin\frac{\pi}{2x}\right)\left(\frac{-1}{x^2}\right)} = \lim_{x\to 1} \frac{2x}{\pi\sin\frac{\pi}{2x}} = \frac{2}{\pi}$$

Example 27: Evaluate $\lim_{x\to 0} x\log\tan x$

$$\lim_{x\to 0} x\log\tan x \quad 0\times\infty \ \ form \ \ = \lim_{x\to 0}\frac{\log\tan x}{(1/x)}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{\sec^2 x/\tan x}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0}\frac{-x^2}{\sin x.\cos x}$$

$$= \lim_{x\to 0}\frac{-2x^2}{\sin 2x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-4x}{2\cos 2x} = \frac{0}{2} = 0$$

Example 28: Evaluate $\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2}$

$$\lim_{x\to 1}(1-x^2)\tan\frac{\pi x}{2} \quad 0\times\infty \ \ form$$

$$= \lim_{x\to 1}\frac{1-x^2}{\cot\frac{\pi x}{2}}\left(\frac{0}{0}\right) = \lim_{x\to 1}\frac{-2x}{-\frac{\pi}{2}\cos ec^2\frac{\pi x}{2}}$$

$$= \frac{2}{\left(\frac{\pi}{2}\right)} = \frac{4}{\pi}$$

Example 29: Evaluate $\lim_{x\to 0}\tan x.\log x$

$$\lim_{x\to 0}\tan x.\log x \quad 0\times\infty \ \ form \ \ = \lim_{x\to 0}\frac{\log x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{-\cos ec^2 x} = \lim_{x\to 0}\frac{-\sin^2 x}{x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{-\sin 2x}{1} = \frac{0}{1} = 0$$

Limits of the form $\infty-\infty$ **:** To evaluate the limits of the form $\infty-\infty$ **,** we take L.C.M. and rewrite the given expression to obtain either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form and then apply the L'Hospital's Rule.

Example 30: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\left[\frac{1}{x}-\frac{\cos x}{\sin x}\right]\ \infty-\infty\ form$$

$$=\lim_{x\to 0}\left[\frac{\sin x-x\cos x}{x\sin x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{\sin x+x\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{x\sin x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{\sin x+x\cos x}{\cos x+\cos x-x\sin x}\right]$$

$$=\frac{0+0}{1+1-0}=0$$

Example 31: Evaluate $\lim_{x\to\frac{\pi}{2}}\sec x-\tan x$

$$\lim_{x\to\frac{\pi}{2}}\sec x-\tan x=\lim_{x\to\frac{\pi}{2}}\left[\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right]\ \infty-\infty\ form$$

$$=\lim_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{\cos x}\right]\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\left[\frac{-\cos x}{-\sin x}\right]=\frac{0}{1}=0$$

Example 32: Evaluate $\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]$

$$\lim_{x\to 1}\left[\frac{1}{\log x}-\frac{x}{x-1}\right]\ \infty-\infty\ form\ =\lim_{x\to 1}\left[\frac{(x-1)-x\log x}{(x-1)\log x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{1-1-\log x}{\frac{x-1}{x}+\log x}\right]=\lim_{x\to 1}\left[\frac{-\log x}{1-\frac{1}{x}+\log x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 1}\left[\frac{-1/x}{\frac{1}{x^2}+\frac{1}{x}}\right]=\frac{-1}{1+1}=\frac{-1}{2}$$

Example 33: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]\ \infty-\infty\ form\ =\lim_{x\to 0}\left[\frac{(e^x-1)-x}{x(e^x-1)}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{e^x-1}{(e^x-1)+xe^x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{e^x}{e^x+e^x+xe^x}\right]$$

$$=\frac{1}{1+1+0}=\frac{1}{2}$$

Example 34: Evaluate $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

$$\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]\ \infty-\infty\ form\ =\lim_{x\to 0}\left[\frac{x-\sin x}{x\sin x}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{1-\cos x}{\sin x+x\cos x}\right]\left(\frac{0}{0}\right)==\lim_{x\to 0}\left[\frac{\sin x}{\cos x+\cos x-x\sin x}\right]=\frac{0}{1+1}=0$$

Example 35: Evaluate $\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]$

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{\log(1+x)}{x^2}\right]=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{1-\frac{1}{1+x}}{2x}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{(1+x)^2}}{2}\right]=\frac{1}{2}$$

Example 36: Evaluate $\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]$

$$\lim_{x\to 0}\left[\frac{a}{x}-\cot\frac{x}{a}\right]=\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\ \infty-\infty\ form\ =\lim_{x\to 0}\left[\frac{a}{x}-\frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\left[\frac{a\sin\frac{x}{a}-x\cos\frac{x}{a}}{x\sin\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{a.\frac{1}{a}\cos\frac{x}{a}-\cos\frac{x}{a}+\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]$$

$$=\lim_{x\to 0}\left[\frac{\frac{x}{a}\sin\frac{x}{a}}{\sin\frac{x}{a}+\frac{x}{a}\cos\frac{x}{a}}\right]\left(\frac{0}{0}\right)=\lim_{x\to 0}\left[\frac{\frac{1}{a}\sin\frac{x}{a}+\frac{x}{a}.\frac{1}{a}.\cos\frac{x}{a}}{\frac{1}{a}\cos\frac{x}{a}+\frac{1}{a}\cos\frac{x}{a}-\frac{x}{a^2}\sin\frac{x}{a}}\right]=\frac{0+0}{\frac{1}{a}+\frac{1}{a}-0}=0$$

Example 37: Find the value of 'a' such that $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ is finite. Also find the value of the limit.

Let A$=\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\cos 2x+a\cos x}{3x^2}=\frac{2+a}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 2+a=0 i.e., a= -2 .

For $a=-2$**,**

$$A=\lim_{x\to 0}\frac{2\cos 2x-2\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-4\sin 2x+2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x+2\cos x}{6}=\frac{-8+2}{6}=-1$$

$\therefore$ *The given* lim *it will have a finite value when* $a=-2$ *and it is* -1.

Example 38: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}=\frac{1}{3}$.

Let $A=\lim_{x\to 0}\frac{x(1-a\cos x)+b\sin x}{x^3}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}=\frac{1-a+b}{0}\neq \textit{finite}$

We can continue to apply the L'Hospital's Rule, if 1-a+b=0 i.e., a-b = 1 .

For $a-b=1$**,**

$$A=\lim_{x\to 0}\frac{(1-a\cos x)+ax\sin x+b\cos x}{3x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2a\sin x+ax\cos x-b\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{3a\cos x - ax\sin x - b\cos x}{6} = \frac{3a-b}{6} = finite$$

This finite value is given as $\frac{1}{3}$. *i.e.,* $\frac{3a-b}{6} = \frac{1}{3} \Rightarrow 3a - b = 2$

Solving the equations $a-b=1$ *and* $3a-b=2$ *we obtain* $a=\frac{1}{2}$ *and* $b=-\frac{1}{2}$.

Example 39: Find the values of 'a' and 'b' such that $\lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = 1$.

$$Let\ A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2} = \frac{a-b}{0} \neq finite$$

We can continue to apply the L'Hospital's Rule, if a-b=0, since the denominator=0 .

For $a-b=0$,

$$A = \lim_{x\to 0}\frac{a\cosh x - b\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{a\sinh x + b\sin x}{2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{a\cosh x + b\cos x}{2} = \frac{a+b}{2}$$

But this is given as 1.

$\therefore a+b=2$

Solving the equations $a-b=0$ *and* $a+b=2$ *we obtain* $a=1$ *and* $b=1$.

Limits of the form $0^0, \infty^0$ *and* 1^∞ **:** To evaluate such limits, where function to the power of function exists, we call such an expression as some constant, then take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

Example 40: Evaluate $\lim_{x\to 0} x^x$

$$Let\ A = \lim_{x\to 0} x^x\ (0^0\ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0} \log x^x = \lim_{x\to 0} x.\log x \; (0\times\infty \;\; form) = \lim_{x\to 0} \frac{\log x}{1/x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{1/x}{(-1/x^2)} = \lim_{x\to 0} \frac{-x}{1} = \frac{0}{1} = 0$$

$$\log_e A = 0 \Rightarrow A = e^0 = 1 \quad \therefore \lim_{x\to 0} x^x = 1$$

Example 41: Evaluate $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

$$Let \; A = \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} \; (1^\infty \;\; form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0} \log(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0} \frac{1}{x^2} \log\cos x \; (\infty\times 0 \;\; form) = \lim_{x\to 0} \frac{\log\cos x}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\tan x}{2x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{-\sec^2 x}{2} = \frac{-1}{2}$$

$$\log_e A = -\frac{1}{2} \Rightarrow A = e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}} \quad \therefore \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$$

Example 42: Evaluate $\lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x}$

$$Let \; A = \lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x} \; (\infty^0 \;\; form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to \frac{\pi}{2}} \log(\tan x)^{\cos x} = \lim_{x\to \frac{\pi}{2}} \cos x \log(\tan x) \; (0\times\infty \;\; form)$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\log\tan x}{\sec x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to \frac{\pi}{2}} \frac{\sec^2 x/\tan x}{\sec x.\tan x} = \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

$$\log_e A = 0 \Rightarrow A = e^0 = 1 \quad \therefore \lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x} = 1$$

Example 43: Evaluate $\lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}$

$$Let\ A = \lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log(\frac{\tan x}{x})^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{1}{x^2}\log(\frac{\tan x}{x})\ (\infty\times 0\ form)$$

$$= \lim_{x\to 0}\frac{\log(\frac{\tan x}{x})}{x^2}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{1}{\sin x.\cos x}-\frac{1}{x}}{2x} = \lim_{x\to 0}\frac{\frac{2}{\sin 2x}-\frac{1}{x}}{2x} = \lim_{x\to 0}\frac{2x-\sin 2x}{2x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{2-2\cos 2x}{4x\sin 2x+4x^2\cos 2x}\left(\frac{0}{0}\right) == \lim_{x\to 0}\frac{-4\sin 2x}{4\sin 2x+16x\cos 2x-8x^2\sin 2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-8\cos 2x}{24\cos 2x-48x\sin 2x-16x^2\cos 2x} = \frac{-8}{24} = \frac{-1}{3}$$

$$\log_e A = -\frac{1}{3} \Rightarrow A = e^{-\frac{1}{3}} \quad \therefore \lim_{x\to 0}(\frac{\tan x}{x})^{\frac{1}{x^2}} = e^{-\frac{1}{3}}$$

Example 44: Evaluate $\lim_{x\to 0}(a^x + x)^{\frac{1}{x}}$

$$Let\ A = \lim_{x\to 0}(a^x + x)^{\frac{1}{x}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A = \lim_{x\to 0}\log(a^x + x)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log(a^x + x)\ (\infty\times 0\ form)$$

$$= \lim_{x\to 0}\frac{\log(a^x + x)}{x}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{(a^x\log a + 1)/(a^x + x)}{1}$$

$$= \log a + 1 = \log a + \log e = \log ae$$

$$\therefore \log_e A = \log ea \Rightarrow A = ea \qquad Hence\ \lim_{x\to 0}(a^x + x)^{\frac{1}{x}} = ea.$$

Example 45: Evaluate $\lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}$

$$Let\ A=\lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}\ (1^{\infty}\ \ form)$$

Take log on both sides to write

$$\log_e A=\lim_{x\to a}\log(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}=\lim_{x\to a}\tan\frac{\pi x}{2a}.\log(2-\frac{x}{a})\ \ \infty\times 0\ \ form$$

$$=\lim_{x\to a}\frac{\log(2-\frac{x}{a})}{\cot\frac{\pi x}{2a}}\left(\frac{0}{0}\right)=\lim_{x\to a}\left(\frac{\frac{(-1/a)}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\cos ec^2\frac{\pi x}{2a}}\right)=\lim_{x\to a}\frac{2}{\pi}.\frac{\sin^2\frac{\pi x}{2a}}{2-\frac{x}{a}}=\frac{2}{\pi}$$

$$\therefore \log_e A=\frac{2}{\pi}\Rightarrow A=e^{\frac{2}{\pi}}\qquad Hence\ \lim_{x\to a}(2-\frac{x}{a})^{\tan\frac{\pi x}{2a}}=e^{\frac{2}{\pi}}.$$

PARTIAL DIFFERENTIATION:

Introduction: We often come across qualities which depend on two or more variables. For e.g. the area of a rectangle of length x and breadth y is given by
Area = A(x,y) = xy. The area A(x, y) is, obliviously, a function of two variables. Similarly, the distances of the point (x, y, z) from the origin in three-dimensional space is an example of a function of three variables x, y, z.

Partial derivatives: Let z = f(x, y) be a function of two variables x and y.

The first order partial derivative of z w.r.t. x, denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or z_x or f_x is defined as

$$\frac{\partial z}{\partial x}=\lim_{\delta x\to 0}\frac{f(x+\delta x,y)-f(x,y)}{\delta x}$$

From the above definition, we understand that $\left(\frac{\partial z}{\partial x}\right)$ is the ordinary derivative of z w.r.t x, treating y as constant.

The first order partial derivative of z w.r.t y, denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or z_y or f_y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

From the above definition, we understand that $\left(\frac{\partial z}{\partial y}\right)$ is the ordinary derivative of z w.r.t y, treating x as constant

The partial derivatives $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or z_{xx} or f_{xx};

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ or $\frac{\partial^2 f}{\partial y^2}$ or z_{yy} or f_{yy};

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or z_{yx} or f_{yx}

and $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or z_{xy} or f_{xy}

are known as second order Partial derivatives.

In all ordinary cases, it can be verified that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

The third and higher order partial derivatives of f(x,y) are defined in an analogous way Also, the second and higher order partial derivatives of more than two independent variables are defined similarly.

A note on rules of partial differentiation:-

All the rules of differentiation applicable to functions of a single independent variable are applicable for partial differentiation also; the only difference is that while differentiating partially w.r.t one independent variable all other independent variables are treated as constants.

Total derivatives, Differentiation of Composite and Implicit functions

In this lesson we learn the concept of total derivatives of functions of two or more variables and, also rules for differentiation of composite and implicit functions.

a) Total differential and Total derivative:-

For a function $z = f(x, y)$ of two variables, x and y the **total differential (or exact differential) dz** is defined by:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy \text{ --------------------------(1)}$$

Further, if $z = f(x, y)$ where $x = x(t)$, $y = y(t)$ i.e. x and y are themselves functions of an independent variable t, then total **derivative of z** is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \text{-------------------------(2)}$$

Similarly, the total differential of a function $u = f(x, y, z)$ is defined by

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \text{-----------------(3)}$$

Further, if $u = f(x, y, z)$ and if $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of u is given by

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \text{---------------(4)}$$

(b) <u>Differentiation of implicit functions:-</u>

An implicit function with x as an independent variable and y as the dependent variable is generally of the form $z = f(x, y) = 0$. This gives $\left(\frac{dz}{dx}\right) = \left(\frac{df}{dx}\right) = 0$.Then, by virtue of expression (2) above, we get

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\text{, and hence}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\text{, so that we get } \frac{dy}{dx} = -\frac{\left(\partial f/\partial x\right)}{\left(\partial f/\partial y\right)} \text{----------- (5)}$$

(c) <u>Differentiation of composite functions:-</u>

Let z be an function of x and y and that $x = \phi(u, v)$ and $y = \varphi(u, v)$ are functions of u and v then,

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \& \quad \frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \text{-------------------(6)}$$

Similarly, if $z = f(u, v)$ are functions of u and v and if $u = \phi(x, y)$ and $v = \varphi(x, y)$ are functions of x and y then,

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ \& \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\end{aligned}\right\} \quad \text{-------------------------(7)}$$

Note:-1) The above formulae can be extended to functions of three are more variables and formulas (6) and(7) are called **Chain rule** for partial differentiation.

2) The second and higher order partial derivatives of $z = f(x, y)$ can be obtained by repeated applications of the above formulas

Evaluate:

1. Find the total differential of

(i) $e^x (x\sin y + y\cos y)$ (ii) e^{xyz}

Sol:- (i) Let $z = f(x, y) = e^x (x\sin y + y\cos y)$ Then

$$\frac{\partial z}{\partial x} = e^x [(1+x)\sin y + y\cos y]$$

and $\frac{\partial z}{\partial y} = e^x [(1+x)\cos y - y\sin y]$ Hence, using formula (1), we get

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

i.e $dz = e^x [(1+x)\sin y + y\cos y]dx + e^x [(1+x)\cos y - y\sin y]dy$

(ii) Let $z = f(x, y, z) = e^{xyz}$ Then

$$\frac{\partial u}{\partial x} = (yz)e^{xyz}; \frac{\partial u}{\partial y} = (xz)e^{xyz}; \frac{\partial u}{\partial z} = (xy)e^{xyz}$$

$\therefore$ Total differential of $z = f(x, y, z)$ is (see formula (3) above)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= e^{xyz}(yzdx + zxdy + xydz)$$

2. Find $\frac{dz}{dt}$ if

(i) $z = xy^2 + x^2 y$,where $x = at^2, y = 2at$

(ii) $u = \tan^{-1}\left(y/x\right)$,where $x = e^t - e^{-t}, y = e^t + e^{-t}$ **(VTU-Jan 2003)**

Sol:- (i) Consider $z = xy^2 + x^2 y$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \quad \& \frac{\partial z}{\partial y} = 2xy + x^2$$

Since $x = at^2$ & $y = 2at$, We have $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

Hence, using formula (2), we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

$$= (y^2 + 2xy)y + 2a(2xy + x^2), \text{ Using } y = 2at$$

$$\frac{dz}{dt} = y^3 + 2xy^2 + 4axy + 2ax^2$$

To get $\left(\frac{dz}{dt}\right)$ explicitly in terms of t, we substitute

$x = at^2$ & $y = 2at$, to get

$$\left(\frac{dz}{dt}\right) = 2a^3(8t^3 + 5t^4)$$

(ii) Consider

$$u = \tan^{-1}\left(y/x\right)$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2 + y^2}, \frac{\partial u}{\partial y} = \frac{x}{x^2 + y^2}$$

Since $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$, we have

$$\frac{dx}{dt} = e^t + e^{-t} = y \quad \frac{dy}{dt} = e^t - e^{-t} = x$$

Hence $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ (see eqn (2))

$$= \left(\frac{-y}{x^2 + y^2}\right)y + \left(\frac{x}{x^2 + y^2}\right)x = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

Substituting $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$, we get

$$\frac{du}{dt} = \frac{-2}{e^{2t} + e^{-2t}}$$

3. Find $\left(\frac{dy}{dx}\right)$ if (i) $x^y + y^x$ =Constant

(ii) $x + e^y = 2xy$

Sol: - (i) Let $z = f(x, y) = x^y + y^x$ =Constant. Using formula (5)

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \text{-----------------(*)}$$

But $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$ and $\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$ Putting those in(*), we get

$$\frac{dy}{dx} = -\left\{\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right\}$$

(ii) Let $z = f(x, y) = e^x + e^y - 2xy$ =Constant

Now, $\frac{\partial f}{\partial x} = e^x - 2y$; $\frac{\partial f}{\partial y} = e^y - 2x$ Using this in (8),

$$\frac{dy}{dx} = -\left\{\frac{\partial f / \partial x}{\partial f / \partial y}\right\} = -\left\{\frac{e^x - 2y}{e^y - 2x}\right\}$$

4. (i) If $z = f(x, y)$,where $x = r\cos\theta$, $y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$ **(VTU July-2005)**

(ii) If $z = f(x, y)$,where $x = e^u + e^{-v}$ & $y = e^{-u} - e^v$,Show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Sol: As $x = r\cos\theta$ and $y = r\sin\theta$, we have

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$; $\frac{\partial y}{\partial r} = \sin\theta$ & $\frac{\partial y}{\partial \theta} = r\cos\theta$.

Using Chain rule (6) & (7) we have

$$\left(\frac{\partial z}{\partial r}\right) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(\cos\theta) + \frac{\partial z}{\partial y}(\sin\theta)$$

$$\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

Squaring on both sides, the above equations, we get

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta - 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

Adding the above equations , we get

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}(\cos^2\theta+\sin^2\theta)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 \text{ as desired.}$$

(ii) As $x=e^u+e^{-v}$ & $y=e^{-u}-e^v$, We have

$$\frac{\partial x}{\partial u}=e^u,\ \frac{\partial x}{\partial v}=-e^{-v},\ \frac{\partial y}{\partial u}=-e^{-u} \ \& \ \frac{\partial y}{\partial v}=-e^v$$

Using Chain rule (6) we get

$$\left(\frac{\partial z}{\partial u}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-u})$$

$$\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^v)$$

$$\therefore\ \left(\frac{\partial z}{\partial u}\right)-\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}(e^u+e^{-v})+\frac{\partial z}{\partial y}(e^{-u}-e^v)$$

$$=\frac{\partial z}{\partial x}x-\frac{\partial z}{\partial y}y$$

5. (i) If $u=f(x,z,y/z)$ Then show that $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=0$ **(VTU-July-2004)**

(ii) If $H=f(x-y,y-z,z-x)$, show that

$$\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0$$ **(VTU-July-2003)**

Sol: - (i) Let $u=f(v,w)$, where $v=xz$ and $w=y/z$

$$\frac{\partial v}{\partial x}=z,\ \frac{\partial v}{\partial y}=0,\ \frac{\partial v}{\partial z}=x \quad \& \quad \frac{\partial w}{\partial x}=0,\ \frac{\partial w}{\partial y}=1/z,\ \frac{\partial w}{\partial z}=-y/z^2$$

Using Chain rule,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial u}{\partial v}(z)+\frac{\partial u}{\partial w}(0)=z\frac{\partial u}{\partial v}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial u}{\partial v}(0)+\frac{\partial u}{\partial w}(1/z)=\frac{1}{z}\frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial u}{\partial v}(x)+\frac{\partial u}{\partial w}\left(-y/z^2\right)=x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}$$

From these, we get

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=xz\frac{\partial u}{\partial v}-\frac{y}{z}\frac{\partial u}{\partial w}-z\left(x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}\right)$$

$$=0$$

(ii) Let $H=f(u,v,w)$ Where $u=x-y, v=y-z, w=z-x$

Now, $\frac{\partial u}{\partial x}=1, \frac{\partial u}{\partial y}=-1, \frac{\partial u}{\partial z}=0$

$\frac{\partial v}{\partial x}=0, \frac{\partial v}{\partial y}=1, \frac{\partial v}{\partial z}=-1$

$\frac{\partial w}{\partial x}=-1, \frac{\partial w}{\partial y}=0, \frac{\partial u}{\partial z}=1$ Using Chain rule,

$$\frac{\partial H}{\partial x}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial H}{\partial u}(1)+\frac{\partial H}{\partial v}(0)+\frac{\partial H}{\partial w}(-1)$$

$$\frac{\partial H}{\partial y}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial H}{\partial u}(-1)+\frac{\partial H}{\partial v}(1)+\frac{\partial v}{\partial w}(0)$$

$$\frac{\partial H}{\partial z}=\frac{\partial H}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial H}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial H}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial H}{\partial u}(0)+\frac{\partial H}{\partial v}(-1)+\frac{\partial H}{\partial w}(1)$$

Adding the above equations, we get the required result $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=0$

Applications to Jacobians:

In this lesson, we study Jacobians, errors and approximations using the concept of partial differentiation.

Jacobians:-

Jacobians were invented by German mathematician C.G. Jacob Jacobi (1804-1851),who made significant contributions to mechanics, Partial differential equations and calculus of variations.

Definition:- Let u and v are functions of x and y, then Jacobian of u and v w.r.t x and denoted by

J or $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

is defined by

$$J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

Similarly, if u, v, w are functions of three independent variables of x, y, z, then

$$J = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

<u>Remark:-</u> In a similar way, Jacobian of n functions in n-variables can be defined

Note:- (i) If $J = \frac{\partial(u,v)}{\partial(x,y)}$, then the "inverse Jacobian" of the Jacobian J,

denoted by J',is defined as

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

(ii) Similarly, "inverse Jacobian" of $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ is defined as $J' = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

<u>Properties of Jacobians</u> :-

Property 1:- If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

Proof:- Consider

$$JJ^1 = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 1$$

<u>Property 2:-</u> (Chain rule for Jacobians):- If u and v are functions of r&s and r,s are functions x&y,then

$$J = \left(\frac{u,v}{x,y}\right) = J\left(\frac{u,v}{r,s}\right) \times J\left(\frac{r,s}{x,y}\right)$$

Proof:- Consider

$$J\left(\frac{u,v}{r,s}\right)\times J\left(\frac{r,s}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial s}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s}\end{vmatrix}\times\begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}\\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial s}\frac{\partial s}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=J\left(\frac{u,v}{x,y}\right)$$

Jacobians in various co-ordinate systems:-

1. In Polar co-ordinates, $x=r\cos\theta$, $y=r\sin\theta$
   we have $\frac{\partial(u,v)}{\partial(r,\theta)}=r$

2. In spherical coordinates, $x=\rho\cos\phi, y=\rho\sin\phi, z=z$, we have
$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}=\rho$

3. In spherical polar co-ordinates, $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$

Proof of 1:- we have, $\frac{\partial x}{\partial r}=\cos\theta$ and $\frac{\partial y}{\partial r}=\sin\theta$

$\frac{\partial x}{\partial\theta}=-r\sin\theta$ and $\frac{\partial y}{\partial\theta}=r\cos\theta$

$$\therefore\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$$

Proof of 2 :- we have $\frac{\partial x}{\partial\rho}=\cos\phi$, $\frac{\partial y}{\partial\rho}=\sin\phi$, $\frac{\partial z}{\partial\rho}=0$

$\frac{\partial x}{\partial\phi}=-\rho\sin\phi$, $\frac{\partial y}{\partial\phi}=\rho\cos\phi$, $\frac{\partial z}{\partial\phi}=0$

$$\frac{\partial z}{\partial \rho} = 0, \frac{\partial z}{\partial \phi} = 0, \frac{\partial z}{\partial z} = 0$$

$$\therefore \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

<u>Proof of 3:-</u> We have

$$\frac{\partial x}{\partial r} = \sin\theta\cos\theta, \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta, \frac{\partial z}{\partial \theta} = -r\sin\theta, \frac{\partial z}{\partial \phi} = 0$$

$$\therefore \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\theta & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

<u>Evaluate</u>

1. If $u = x^2 - 2y^2, v = 2x^2 - y^2$, where $x = r\cos\theta, y = r\sin\theta$ show that

$$\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3\sin 2\theta$$ **(VTU-Jan-2006)**

Consider $u = x^2 - 2y^2 = r^2\cos^2\theta - 2r^2\sin^2\theta$

$$v = 2x^2 - y^2 = 2r^2\cos^2\theta - r^2\sin^2\theta$$

$$\therefore \frac{\partial u}{\partial r} = 2r\cos^2\theta - 4r\sin^2\theta, \frac{\partial v}{\partial r} = 4r\cos^2\theta - 2r\sin^2\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta$$

$$\frac{\partial v}{\partial \theta} = -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta$$

$$\frac{\partial(u, v)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos^2\theta - 4r\sin^2\theta & -2r^2\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta \\ 4r\cos^2\theta - 2r\sin^2\theta & -4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta \end{vmatrix}$$

$$= \left(2r\cos^2\theta - 4r\sin^2\theta\right)\left(4r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta\right)$$
$$= \left(-2r\cos\theta\sin\theta - 4r^2\sin\theta\cos\theta\right)\left(4r\cos^2\theta - 2r\sin^2\theta\right)$$
$$= 6r^3\sin 2\theta$$

2. If $x = u(1-v),\ y = uv,$ Prove that $J\left(\frac{x,y}{u,v}\right)\times J^1\left(\frac{x,y}{u,v}\right) = 1$ **(VTU-2001)**

Consider $\frac{\partial x}{\partial u} = 1-v,\ \frac{\partial x}{\partial v} = -u$

$\frac{\partial y}{\partial u} = v,\ \frac{\partial y}{\partial v} = u$

$$\therefore J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u - (-uv) = u - uv + uv = u$$

$$\therefore J\left(\frac{x,y}{u,v}\right) = u \quad ---(1)$$

Further, as $x = u(1-v),\ y = uv,$

$= u - uv$

We write, $x = u - y \quad \therefore u = x + y$ and

$v = \frac{y}{u} = \left(\frac{y}{x+y}\right) \qquad \therefore v = \left(\frac{y}{x+y}\right)$

$\therefore \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 1$ and

$\frac{\partial v}{\partial x} = -\frac{y}{(x+y)^2},\ \frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$

$$\therefore J^1\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \left(\frac{1}{x+y}\right) = \frac{1}{u} \qquad \therefore JJ^1 = u\frac{1}{u} = 1$$

3. If $x = e^u\cos v,\ y = e^u\sin v,$ Prove that

$$\frac{\partial(x,y)}{\partial(u,v)}\times\frac{\partial(u,v)}{\partial(x,y)} = 1$$

Consider $x = e^u\cos v \qquad y = e^u\sin v$

$$\frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial x}{\partial v} = -e^u \sin v \qquad \frac{\partial y}{\partial v} = e^u \cos v$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$i.e \frac{\partial(x, y)}{\partial(u, v)} = e^{2u} \text{ -----}(1)$$

Again Consider $x = e^u \cos v,\ y = e^u \sin v,$

$$\therefore x^2 + y^2 = e^{2u} \quad \text{or} \quad u = \frac{1}{2}\log(x^2 + y^2)$$

$$\& \ \frac{y}{x} = \tan v \quad \text{or} \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Hence $\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$,; $\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$,;

$$\& \ \frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2};\ \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} = \frac{1}{x^2 + y^2} = (x^2 + y^2)^{-1}$$

$$\text{i.e } \frac{\partial(u, v)}{\partial(x, y)} = e^{-2u} \text{ -------}(2)$$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = e^{2u} \times e^{-2u} = 1$$

4. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, Show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

$$\text{Now, } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

i.e $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \left(-\frac{yz}{x^2}\right)\left\{\left(\frac{-zx}{y^2}\right)\left(\frac{-xy}{z^2}\right)\right\} - \left(\frac{z}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{y}{z}\right)\left(\frac{x}{y}\right)\right\}$$
$$+\left(\frac{y}{x}\right)\left\{\left(\frac{z}{y}\right)\left(\frac{x}{z}\right) - \left(\frac{y}{z}\right)\left(\frac{-zx}{y^2}\right)\right\}$$

= 4, as desired.

5. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Now, by definition

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

i.e $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,\left(0 - (r^2\sin^2\theta\cos\phi)\right)$$
$$- r\cos\theta\cos\phi\,\left(0 - (r\sin\theta\cos\theta\cos\phi)\right)$$
$$- r\sin\theta\sin\phi\,\left(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right)$$
$$= (r^2\sin^2\theta)\sin\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi$$
$$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta)\cos^2\phi + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta\,(\cos^2\phi + \sin^2\phi)$$
$$= r^2\sin\theta\text{, as required}$$

MODULE III

VECTOR CALCULUS

CONTENTS:

Introduction:

Basically vector is a quantity having both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of reference in physical and engineering problems. We are familiar with vector algebra which gives an exposure to all the basic concepts related to vectors.

Differentiation and Integration are well acquainted topics in calculus. In the background of all these we discuss this chapter vector calculus comprising vector Differentiation. Many concepts are highly significant in various branches of engineering.

Basic Concepts – Vector function of a single variable and the derivative of a vector

Let the position vector of a point p(x, y, z) in space be

$$\vec{r} = xi + yi + zk$$

If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a vector point function usually denoted as $\vec{r} = \vec{r}$ (t). As the parameter t varies, the point P traces in space. Therefore

$\vec{r}$ = x(t) i+y (t) j+z(t) k

is called as the vector equation of the curve.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

Is a vector along the tangent to the curve at P.

If t is the time variable,

$\vec{\upsilon} = \frac{d\vec{r}}{dt}$ gives the velocity of the particle at time t.

Further $\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{\upsilon}$ and is called the acceleration of the particle at time t.

1. $\frac{d}{dt}\; c_1\vec{r}_1(t) \pm c_2\vec{r}_2(t) \; = c_1\vec{r}_1{}'(t) \pm c_2\vec{r}_2(t)$ where c_1 , c_2 are constants.

2. $\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = \vec{F}.\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}.\vec{G}$

3. $\frac{d}{dt}\left(\vec{F}\times\vec{G}\right)=\vec{F}.\frac{d\vec{G}}{dt}+\frac{d\vec{F}}{dt}\times\vec{G}$

$Where \quad \vec{F}=\vec{F}(t) \ and \ \vec{G}=\vec{G}(t).$

Gradient, Divergence, Curl and Laplacian:

If ϕ is scalar function $\vec{A}$ is vector function $\vec{A}=a_1 i+a_2 j+a_3 k$ then

1. ie., grad $\phi=\nabla\phi=\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k$

2. If $\vec{A}=a_1 i+a_2 j+a_3 k,$

$$div\,\vec{A}=\nabla.\vec{A}=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right).\ a_1 i+a_2 j+a_3 k$$

$$div\,\vec{A}=\nabla.\vec{A}=\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}+\frac{\partial a_3}{\partial z}$$

3. If $\vec{A}=a_1 i+a_2 j+a_3 k,$

$$div\,\vec{A}=\nabla\times\vec{A}=\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

$$=i\left(\frac{\partial a_3}{\partial y}-\frac{\partial a_2}{\partial z}\right)-j\left(\frac{\partial a_3}{\partial x}-\frac{\partial a_1}{\partial z}\right)+k\left(\frac{\partial a_2}{\partial x}-\frac{\partial a_1}{\partial y}\right)$$

4. Laplacian of $\phi=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

5. Laplacian of $\vec{A}=\nabla^2\vec{A}=\frac{\partial^2\vec{A}}{\partial x^2}+\frac{\partial^2\vec{A}}{\partial y^2}+\frac{\partial^2\vec{A}}{\partial z^2}$

Important points:

1. If $\vec{r}=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}, then\, \frac{d\vec{r}}{dt}$ is velocity and $\frac{d^2\vec{r}}{dt^2}$ is acceleration.

2. The unit tangent vector $\hat{T} = \dfrac{\left(\dfrac{d\vec{r}}{dt}\right)}{\left|\dfrac{d\vec{r}}{dt}\right|}$ and unit normal vector is $\hat{n} = \dfrac{\dfrac{d\vec{T}}{ds}}{|\hat{T}|}$ where

$\hat{T} = \dfrac{ds}{dt}$.

3. If $\vec{A}$ and $\vec{B}$ are any two vector and θ is angle between two vectors, then

$$\theta = \cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{|\vec{A}|\,.\,|\vec{B}|}\right).$$

4. Component of a vector (velocity or acceleration) $\vec{F}$ along a given vector $\vec{C}$ is the resolved part of $\vec{F}$ given by $\vec{F}\,.\,\hat{n}$ where $\hat{n} = \dfrac{\vec{c}}{|\vec{c}|}$.

5. Component of a vector $\vec{F}$ along normal to the $\vec{c}$ is given by

$$\left|\vec{F} - \text{resolved part of acceleration along } \vec{c}\right| = \left|F - \left(\vec{F}\,.\,\vec{c}\right).\hat{c}\right|.$$

1. Find the unit tangent vector to the curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$.

Soln: Given the space curve $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$

$$\therefore \vec{T} = \frac{d\vec{r}}{dt} = -\sin t\hat{i} + \cos t\hat{j} + \hat{k}$$

$$|\vec{T}| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

Therefore, the unit tangent vector to the given curve at any point is

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{-\sin t\hat{i} + \cos t\hat{j} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\ -\sin t\hat{i} + \cos t\hat{j} + \hat{k}$$

2. Find the unit normal vector to the curve $\vec{r} = 4\sin t\hat{i} + 4\cos t\hat{j} + 3t\hat{k}$.

Soln: Given $\vec{r} = 4\sin t\hat{i} + 4\cos t\hat{j} + 3t\hat{k}$

$$\therefore \vec{T} = \frac{d\vec{r}}{dt} = 4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}$$

$$\left|\vec{T}\right| = \sqrt{16\ \cos^2 t + \sin^2 t\ \ + 9} = \sqrt{25} = 5$$

Therefore, the unit tangent vector to the given curve at any point t is

$$\hat{T} = \frac{\vec{T}}{\left|\vec{T}\right|} = \frac{4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}}{5} = \frac{1}{5}\ 4\cos t\hat{i} - 4\sin t\hat{j} + 3\hat{k}$$

$$\frac{d\hat{T}}{ds} = \left(\frac{\frac{d\hat{T}}{dt}}{\frac{ds}{dt}}\right) = \left(\frac{\frac{d\hat{T}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}\right) = \frac{1}{5}\left[\frac{1}{5}\ -4\sin t\hat{i} - 4\cos t\hat{j}\ \right]$$

$$\frac{d\hat{T}}{ds} = -\frac{4}{25}\ \sin t\hat{i} + \cos t\hat{j}$$

$$and\ \left|\frac{d\hat{T}}{ds}\right| = \sqrt{\left(\frac{4}{25}\right)^2\ \sin^2 t + \cos^2 t\ } = \frac{4}{25}$$

The unit normal vector to the given curve is

$$\hat{n} = \frac{\left(\frac{d\hat{T}}{ds}\right)}{\left|\frac{d\hat{T}}{ds}\right|} = \frac{(-4/25)\ \sin t\hat{i} + \cos t\hat{j}}{(4/25)} = -\sin t\hat{i} - \cos t\hat{j}$$

3. Find the angle between tangents to the curve $x = t^2, y = t^3, z = t^4$ at t=2 and t=3.

Soln: Define the position vector $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

$$\vec{r} = r^2\hat{i} + t^3\hat{j} + t^4\hat{k}$$

i.e.,
$$\vec{T} = \frac{d\vec{r}}{dt} = 2t\hat{i} + 3t^2\hat{j} + 4t^3\hat{k}$$

$$\vec{A} = \vec{T}\,\Big|_{t=2} = 4\hat{i} + 12\hat{j} + 32\hat{k} = 4(\hat{i} + 3\hat{j} + 8\hat{k})$$

$$\therefore \left|\vec{A}\right| = \sqrt{16(1+9+64)} = 4\sqrt{74}$$

$$\vec{B} = \vec{T}\big|_{t=3} = 6\hat{i} + 27\hat{j} + 108\hat{k} = 3(2\hat{i} + 9\hat{j} + 36\hat{k})$$

$$\therefore \left|\vec{A}\right| = \sqrt{9(4+81+1296)} = 3\sqrt{1381}$$

Let θ be the angle between two vectors $\vec{A}$ and $\vec{B}$, then

$$\theta = \cos^{-1}\left(\frac{\vec{A}\,.\,\vec{B}}{\left|\vec{A}\right|.\left|\vec{B}\right|}\right) = \cos^{-1}\left(\frac{4\ \hat{i}+3\hat{j}+8\hat{k}\ .3\ 2\hat{i}+9\hat{j}+36\hat{k}}{4\sqrt{74}\times 3\sqrt{1381}}\right).$$

$$= \cos^{-1}\left(\frac{2+27+248}{\sqrt{74}\times\sqrt{1381}}\right) = \cos^{-1}\ 0.8665$$

$$\theta = \ 30^\circ$$

4. A particle moves along the curve $x = 1 - t^3, y = 1 + t^2, z = 2t - 5.$ Determine its velocity and acceleration. Find the component of velocity and acceleration at t=1 in the direction $2\hat{i} + \hat{j} + 2\hat{k}$

Soln: Given the position vector

$$\vec{r} = (1-t^3)\hat{i} + (1+t^2)\hat{j} + (2t-5)\hat{k}$$

$$the\, velocity,\ \vec{v} = \frac{d\vec{r}}{dt} = (-3t^2)\hat{i} + (2t)\hat{j} + (2)\hat{k}$$

$$and\ \ acceleration\ \vec{a} = \frac{d^2\vec{r}}{dt^2} = (-6t^3)\hat{i} + 2\hat{j}$$

$$at\, t = 1, \qquad \vec{v} = -3\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{a} = -6\hat{i} + 2\hat{j}$$

Therefore, the component of velocity vector in the given direction

$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}.$

$$\vec{v}.\hat{n} = (-3\hat{i} + 2\hat{j} + 2\hat{k}).\frac{(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{4+1+4}} = \frac{-6+2+4}{3} = 0$$

The normal component of acceleration in the given direction

$$\vec{a}.\hat{n} = (-6\hat{i} + 2\hat{j}).\frac{(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{4+1+4}} = \frac{-12+2}{3} = -\frac{10}{3}.$$

5. A particle moves on the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5,$ where t is the time. Find the components of velocity and acceleration at time t=1 in the direction $\hat{i} + 3\hat{j} + 2\hat{k}.$

Soln: The position vector at any point (x,y,z) is given $\vec{r} = xi + yj + zk,$ but

$\vec{r} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

Therefore, the velocity and acceleration are

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j}$$

$$at\ t = 1, \quad \vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

Therefore the component of velocity in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{v}.\hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{16}{\sqrt{14}}$$

Since dot product of two vector is a scalar.

The components of acceleration in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\vec{a}.\hat{n} = (4\hat{i} + 2\hat{j}).\frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}} = \frac{-2}{\sqrt{14}}$$

Unit Normal Vector:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{where } \hat{n} = \nabla\phi = \text{Normal vector}$$

Directional Derivative (D.D.):

If $\vec{a}$ is any vector and ϕ is any scalar point function then Directional Derivative

(D.D.)= $\nabla\phi.\vec{a} = \dfrac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Maximum directional derivative (Normal derivative)

The directional derivative will be the maximum in the direction $\nabla\phi (i.e., \vec{a} = \nabla\phi)$

and the maximum value of directional derivative = $\nabla\phi. = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{|\nabla\phi|^2}{|\nabla\phi|} = |\nabla\phi|.$

Note : The maximum directional derivative is also called normal derivative i.e., Normal derivative $= |\nabla\phi|$

Equations of Tangent plane and Normal Line:

Let $\phi(x, y, z) = c$ be any given surface and (x_1, y_1, z_1) be a point on it, then

(i) Equation of tangent plane to $\phi(x, y, z) = c$ at $P(x_1, y_1, z_1)$ is

$$(x - x_1)\left(\frac{\partial\phi}{\partial x}\right)_p + (y - y_1)\left(\frac{\partial\phi}{\partial y}\right)_p + (z - z_1)\left(\frac{\partial\phi}{\partial z}\right)_p = 0$$

(ii) Equation of normal line to $\phi(x, y, z) = c$ at $P(x_1, y_1, z_1)$ is

$$\frac{x - x_1}{\left(\frac{\partial\phi}{\partial x}\right)_p} = \frac{y - y_1}{\left(\frac{\partial\phi}{\partial y}\right)_p} = \frac{z - z_1}{\left(\frac{\partial\phi}{\partial z}\right)_p}$$

Ex. 1: If $\phi = x^3y^3z^3$, find $\nabla\phi$ at (1,2,1) along $\hat{i}+2\hat{j}+2\hat{k}$

Soln: Given $\phi = x^3y^3z^3$, then

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= (3x^2y^3z^3)\hat{i} + (3x^3y^2z^3)\hat{j} + (3x^3y^3z^2)\hat{k}$$

$$\nabla\phi\big|_{(1,2,1)} = 24\hat{i} + 12\hat{j} + 24\hat{k}$$

$$Let\ \vec{a} = \hat{i} + 2\hat{j} + 2\hat{k} \Rightarrow |\vec{a}|$$

$$= \sqrt{1+4+4} = 3$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$\nabla\phi$ at (1,2,1) along the vectors $\vec{a}$ is

$$\nabla\phi.\vec{a} = 24\hat{i} + 12\hat{j} + 24\hat{k} \ . \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

$$= \frac{1}{3}(24 + 24 + 48) = \frac{96}{3}$$

$$\nabla\phi.\vec{a} = 32$$

Ex. 2: Find the unit normal vector to the surface $xy^3z^2 = 4$ at the point (-1,-1,2).

$$\phi = xy^3z^2$$

Soln: Let $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k} = y^3z^2)\hat{i}+3x^2z^2\hat{j}+2xy^3z\hat{k}$

$$and\ \nabla\phi\ |_{(-1,-1,2)} = -4\hat{i}-12\hat{j}+4\hat{k}$$

Therefore, the normal vector to the given surface is

$$\vec{n}\ \nabla\phi\ |_{(-1,-1,2)} = -4\hat{i}-12\hat{j}+4\hat{k}$$

$$\left|\vec{n}\right| = \sqrt{16+144+16} = \sqrt{176} = 4\sqrt{11}$$

$$\therefore \vec{n} = \frac{\vec{n}}{\left|\vec{n}\right|} = \frac{1}{\sqrt{11}}\ \ -\hat{i}-3\hat{j}+\hat{k}$$

Ex. 3: Find the angle between the normals to the surface $xy = z^2$ at the points (1, 4, 2) and (-3,-3, 3).

Soln: Let $\phi\ \ x, y, z\ \ = xy - z^2$ be the surface

The normal to the surface is

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy\text{-}z^2)+\hat{j}\frac{\partial}{\partial y}(xy\text{-}z^2)+\hat{k}\frac{\partial}{\partial z}(xy\text{-}z^2)$$

$$\nabla\phi = y\hat{i} - x\hat{j} + 2z\hat{k}$$

$$\nabla\phi\ |_{(1,4,2)} = 4\hat{i}+\hat{j}\text{-}4\hat{k}$$

$$\nabla\phi\ |_{(-3,-3,3)} = -3\hat{i}\text{-}3\hat{j}+6\hat{k}$$

Are normals to the surface at (1, 4, 2) and (-3,-3,3).

Let θ be the angle between the normals

$$\therefore \cos\theta = \frac{\vec{n_1}\ .\vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|} = \frac{4\hat{i}+\hat{j}\text{-}4\hat{k}\ \ .\ \ \text{-}3\hat{i}\text{-}3\hat{j}+6\hat{k}}{\sqrt{16+1+16}\ \ \sqrt{9+9+36}}$$

$$= \frac{-12-3-24}{\sqrt{33}\sqrt{54}}$$

$$= \frac{-39}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(-\frac{-39}{\sqrt{33}\sqrt{54}}\right)$$

Ex.4: Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (-1,-2,-1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}.$

Soln: Given

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz+4xz^2)\hat{i} + \frac{\partial}{\partial y}(x^2yz+4xz^2)\hat{j} + \frac{\partial}{\partial z}(x^2yz+4xz^2)\hat{k}$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y+8xz)\hat{k}$$

$$\nabla\phi\big|_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

The directional derivative of ϕ at the point (1,-2,-1) in the direction of vector $2\hat{i} - \hat{j} - 2\hat{k}.$ is

$$= (8\hat{i} - \hat{j} - 10\hat{k}).\left(\frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}\right) = \frac{1}{3}\ 16+1+20\ = \frac{37}{3}$$

Ex.5: Find the directional derivative of $\phi = xy^2 + yz^3$ at the point (1,-2,-1) in the direction of the normal to the surface x log $z - y^2 = -4 at (-1,2,1).$

Soln: Given

$$\phi = xy^2 + yz^3$$

$$\therefore \nabla\phi = y^2\hat{i} + \ 2xy + z^3\ \hat{j} + \ 3yz^2\ \hat{k}$$

$$\therefore \nabla\phi\big|_{(1,-2,-1)} = 4\hat{i} - 5\hat{j} - 5\hat{k}$$

$$Let\ \phi_1 = x\log z - y^2$$

Therefore the normal vector to the surface x log $z - y^2 = -4\ is\ \nabla\phi_1$

$$\therefore \nabla\phi_1 = \log z\hat{i} - +(-2y)\hat{j} + \left(\frac{x}{z}\right)\hat{k}$$

and $\vec{n} = \nabla\phi_1\big|_{(-1,2,1)} = \log(1)\hat{i} - (2\times 2)\hat{j} + \left(-\frac{1}{1}\right)\hat{k} = -4\hat{j} - \hat{k}$

$$\left|\vec{n}\right| = \sqrt{16+1}\ = \sqrt{17}$$

$$\hat{n} = \frac{\vec{n}}{\left|\vec{n}\right|} = \frac{1}{\sqrt{17}}\ -4\hat{j} - \hat{k}$$

Therefore, the directional derivative of the surface $\phi = xy^2 + yz^3$ at (1,-2,-1) in the direction normal to the surface x log $z - y^2 = -4 at (-1,2,1).$ is

$$\therefore \nabla\phi.\hat{n} = \ 4\hat{i} - 5\hat{j} - 5\hat{k}\ .\frac{1}{\sqrt{17}}\ -4\hat{j} - \hat{k}\ = \frac{1}{\sqrt{17}}\ 20+5\ = \frac{25}{\sqrt{17}}.$$

Ex.6: Find the equation of tangent plane and normal to line to the surface x log $z - y^2 = -4$at the point $(-1,2,1)$.

Soln: Let $\phi = x\log z - y^2$.

$$\left.\frac{\partial \phi}{\partial x}\right| = \log z \Rightarrow \left.\frac{\partial \phi}{\partial x}\right|_{(-1,2,1)} = 0$$

Therefore $$\left.\frac{\partial \phi}{\partial y}\right| = -2y \Rightarrow \left.\frac{\partial \phi}{\partial y}\right|_{(-1,2,1)} = -4$$

$$\left.\frac{\partial \phi}{\partial z}\right| = \frac{x}{z} \Rightarrow \left.\frac{\partial \phi}{\partial z}\right|_{(-1,2,1)} = -1$$

Thertefore, equation of the tangent plane is

$$(x-x_1)\left(\frac{\partial \phi}{\partial x}\right)_p + (y-y_1)\left(\frac{\partial \phi}{\partial y}\right)_p + (z-z_1)\left(\frac{\partial \phi}{\partial z}\right)_p = 0$$

$$Hence\,(x_1, y_1, z_1) = (-1,2,1),\ then$$

$$(x+1)0 + (y-2)(-4) + (z-1)(-1) = 0$$

$$-4y + 8 - z + 1 = 0$$

$$4y + z + 9 = 0$$

And the equation of the normal line is

$$\frac{x-x_1}{\left(\frac{\partial \phi}{\partial x}\right)} = \frac{y-y_1}{\left(\frac{\partial \phi}{\partial y}\right)} = \frac{z-z_1}{\left(\frac{\partial \phi}{\partial z}\right)} \Rightarrow \frac{x+1}{0} = \frac{y-2}{-4} = \frac{z-1}{-1}$$

Properties of divergence:

1.Prove that $div\left(\vec{A} + \vec{B}\right) = div\,\vec{A} + div\,\vec{B}$

$Or\ \nabla.\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$

1. Proof: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$$\therefore \left(\vec{A} + \vec{B}\right) = \left(A_1 + B_1\right)\hat{i} + \left(A_2 + B_2\right)\hat{j} + \left(A_3 + B_3\right)\hat{k}$$

$$\nabla.\left(\vec{A} + \vec{B}\right) = \frac{\partial}{\partial x}\left(A_1 + B_1\right) + \frac{\partial}{\partial y}\left(A_2 + B_2\right) + \frac{\partial}{\partial z}\left(A_3 + B_3\right)$$

$$= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$\nabla.\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$$

2. **Prove that**

$$div\left(\phi\vec{A}\right)=\left(grad\phi\right).\vec{A}+\phi\left(div\vec{A}\right)$$

$$Or\ \nabla.\left(\phi\vec{A}\right)=\left(\nabla\phi\right).\vec{A}+\phi\left(\nabla.\vec{A}\right)$$

Proof: Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$

$$then\ ,\ \phi\vec{A}=\phi A_1\hat{i}+\phi A_2\hat{j}+\phi A_3\hat{k}$$

$$\nabla.\left(\phi\vec{A}\right)=\frac{\partial}{\partial x}\left(\phi A_1\right)+\frac{\partial}{\partial y}\left(\phi A_2\right)+\frac{\partial}{\partial z}\left(\phi A_3\right)$$ by the property,

$$=\phi\frac{\partial A_1}{\partial x}+A_1\frac{\partial\phi}{\partial x}+\phi\frac{\partial A_2}{\partial y}+A_2\frac{\partial\phi}{\partial y}+\phi\frac{\partial A_3}{\partial z}+A_3\frac{\partial\phi}{\partial z}$$

$$=\phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)+A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}$$

$$=\phi\left(\nabla.\vec{A}\right)+\left(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}\right).\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial y}\hat{k}\right)$$

$$\nabla.\left(\phi\vec{A}\right)=\left(\nabla\phi\right).\vec{A}+\phi\left(\nabla.\vec{A}\right)$$

<u>Properties of curl</u> :

1. Prove that curl:

$$\left(\vec{A_1}+\vec{B_1}\right)=curl\ \vec{A}+curl\ \vec{B}$$

$$Or\ \nabla\times\left(\vec{A_1}+\vec{B_1}\right)=\nabla\times\vec{A}+\nabla\times\vec{B}$$

Proof: Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k},\quad \vec{B}=B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$

$$Therefore,\vec{A}+\vec{B}=\left(A_1+B_1\right)\hat{i}+\left(A_2+B_2\right)\hat{j}+\left(A_3+B_3\right)\hat{k}$$

$$\therefore\nabla\times\left(\vec{A}+\vec{B}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\A_1+B_1&A_2+B_2&A_3+B_3\end{vmatrix}$$

$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\A_1&A_2&A_3\end{vmatrix}+\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\B_1&B_2&B_3\end{vmatrix}$$

$$\therefore\nabla\times\left(\vec{A_1}+\vec{B_1}\right)=\nabla\times\vec{A}+\nabla\times\vec{B}$$

2. If $\vec{A}$ is a vector function and ϕ is a scalar function then

$$curl\left(\phi\,\vec{A}\right)=\phi\left(curl\,\vec{A}\right)+grad\phi\times\vec{A}$$

$$Or\ \nabla\times\left(\phi\,\vec{A}\right)=\phi\left(\nabla\times\vec{A}\right)+\nabla\phi\times\vec{A}$$

Proof: Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$

$$then\ ,\ \ \phi\,\vec{A}=\phi A_1\hat{i}+\phi A_2\hat{j}+\phi A_3\hat{k}$$

$$\nabla\times\phi\,\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \phi A_1 & \phi A_2 & \phi A_3\end{vmatrix}=\sum\left[\frac{\partial}{\partial y}\ \phi A_3\ -\frac{\partial}{\partial z}\ \phi A_3\ \right]\hat{i}$$

$$=\sum\left[\phi\frac{\partial A_3}{\partial y}+A_3\frac{\partial\phi}{\partial y}-\phi\frac{\partial A_2}{\partial z}-A_2\frac{\partial\phi}{\partial z}\right]\hat{i}$$

$$=\sum\left[\phi\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+A_3\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial z}\right]\hat{i}$$

$$=\phi\sum\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\sum\left(A_3\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial z}\right)\hat{i}$$

$$=\phi\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$\nabla\times\phi\,\vec{A}=\phi\left(\nabla\times\vec{A}\right)+\nabla\phi\times\vec{A}$$

<u>Laplacian:</u> The Laplacian operator ∇^2 is defined by

$$\nabla\cdot\nabla=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

<u>Irrotational Vector (or Conservative Force Field):</u> A vector field $\vec{F}$ is said to be irrotational vector or conservative force field or curl free vector if $\nabla\times\vec{F}=0$ or curl $\vec{F}=0$.

<u>Scalar Potential</u>: A vector field $\vec{F}$ which can be derived from the scalar field ϕ such that $\vec{F}=\nabla\phi$ is called conservative force field and ϕ is called scalar potential.

Solenoidal Vector Function: **A vector** $\vec{A}$ is said to be solenoidal vector or divergence free vector if div $\vec{A} = \nabla . \vec{A} = 0$.

Curl of a vector function: If $\vec{A}$ is any vector function differentiable at each point (x,y,z) then curl of $\vec{A}$ is denoted by curl $\vec{A}$ *or* $\nabla \times \vec{A}$ and it is defined by

$$curl\ \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$curl\ \vec{A} = \nabla \times \vec{A} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\hat{k}$$

Hence, curl of a vector function is a vector.

Ex.1: If $\vec{F} = grad\ \ x^3y + y^3z + z^3x - x^2y^2z^2$ then find div $\vec{F}$ *at* (1,2,3).

Soln: Given

$$\vec{F} = grad\ \ x^3y + y^3z + z^3x - x^2y^2z^2$$

$$= \nabla\ \ x^3y + y^3z + z^3x - x^2y^2z^2$$

$$\vec{F} = \ 3x^3y + z^3 - 2xy^2z^2\ \ \hat{i} + \ x^3 + 3y^2z - 2x^2yz^2\ \ \hat{j} + \ y^3 + 3z^2x - 2x^2y^2z\ \ \hat{k}$$

$$div\ \vec{F} = 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2$$

$$\therefore \nabla \vec{F}\Big|_{1,2,3} = 12 - 72 + 36 - 18 + 18 - 8 = -32.$$

Ex.2 : If $\vec{F}$ =(x+y+1) $\hat{i} + \hat{j} - (x+y)\hat{k}$, then show that $\vec{F}$ .curl $\vec{F}$ =0.

Soln: Given $\vec{F}$ =(x+y+1) $\hat{i} + \hat{j} - (x+y)\hat{k}$,

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$=(-1-0)\hat{i}-(-1-0)\hat{j}+(0-1)\hat{k}$$

$$curl\ \vec{F}=-\hat{i}+\hat{j}-\hat{k}$$

$$\therefore\ \vec{F}.curl\ \vec{F}=\left[(x+y+1)\hat{i}+\hat{j}-(x+y)\hat{k}\right].\left[-\hat{i}+\hat{j}-\hat{k}\right]$$

$$=-(x+y+1)+1+(x+y)$$

$$=0$$

Ex.3: If $\vec{F}=(ax+3y+4z)\hat{i}+(x-2y+3z)\hat{j}+(3x+2y-z)\hat{k}$ is solenoidal, find ;'a'.

Soln: Given

$$\vec{F}=(ax+3y+4z)\hat{i}+(x-2y+3z)\hat{j}+(3x+2y-z)\hat{k},\ then$$

$$\nabla.\vec{F}=\frac{\partial}{\partial x}(ax+3y+4z)+\frac{\partial}{\partial y}(x-2y+3z)+\frac{\partial}{\partial z}(3x+2y-z)=a-2-1$$

$$\nabla.\vec{F}=a-3$$

Since the vector field is solenoidal therefore $\nabla.\vec{F}=0,\ then$

$$a-3=0\Rightarrow a=3$$

Ex.4: Find the constants a,b,c such that the vector

$\vec{F}=(x+y+az)\hat{i}+(x+cy+2z)\hat{k}+(bx+2y-z)\hat{j}$ is irrotational.

Soln: Given $\vec{F}=(x+y+az)\hat{i}+(x+cy+2z)\hat{k}+(bx+2y-z)\hat{j}$

Since the vector field is irrotational, therefore $\nabla\times\vec{F}=0$.

$$curl\ \vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+az & bx+2y-z & x+cy+2z\end{vmatrix}$$

$$=(c+1)\hat{i}-(1-a)\hat{j}+(b-1)\hat{k}$$

$i.e.,\ (c+1)\hat{i}-(1-a)\hat{j}+(b-1)\hat{k}=0$

This is possible only when,

c-1=0, 1-a=0, b-1=0 $\Rightarrow$ a=1, b=1, c=1.

Ex.5: Prove that 1. $\nabla r^n = nr^{n-2}\vec{r},$

Soln: We have by the relation x=rcosθ, y=r sinθ.

By definition

$$\nabla r^n = \frac{\partial}{\partial x}\ r^n\ \hat{i} + \frac{\partial}{\partial y}\ r^n\ \hat{j} + \frac{\partial}{\partial z}\ r^n\ \hat{k}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\frac{\partial r}{\partial z}\hat{k}$$

$$But\ \ r^2 = x^2 + y^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \therefore \frac{\partial r}{\partial x} = \frac{x}{r}\ \ lll^{ly}\ \ \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= nr^{n-1}\left(\frac{x}{r}\right)\hat{i} + nr^{n-1}\left(\frac{y}{r}\right)\hat{j} + nr^{n-1}\left(\frac{z}{r}\right)\hat{k}$$

$$= nr^{n-2}x\hat{i} + nr^{n-2}y\hat{j} + nr^{n-2}xz\hat{k}$$

$$= nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= nr^{n-2}\vec{r}$$

Ex.6: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r),\ where\, r^2 = x^2 + y^2 + z^2.$

Soln: Given $r^2 = x^2 + y^2 + z^2.$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\ \text{ similarly }\ \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$Let\,\phi = f(x) \Rightarrow \nabla^2\phi = \sum\frac{\partial^2\phi}{\partial x^2} = \sum\frac{\partial^2}{\partial x^2}\ f(x)\ = \sum\frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right]$$

$$Where\sum\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \sum\frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = \sum\frac{\partial}{\partial x}\left[\frac{xf'(r)}{r}\right]$$

$$= \sum\frac{1}{r^2}\left[r\frac{\partial}{\partial x}\ xf'(r)\ - xf'(r)\frac{\partial r}{\partial x}\right]$$

$$= \sum\frac{1}{r^2}\left[r\left\{xf''(r)\frac{\partial r}{\partial x} + 1.f'(r)\right\} - xf'(r)\frac{x}{r}\right]$$

$$= \sum\frac{1}{r^2}\left[r\left\{xf''(r)\frac{x}{r} + f'(r)\right\} - \frac{x^2}{r}f'(r)\right]$$

$$= \sum \frac{1}{r^2}\left[x^2 f''(r) + rf'(r) - \frac{x^2}{r} f'(r) \right]$$

$$= \sum \frac{1}{r^2}\left[x^2 f''(r) + \left(r - \frac{x^2}{r} \right) f'(r) \right]$$

$$= \frac{1}{r^2}\left[x^2 f''(r) + \left(r - \frac{x^2}{r} \right) f'(r) \right] + \frac{1}{r^2}\left[x^2 f''(r) + \left(r - \frac{y^2}{r} \right) f'(r) \right]$$

$$+ \frac{1}{r^2}\left[x^2 f''(r) + \left(r - \frac{z^2}{r} \right) f'(r) \right]$$

$$= \frac{1}{r^2}\left[x^2 + y^2 + z^2 \right] f''(r) + \frac{1}{r^2}\left[r - \frac{x^2}{r} + r - \frac{y^2}{r} + r - \frac{z^2}{r} \right] f'(r)$$

$$= \frac{1}{r^2} r^2 f''(r) + \frac{1}{r^2}\left[3r - \frac{1}{r}\left(x^2 + y^2 + z^2 \right) \right] f'(r)$$

$$= \frac{1}{r^2}\left[r^2 f''(r) \right] + \frac{1}{r^2}\left[3r - \frac{1}{r}\left(r^2 \right) \right] f'(r)$$

$$= \frac{1}{r^2}\left[r^2 f''(r) \right] + \frac{1}{r^2}\left[3r - r \right] f'(r)$$

$$\therefore \nabla^2 f(r) = f''(r) + \frac{3}{r} f'(r)$$

Ex.7: Find the constants 'a' and 'b' so that $\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$

irrotational and find ϕ such that $\vec{F} = \nabla\phi$.

Soln: Given $\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$

Since $\vec{F}$ is irrotational i.e., $\nabla \times \vec{F} = 0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$i.e., \ (-1+1)\hat{i} - (bz^2 - 3z^2)\hat{j} + (6x - ax)\hat{k} = 0$$

$$i.e., \ -z^2(b-3)\hat{j} + (6-a) = 0$$

which holds good if any only if b-3=0 and 6-a=0 $\Rightarrow$ a=6 and b=3.

Also given that $\nabla\phi = \vec{F}$

$$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = 6xy + z^3\ \hat{i} + +(3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = 3x^2y + xz^3 + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2y - yz + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + f_2(x,y)$$

Hence $\phi = 3x^2y + xz^3 - yz.$

Vector Identities-

These are some properties relating to various meaningful combinations of *gradient, divergence, curl and laplacian*. These are established by taking a general scalar point function or a vector point function.

V.I - 1 curl (grad ϕ) = $\vec{0}$. or $\nabla \times (\nabla\phi) = \vec{0}$

Proof : Let ϕ be a scalar point function of x, y, z. grad $\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$$\text{curl (grad }\phi) = \nabla \times (\nabla\phi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

ie., $$= \Sigma\left\{\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right\} i = \Sigma\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right) i = \vec{0}$$

Thus **curl (grad ϕ) = 0**, for any scalar function ϕ

V.I - 2 div (curl $\vec{A}$) = 0 . or $\nabla \cdot (\nabla \times \vec{A}) = 0$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)$$

Now div (curl $\vec{A}$) = $\nabla \cdot (\nabla \times \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) = \Sigma \left(\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z} \right)$$

On expanding we get,

$$\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z} + \frac{\partial^2 a_1}{\partial y \partial z} - \frac{\partial^2 a_3}{\partial y \partial x} + \frac{\partial^2 a_2}{\partial z \partial x} - \frac{\partial^2 a_1}{\partial z \partial y} = 0$$

Thus **div (curl $\vec{A}$) = *0***, for any vector function $\vec{A}$

V.I - 3 curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$ or $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z

$$\therefore \quad \text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now curl (curl $\vec{A}$) $= \nabla \times (\nabla \times \vec{A})$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) & \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) & \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) \end{vmatrix}$$

$$= \Sigma i \left\{ \frac{\partial}{\partial y} \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) \right\}$$

$$= \Sigma i \left(\frac{\partial^2 a_2}{\partial y \partial x} + \frac{\partial^2 a_3}{\partial z \partial x} \right) - \Sigma i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right), \text{ by rearranging.}$$

Adding and subtracting $\Sigma i \frac{\partial^2 a_1}{\partial x^2}$ we get

$$\Sigma i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial y \partial x} + \frac{\partial^2 a_3}{\partial z \partial x} \right) - \Sigma i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right)$$

$$= \Sigma i \frac{\partial}{\partial x} \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \right) - \Sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) a_1 i$$

$$= \Sigma \frac{\partial}{\partial x} (\text{div } \vec{A})\, i - \nabla^2 \Sigma a_1 i = \text{grad}(\text{div } \vec{A}) - \nabla^2 \vec{A}$$

Thus $\textbf{curl}(\textbf{curl } \vec{A}) = \textbf{grad}(\textbf{div } \vec{A}) - \nabla^2 \vec{A}$

V.I - 4 $\text{div}\left(\phi \vec{A}\right) = \phi(\text{div } \vec{A}) + \text{grad}\,\phi \cdot \vec{A}$ or $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and ϕ be a scalar point function of x, y, z

$\therefore \quad \phi \vec{A} = \phi(a_1 i + a_2 j + a_3 k) = \Sigma (\phi a_1)\, i$

Now $\text{div}(\phi \vec{A}) = \nabla \cdot (\phi \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \Sigma (\phi a_1)\, i$$

$$= \Sigma \frac{\partial}{\partial x} (\phi a_1) = \Sigma \left(\phi \frac{\partial a_1}{\partial x} + \frac{\partial \phi}{\partial x} a_1\right)$$

ie., $\quad \text{div}(\phi \vec{A}) = \phi \Sigma \frac{\partial a_1}{\partial x} + \Sigma \frac{\partial \phi}{\partial x} i \cdot \Sigma a_1 i$

Thus $\textbf{div}(\phi \vec{A}) = \phi(\textbf{div } \vec{A}) + \textbf{grad}\,\phi \cdot \vec{A}$

V.I - 5 $\text{curl}\left(\phi \vec{A}\right) = \phi(\text{curl } \vec{A}) + \text{grad}\,\phi \times \vec{A}$ or

$\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$

Proof : Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z

$\therefore \quad \phi \vec{A} = (\phi a_1)\, i + (\phi a_2)\, j + (\phi a_3)\, k$

Now $\text{curl}(\phi \vec{A}) = \nabla \times (\phi \vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$

ie.,
$$= \Sigma i \left\{\frac{\partial}{\partial y}(\phi a_3) - \frac{\partial}{\partial z}(\phi a_2)\right\}$$

$$= \Sigma i \left\{\left(\phi \frac{\partial a_3}{\partial y} + \frac{\partial \phi}{\partial y} a_3\right) - \left(\phi \frac{\partial a_2}{\partial z} + \frac{\partial \phi}{\partial z} a_2\right)\right\}$$

$$= \phi \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i + \Sigma \left(\frac{\partial \phi}{\partial y} a_3 - \frac{\partial \phi}{\partial z} a_2\right) i$$

$$= \phi \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi \, (\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$$

Thus **curl $(\phi \vec{A}) = \phi$ (curl $\vec{A}$) $+ \nabla\phi \times \vec{A}$**

V.I - 6 div $(\vec{A} \times \vec{B}) = \vec{B} \cdot$ curl $\vec{A} - \vec{A} \cdot$ curl $\vec{B}$ or

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$, be two vector point functions of x, y, z

$$\therefore \quad \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \Sigma \, i \, (a_2 b_3 - a_3 b_2)$$

Now div $(\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A} \times \vec{B})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma \, i \, (a_2 b_3 - a_3 b_2)$$

$$= \Sigma \frac{\partial}{\partial x} (a_2 b_3 - a_3 b_2)$$

$$= \Sigma \left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x} \right)$$

On expanding we get

$$\left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x} \right) + \left(a_3 \frac{\partial b_1}{\partial y} + b_1 \frac{\partial a_3}{\partial y} - a_1 \frac{\partial b_3}{\partial y} - b_3 \frac{\partial a_1}{\partial y} \right)$$

$$+ \left(a_1 \frac{\partial b_2}{\partial z} + b_2 \frac{\partial a_1}{\partial z} - a_2 \frac{\partial b_1}{\partial z} - b_1 \frac{\partial a_2}{\partial z} \right)$$

ie., $$= \Sigma \, b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - \Sigma \, a_1 \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z} \right)$$

$$= (\Sigma \, b_1 i) \cdot \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) i - (\Sigma \, a_1 i) \cdot \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z} \right) i$$

$(\because \quad \Sigma \, A_1 B_1 = \Sigma \, A_1 i \cdot \Sigma \, B_1 i)$

$$= (\Sigma\, b_1 i) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - (\Sigma\, a_1 i) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Thus $\textbf{div}\ (\vec{A} \times \vec{B}) = \vec{B} \cdot \textbf{curl}\ \vec{A} - \vec{A} \cdot \textbf{curl}\ \vec{B}$

1. *If $\vec{F_1}$ and $\vec{F_2}$ are irrotational, prove that $\vec{F_1} \times \vec{F_2}$ is solenoidal.*

>> $\vec{F_1}$ and $\vec{F_2}$ are irrotational by data.

$\Rightarrow$ curl $\vec{F_1} = \vec{0}$ and curl $\vec{F_2} = \vec{0}$ ...(1)

We have to prove that div $(\vec{F_1} \times \vec{F_2}) = 0$

We have the vector identity (V.I - 6),

div $(\vec{A} \times \vec{B}) = \vec{B} \cdot$ curl $\vec{A} - \vec{A} \cdot$ curl $\vec{B}$ *(assumed)*

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot$ curl $\vec{F_1} - \vec{F_1} \cdot$ curl $\vec{F_2}$

ie., div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot \vec{0} - \vec{F_1} \cdot \vec{0} = 0$, by using (1)

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = 0 \Rightarrow \vec{F_1} \times \vec{F_2}$ **is solenoidal.**

2. *If $u\vec{F} = \nabla v$, prove that $\vec{F}$ and curl $\vec{F}$ are at right angles.*

>> $\vec{F} = \frac{1}{u} \nabla v$ and we have to prove that $\vec{F} \cdot$ curl $\vec{F} = 0$

We shall first find curl $\vec{F}$ where $\vec{F}$ is of the form $\phi \vec{A}$

Let us consider the vector identity (V.I - 5)

$$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$$

$$\therefore \quad \nabla \times \left(\frac{1}{u} \nabla v\right) = \frac{1}{u} \{\nabla \times (\nabla v)\} + \nabla \left(\frac{1}{u}\right) \times \nabla v$$

The first term in the R.H.S of this equation is zero by a vector identity curl (grad ϕ) = 0 (V.I - 1)

$$\therefore \quad \nabla \times \left(\frac{1}{u} \nabla v\right) = \nabla \left(\frac{1}{u}\right) \times \nabla v \quad ie., \text{ curl } \vec{F} = \nabla \left(\frac{1}{u}\right) \times \nabla v$$

Now $\vec{F} \cdot$ curl $\vec{F} = \left(\frac{1}{u} \nabla v\right) \cdot \left\{\nabla \left(\frac{1}{u}\right) \times \nabla v\right\}$

R.H.S of this equation is a scalar triple product or the box product of three vectors.

ie., $\vec{F} \cdot \text{curl } \vec{F} = \left[\frac{1}{u} \nabla v,\ \nabla\left(\frac{1}{u}\right),\ \nabla v \right]$

ie., $\vec{F} \cdot \text{curl } \vec{F} = \frac{1}{u}\left[\nabla v,\ \nabla\left(\frac{1}{u}\right),\ \nabla v \right] = 0$

Hence $\vec{F} \cdot \text{curl } \vec{F} = 0$, since two vectors are identical in the box product.

Thus $\vec{F}$ **is perpendicular to curl** $\vec{F}$

MODULE IV

INTEGRAL CALCULUS

CONTENTS:

DIFFERENTIAL EQUATIONS

Reduction formula:

1. Reduction formula for $\int \sin^n x \, dx$ and $\int_0^{\pi/2} \sin^n x \, dx$, n is a positive integer.

Let
$$I_n = \int \sin^n x \, dx$$
$$= \int \sin^{n-1} x . \sin x \, dx = \int u \; v \, dx \, (say)$$

We have the rule of integration by parts,

$\int u \; v \, dx = u \int v \, dx - \int \int v \, dx . u' \, dx$

$$\therefore I_n = \sin^{n-1} x(-\cos x) - \int -\cos x \quad n-1 \; \sin^{n-2} x . \cos x \, dx$$
$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x . \cos^2 x \, dx$$
$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x(1 - \sin^2 x) \, dx$$
$$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$
$$i.e., \; I_n = \sin^{n-1} x . \cos x + (n-1) I_{n-2} - (n-1) I_n$$
$$i.e., \; I_n \; 1 + (n-1) = -\sin^{n-1} x . \cos x + (n-1) I_{n-2}$$
$$\therefore \; I_n = \int \sin^n x \, dx = \frac{-\sin^{n-1} x . \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

This is the required reduction formula.

2. . Reduction formula for $\int \cos^n x$ and $\int_0^{\pi/2} \cos^n x \, dx$,

Where n is a positive integer.

Let
$$I_n = \int \cos^n x \, dx$$
$$= \int \cos^{n-1} x . \cos x \, dx$$

$$\therefore I_n = \cos^{n-1}x.\sin x - \int \sin x \; (n-1) \; \cos^{n-2} x(-\sin x) \; dx$$

$$= -\cos^{n-1}x.\cos x + (n-1)\int \cos^{n-2}x.\sin^2 x \; dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x) \; dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int \cos^{n-2}x \; dx - (n-1) \int \cos^n x \; dx$$

$$I_n = \cos^{n-1}x.\sin x + (n-1)I_{n-2} - (n-1)\, I_n$$

$$i.e., \; I_n \left[1 + (n-1)\right] = \cos^{n-1}x.\sin x + (n-1)I_{n-2}$$

$$\therefore \; I_n = \int \cos^n x \; dx = \frac{\cos^{n-1}x.\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Next, let $I_n = \int_0^{\pi/2} \cos^n x dx$

$$\therefore \; from(1), \; I_n = \left[\frac{\cos^{n-1}x.\sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

$But \;\; \cos(\pi/2) = 0 = \sin 0.$

$$Thus \;\; I_n = \frac{n-1}{n} I_{n-2}$$

1. Reduction formula for $\int \sin^m x \cos^n x \, dx$ and $\int_0^{\pi/2} \sin^m \cos^n x \, dx$ where m and n are positive integers.

$$I_{m,n} = \int \sin^m x \;\; \cos^n x dx$$

$$= \int \sin^{m-1} x \;\; \sin x \cos^n x \;\; dx = \int u \; v \; dx \,(say)$$

we have $\int u\, v\ dx = u\int v\, dx - \iint v\, dx . u'\ dx$

$Here\ \int u\, dx = \int \sin x \cos^n dx$

$Put\quad \cos x = t \quad \therefore -\sin x dx = dt$

$$Hence\ \int v\ dx = \int -t^n dt = -\frac{t^{n+1}}{n+1} = -\frac{\cos^{n+1} x}{n+1}$$

$$Now\, I_{m,n} = \sin^{m-1} x \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int \frac{-\cos^{n+1} x}{n+1}(m-1)\sin^{m-2} x \cos x dx$$

$$i.e.,\ = -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n+2} x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x . \cos^n x . \cos^2 x dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x (1-\sin^2 x) dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x dx$$

$$I_{m,n} = -\frac{-\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$i.e.,\ I_{m,n}\left[1 + \frac{m-1}{n+1}\right] = \frac{1}{n+1}\left[-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}\right]$$

$$I_{m,n}\left[\frac{m-1}{n+1}\right] = \frac{1}{n+1}\left[-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}\right]$$

$$\therefore\ I_{m,n} = \int \sin^m x \cos^n x dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad ......(1)$$

PROBLEMS:

1. $Let\ \ I = \int_0^{\pi} \sin^4 x\, dx$

$f(x) = \sin^4 x\, and\ \ 2a = \pi\ \ or\ \ a = \pi/2$

$f(2a-x) = \sin^4(\pi - x) = \sin^4 x = f(x) \quad ie.,\ \ f(2a-x) = f(x)$

Thus by the property $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ *we have,*

$I = 2\int_0^{\pi/2} \sin^4 x\, dx = 2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ by reduction formula.

Thus $\ I = 3\pi/8$

2. *Let* $I = \int_0^{\pi} x \sin^8 x \, dx$

We have the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi} (\pi - x)\sin^8(\pi - x)dx = \int_0^{\pi}(\pi - x)\sin^8 x dx$$

$$= \pi\int_0^{\pi}\sin^8 x dx - \int_0^{\pi} x\sin^8 x dx$$

$$I = \pi\int_0^{\pi}\sin^8 x dx - I$$

$$or\ 2I = \pi.2\int_0^{\pi/2}\sin^8 x dx$$

$$Hence\ I = \pi.\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2},\ .$$

Thus by reduction formula

$$I = \frac{35\pi^2}{256}$$

3. *Let* $I = \int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

$$I = \int_0^{\pi}(\pi - x)\sin^2(\pi - x)\cos^4(\pi - x)dx, \text{ by a property.}$$

$$= \int_0^{\pi}(\pi - x)\sin^2 x \cos^4 x dx$$

$$= \pi\int_0^{\pi}\sin^2 x\cos^4 x\, dx - \int_0^{\pi} x\sin^2 x\cos^4 x\, dx$$

$$= \pi\int_0^{\pi}\sin^2 x\cos^4 x\, dx - I$$

$$2I = \pi.2\int_0^{\pi/2}\sin^2 x\cos^4 x\, dx$$

$$I = \pi.\frac{(1).(3).(1)}{6\times 4\times 2}.\frac{\pi}{2} \text{ by reduction formula.}$$

Thus $I = \pi^2/32$

2. *Evaluate* $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

>> Let $I = \int_0^1 x^{3/2}(1-x)^{3/2}\,dx$

Put $x = \sin^2\theta,\ dx = 2\sin\theta\cos\theta\,d\theta$ and θ varies from 0 to $\pi/2$.

Also $(1-x)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

$\therefore\quad I = \int_{\theta=0}^{\pi/2} \sin^3\theta\cos^3\theta\cdot 2\sin\theta\cos\theta\,d\theta$

i.e., $I = 2\int_0^{\pi/2}\sin^4\theta\cos^4\theta\,d\theta$

Hence $I = 2\cdot\dfrac{[(3)(1)][(3)(1)]}{8\times6\times4\times2}\cdot\dfrac{\pi}{2}$ by reduction formula.

Thus $I = 3\pi/128$

Introduction:

Many problems in all branches of science and engineering when analysed for putting in a mathematical form assumes the form of a differential equation.

An engineer or an applied mathematician will be mostly interested in obtaining a solution for the associated equation without bothering much on the rigorous aspects. Accordingly the study of differential equations at various levels is focused on the methods of solving the equations.

Preliminaries:

Ordinary Differential Equation (O.D.E)

If $y = f(x)$ is an unknown function, an equation which involves atleast one derivative of y, *w.r.t.* x is called an **ordinary differential equation** which in future will be simply referred to as **Differential Equation (D.E)**.

The order of D.E is the order of the highest derivative present in the equation and the degree of the D.E. is the degree of the highest order derivative after clearing the fractional powers.

Finding y as a function of x explicitly [$y = f(x)$] or a relationship in x and y satisfying the D.E. [$f(x, y) = c$] constitutes the solution of the D.E.

Observe the following equations along with their order and degree.

1 $\frac{dy}{dx} = 2x$ [order = 1, degree = 1]

2 $\left(\frac{dy}{dx}\right)^2 + 3\frac{dy}{dx} + 2 = 0$ [order = 1, degree = 2]

General solution and particular solution:

A solution of a D.E. is a relation between the dependent and independent variables satisfying the given equation identically.

The general solution will involve arbitrary constants equal to the order of the D.E.

If the arbitrary constants present in the solution are evaluated by using a set of given conditions then the solution so obtained is called a **particular solution**. In many physical problems these conditions can be formulated from the problem itself.

Note : Basic integration and integration methods are essential prerequisites for this chapter.

Solution of differential equations of first order and first degree

Recollecting the definition of the order and the degree of a D.E., a first order and first degree equation will be the form

$\frac{dy}{dx} = f(x, y)$ or M(x,y)dx+N(x,y)dy=0

We discuss mainly classified four types of differential equations of first order and first degree. They as are as follows:

- Variables separable equations
- Homogenous equations
- Exact equations
- Linear equations

Variables separable Equations:

If the given D.E. can be put in the form such that the coefficient of dx is a function of the variable x only and the coefficient of dy is a function of y only then the given equation is said to be in the separable form.

The modified form of such an equation will be,

$P(x)\ dx + Q(y)\ dy = 0$

This is the general solution of the equation.

Example 1: $Solve\ \frac{dy}{dx} = xe^{y-x^2}$ given that y(0)=0

Soln: $\frac{dy}{dx} = xe^{y-x^2} \quad or \quad \frac{dy}{dx} = xe^{y}e^{-x^2}$

put $-x^2 = t \ \therefore -2xdx = dt \, or - xdx = dt$

Hence we have, $-e^{-y} + \int e^{t}\, dt = c$

$$i.e.\ \frac{dy}{e^y} = xe^{-x^2} dx\ by\ separating\ the Variables$$

$$\Rightarrow \int e^{-y} dy - \int xe^{-x^2} dx = 0$$

$$i.e\ \ -e^{-y} - \int xe^{-x^2} dx = c$$

The general solution becomes

$$i.e. \quad -e^{-y} + \frac{e^t}{2} = c$$

$or\ \frac{e^{-x^2}}{2} - e^{-y} = c$ is the general solution.

Now we consider $y(0) = 0$ That is $y=0$ when $x=0$,

$$\frac{1}{2} - 1 = c \text{ or } c = -\frac{1}{2}$$

Now the general solution becomes

$$\frac{e^{-x^2}}{2} - e^{-y} = -\frac{1}{2}$$

This is the required solution.

Example –2 $solve: xy\dfrac{dy}{dx} = 1 + x + y + xy$

□ $xy\dfrac{dy}{dx} = 1 + x + y + xy$

$i.e., \quad xy\dfrac{dy}{dx} = (1+x) + y(1+x)$

$i.e., \quad xy\dfrac{dy}{dx} = (1+x)(1+y)$

$or \quad \dfrac{ydy}{1+y} = \dfrac{1+x}{x}dx$ by separating the variables.

$\Rightarrow \quad \int \dfrac{y}{1+y}dy - \int \dfrac{1+x}{x}dx = c$

$or \quad \int \dfrac{(1+y)-1}{1+y}dy - \int \dfrac{1}{x}dx - \int 1dx = c$

$ie., \quad \int 1dy - \int \dfrac{1}{1+y}dy - \log x - x = c$

$i.e \quad y - \log y - \log x - x = c$

$or \quad y - x \; - \log yx = c \; isthe\, required \; solution$

Example – 4 : Solve : $y - x \; \dfrac{dy}{dx} = y^2 + \dfrac{dy}{dx}$

>> Rearranging the given equation we have,

$y - y^2 = \dfrac{dy}{dx} \;\; x+1$

$or \; \int \dfrac{dx}{x+1} = \int \dfrac{dy}{y-y^2}$

$i.e \; \log \;\; x+1 \;\; = \int \dfrac{dy}{y \;\; 1-y} ----------(1)$

We have to employ the method of partial fractions for the second term of the above.

$Let \; \dfrac{1}{y \;\; 1-y} = \dfrac{A}{y} + \dfrac{B}{y-1}$

$\Rightarrow 1 = A \;\; y-1 \;\; + By$

Put y=0,1 A=-1 and B=1

$$\therefore \int \frac{dy}{y(1-y)} = -\int \frac{dy}{y} + \int \frac{dy}{1-y}$$

$$\int \frac{dy}{y(1-y)} = -\log y - \log(1-y) = \log\frac{1-y}{y}$$

Using this result in (1) we get,

$$\log(x+1) + \log\left(\frac{1-y}{y}\right) = c$$

$$or\ \log\left[\frac{(x+1)(1-y)}{y}\right] = \log k$$

$$\therefore (x+1)(1-y) = ky \textit{ is the required solution.}$$

Example – 5 :

Solve:

$$\tan y\frac{dy}{dx} = \cos(x+y) + \cos(x-y)$$

 The given equation on expanding terms in the R.H.S. becomes

$$\tan y\frac{dy}{dx} = \cos x\cos y - \sin x\sin y + \cos x\cos y + \sin x\sin y$$

$$ie.,\ \tan y\frac{dy}{dx} = 2\cos x\cos y$$

$$or\ \frac{\tan y}{\cos y}dy = 2\cos x\,dx$$ by separating the variables.*

$$\Rightarrow \int \tan y.\sec y\,dy - \int 2\cos x\,dx = c$$

$\therefore \sec y - 2\sin x = c$ is the required solution.

Exact Differential Equations:

The differential equation M(x , y) dx + N(x , y)dy=0 to be an exact equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further the solution of the exact equation is given by

$$\int M\,dx + \int N(y)\,dy = c$$

Where, in the first term we integrate M(x,y) w.r.t x keeping y fixed and N(y) indicate the terms in N with out x
(not containing x)

1. Solve: $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

(Though it is evident that the equation is a homogeneous one, before solving by putting y=vx we should check for exactness)

 $Let\ M = 5x^4 + 3x^2y^2 - 2xy^3 and\ \ N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 and\ \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\ dx + \int N\ (y)\, dy = c$

$ie., \int\ 5x^4 + 3x^2y^2 - 2xy^3\ \ dx + \int -5y^4 dy = c$

$Thus\ \ x^5 + x^3y^2 - x^2y^3 - y^5 = c$, is the required solution.

2. Solve: $\cos x \tan y + \cos(x+y)\ \ dx + \left[\sin x \sec^2 y + \cos(x+y)\right] dy = 0$

 $Let\ M = \cos x \tan y + \cos(x+y) : N = \sin x \sec^2 y + \cos(x+y)$

$\therefore\ \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y);\ \frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\ dx + \int N\ (y)\, dy = c$

$ie., \int\ \cos x \tan y + \cos(x+y)\ dx + \int 0\ \ dy = c$

Thus $\sin x \tan y + \sin(x+y) = c$, is the required solution.

3. Solve: $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

>> The given equation is put in the form,

$y\cos x + \sin y + y\ \ dx + \ \sin x + x\cos y + x\ \ dy = 0.$

$Let\ \ M = y\cos x + \sin y + y\ \ and\ \ N = \sin x + x\cos y + x$

$\therefore\ \frac{\partial M}{\partial y} = \cos x + \cos y + 1 and\ \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$Since\ \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\ dx + \int N\ (y)\, dy = c$

$ie., \int\ \ y\cos x + \sin y + y\ \ dx + \int 0\, dy = c$

$Thus\ \ y\sin x + x\sin y + xy = c$, is the required solution.

4. Solve: $ye^{xy}dx + (xe^{xy} + 2y)dy = 0$

 $Let\ M = ye^{xy}, N = xe^{xy} + 2y$

$$\frac{\partial M}{\partial y} = ye^{xy}x + e^{xy}; \frac{\partial N}{\partial x} = xe^{xy}y + e^{xy}$$

$Since\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

solution is given by $\int M\, dx + \int N(y)\, dy = c$

$ie., \int ye^{xy}dx + \int 2y\, dy = c$

$ie., y\frac{e^{xy}}{y} + y^2 = c$

Thus $e^{xy} + y^2 = c$, is the required solution.

5. Solve: $[y(1+1/x) + \cos y]dx + [x + \log x - x\sin y]dy = 0$

 $Let\ M = y(1+1/x) + \cos y\ and\ N = x + \log x - x\sin y$

$\therefore \frac{\partial M}{\partial y} = 1 + 1/x - \sin y\ and\ \frac{\partial N}{\partial x} = 1 + 1/x - \sin y$

$Since\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

The solution is $\int M\, dx + \int N(y)\, dy = c$

$ie., \int [y(1+1/x) + \cos y]dx + \int 0\, dy = c$

$Thus\ y(x + \log x) + x\cos y = c$, is the required solution.

Equations reducible to the exact form:

Integrating factor: Type-1:

Suppose that, for the equation M dx + N dy =0

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then we take their difference.

The difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ should be close to the expression of M or N.

If it is so, then we compute $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ or $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ or $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$

Then $e^{\int f(x)dx}$ or $e^{-\int g(y)dy}$ is an integrating factor.

The following basic results will be useful :

(i) $e^{\log x} = x$ $(ii) e^{n\log x} = x^n$

1. Solve: $4xy + 3y^2 - x \ dx + x \ x + 2y \ dy = 0$

 $Let\, M = 4xy + 3y^2 - x \ and \ N = x \ x + 2y \ = x^2 + 2xy$

$\frac{\partial M}{\partial y} = 4x + 6y \ and \ \frac{\partial N}{\partial x} = 2x + 2y.$ The equation is not exact

$Consider \ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)....close \ to \ N.$

$Now \ \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) - \frac{2(x+2y)}{x(x+2y)} = \frac{2}{x} = f(x)$

$Hence \ e^{\int f(x)dx}$ is an integrating factor.

ie., $e^{\int f(x)dx} = e^{\int f \frac{2}{x} dx} = e^{2\log x} = e^{\log(x^2)} = x^2$

Multiplying the given equation by x^2 we now have,

$M = 4x^3y + 3x^2y^2 - x^3 \ and \ N = x^4 + 2x^3y$

$\frac{\partial M}{\partial y} = 4x^3 + 6x^2y \ and \ \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$

Solution of the exact equation is $\int M\, dx + \int N(y)\, dy = c$

$ie., \int 4x^3y + 3x^2y^2 - x^3 \ dx + \int 0\, dy = c$

$Thus \ x^4y + x^3y^2 - \frac{x^4}{4} = c,$ is the required solution.

2. Solve: y(2x-y+1)dx+x(3x-4y+3)dy=0

 $Let\, M = y \ 2x - y + 1 \ and \ N = x \ 3x - 4y + 3$

$ie., M = 2xy - y^2 + y \ and \ N = 3x^2 - 4xy + 3x$

$\frac{\partial M}{\partial y} = 2x - 2y + 1, \ \frac{\partial N}{\partial x} = 6x - 4y + 3$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x + 2y - 2 = -2(2x - y + 1)....near \ to \ M.$

$Now \ \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2(2x-y+1)}{y(2x-y+1)} = -\frac{2}{y} = g(y)$

$Hence\, I.F = e^{-\int g(y)dy} = e^{\int \frac{2}{y} dy} = e^{2\log y} = e^{\log(y^2)} = y^2$

Multiplying the given equation by y^2 we now have,

$$M = 2xy^3 - y^4 + y^3 \;\; and \;\; N = 3x^2y^2 - 4xy^3 + 3xy^2$$

$$\frac{\partial M}{\partial y} = 6xy^2 - 4y^3 + 3y^2 \;\; and \;\; \frac{\partial N}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

The Solution is $\int M\,dx + \int N(y)\,dy = c$

$$ie., \int \; 2xy^3 - y^4 + y^3 \;\; dx + \int 0\,dy = c$$

$Thus \; x^2y^3 - xy^4 + xy^3 = c,$ is the required solution.

Integrating Factor: Type-2:

If the given equation M dx +N dy=0 is of the form
yf (xy) dx+xg(xy)dy=0
then $\frac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$

1. Solve: $y \;\; 1 + xy + x^2y^2 \;\; dx + x \;\; 1 - xy + x^2y^2 \;\; dy = 0$

>> The equation is of the form yf(xy)dx+xg(xy)dy=0 where,

$$M = yf(xy) = y + xy^2 + x^2y^3 \;\; and$$

$$N = xg(xy) = x - x^2y + x^3y^2$$

$$Now \; Mx - Ny = \;\; xy + x^2y^2 + x^3y^3 \;\; - \;\; xy - x^2y^2 + x^3y^3 \;\; = 2x^2y^2$$

$$\therefore \; \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \; is \; the \; I.F.$$

Multiplying the given equation with $1/2x^2y^2$ it becomes an exact equation where we now have,

$$M = \frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2} \;\; and \;\; N = \frac{1}{2xy^2} - \frac{1}{2y} + \frac{x}{2}$$

The solution is given by $\int M\,dx + \int N(y)dy = c$

$$ie., \; \int\left(\frac{1}{2x^2y} + \frac{1}{2x} + \frac{y}{2}\right)dx + \int -\frac{1}{2y}\,dy = c$$

$$ie., \; \frac{1}{2xy} + \frac{1}{2}\log x + \frac{xy}{2} - \frac{1}{2}\log y = c$$

2. Solve: $y \;\; xy + 2x^2y^2 \;\; dx + x \;\; xy - x^2y^2 \;\; dy = 0$

>> The equation is of the form yf(xy)dx+xg(xy)dy=0 where,

$M = xy^2 + 2x^2y^3$ *and*

$N = x^2y - x^3y^2$

$Now\ Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3$

Thus $1/3x^3y^3$ is the I.F. Multiplying the given equation by this I.F we have an exact equation where we now have,

$$M = \frac{1}{3x^2y} + \frac{1}{3x} \text{ and } N = \frac{1}{3xy^2} - \frac{1}{3y}$$

The solution is $\int M\,dx + \int N(y)dy = c$

$$ie., \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}dy = c$$

$$ie., -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

Integrating factor: Type-3:

If the given equation Mdx+Ndy =0 is of the form

$$x^{k_1}y^{k_2}(c_1ydx + c_2x\,dy) + x^{k_3}y^{k_4}(c_3ydx + c_4x\,dy) = 0$$

Where k_i and c_i (i=1 to 4) are constants then x^ay^b is an integrating factor. The constants a and b are determined such that the condition for an exact equation is satisfied.

1. Solve: x(4y dx+2x dy)+ y^3(3y dx+5x dy) = 0

\>> We have $(4xy + 3y^4)dx + (2x^2 + 5xy^3)dy = 0$

Multiplying the equation by x^ay^b we have,

$\Rightarrow 4(b+1) = 2(a+2)\ and\ 3(b+4) = 5(a+1)$

$ie.,\ a = 2b\ \ and\ \ 5a - 3b = 7$

By solving we get a=2and b=1

We now have, $M = 4x^3y^2 + 3x^2y^5$ *and* $N = 2x^4y + 5x^3y^4$

The solution is $\int Mdx + \int N(y)dy = c$

$M = 4x^{a+1}y^{b+1} + 3x^{a}y^{b+4}$ *and*

$N = 2x^{a+2}y^{b} + 5x^{a+1}y^{b+3}$

$\frac{\partial M}{\partial y} = 4(b+1)x^{a+1}y^{b} + 3(b+4)x^{a}y^{b+3}$

$\frac{\partial N}{\partial x} = 2(a+2)x^{a+1}y^{b} + 5(a+1)x^{a}y^{b+3}$

We have to find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

ie., $\int 4x^{3}y^{2} + 3x^{2}y^{5}\ dx + \int 0\ dy = c$

Thus $x^{4}y^{2} + 3x^{2}y^{5}$=c, is the required solution.

2. Solve: $(y^{2} + 2x^{2}y)dx + (2x^{3} - xy)\,dy = 0$

>>Multiplying the given equation by $x^{a}y^{b}$ we have,

$M = x^{a}y^{b+2} + 2x^{a+2}y^{b+1}$ *and*

$N = 2x^{a+3}y^{b} - x^{a+1}y^{b+1}$

$\frac{\partial M}{\partial y} = (b+2)x^{a}y^{b+1} + 2(b+1)x^{a+2}y^{b}$

$\frac{\partial N}{\partial x} = 2(a+3)x^{a+2}y^{b} - (a+1)x^{a}y^{b+1}$

Let us find a and b such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow (b+2) = -(a+1)\ and\ 2(b+1) = 2(a+3)$

$ie.,\ a+b = -3\ \ and\ \ a-b = -2$

By solving we get a = -5/2 and b=-1/2.

We now have, $M = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}$ *and*

$N = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$

The solution is $\int M dx + \int N(y)dy = c$

ie., $\int \left(x^{-5/2} y^{3/2} + 2x^{-1/2} y^{1/2} \right) dx + \int 0 \ dy = c$

ie., $\dfrac{x^{-3/2}}{-3/2} y^{3/2} + \left(2 \dfrac{x^{1/2}}{1/2} y^{1/2} \right) = c$

ie., $\dfrac{-2}{3} x^{-3/2} y^{3/2} + 4x^{1/2} y^{1/2} = c$

Thus $6\sqrt{xy} - \sqrt{y^3 / x^3} = k$, is the required solution, where k=3c/2

Type-4 Exactness by inspection:

1. Solve : $\left(1 + y\tan(xy)\right) dx + \left(x\tan(xy) \right) dy = 0$

>> The given equation can be put in the form

$dx + \tan(xy) \left(y\,dx + x\,dy \right) = 0$

ie., $dx + \tan(xy) d(xy) = 0$

Integrating we get, $x + \log\sec(xy) = c$, being the required solution.

2. Solve: $\dfrac{y\,dx - x\,dy}{y^2} + (x\,dx + y\,dy) = 0$

>> The given equation is equivalent to the form,

$d\left(\dfrac{x}{y}\right) + x\,dx + y\,dy = 0$

$\Rightarrow \dfrac{x}{y} + \dfrac{x^2}{2} + \dfrac{y^2}{2} = c,$ on integration.

Thus $\dfrac{x}{y} + \dfrac{1}{2}\left(x^2 + y^2 \right) = c$, is the required solution.

Orthogonal trajectories

Definition: If two family of curves are such that every member of one family intersect every member of the other family at right angles then they are said to be orthogonal trajectories each other

Method of finding the orthogonal trajectories

Case - (i) Cartesian family $f(x, y, c) = 0$

We differentiate *w.r.t* x and eliminate the parameter c. The equation so obtained is called as the differential equation of the given family.

We know that if $\tan\psi = \frac{dy}{dx}$ is the slope of a given line then the slope of the line perpendicular to it is $\frac{-1}{\tan\psi} = -\frac{dx}{dy}$. Accordingly in the differential equation of the given family we shall replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to arrive at a new differential equation. Solving this new differential equation we get the orthogonal trajectories of the given family of curves.

Self orthogonal family : If the differential equation of the given family remains unaltered after replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ then the given family of curves is said to be *self orthogonal.*

Case - (ii) : Polar family $f(r, \theta, c) = 0$

We know that $\tan\phi = r\frac{d\theta}{dr}$ for a polar curve where ϕ is the angle between the radius vector and the tangent. $\phi_2 - \phi_1 = 90^0$ is the condition for two polar curves to be orthogonal.

$\therefore \quad \phi_2 = 90^0 + \phi_1 \Rightarrow \tan\phi_2 = \tan(90^0 + \phi_1)$

ie., $\quad \tan\phi_2 = -\cot\phi_1 \quad$ or $\quad \tan\phi_2 = \frac{-1}{\tan\phi_1}$

But $\tan\phi_1 = r\frac{d\theta}{dr}$ for the given curve and $\tan\phi_2 = r\frac{d\theta}{dr}$ for the orthogonal curve at the same point.

$\therefore \quad r\frac{d\theta}{dr}$ for the curve to be replaced by $\frac{-1}{r\frac{d\theta}{dr}}$

ie., $\quad -r^2\frac{d\theta}{dr}$ to be replaced by $\frac{dr}{d\theta}$ or vice - versa.

In otherwords, we have to differentiate $f(r, \theta, c) = 0$ *w.r.t* θ and eliminate c to obtain the D.E of the given family. We have to replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ to obtain the new D.E and solve the same to obtain the required orthogonal trajectories.

1. *Find the O.T of the family of parabolas* $y^2 = 4ax$.

>> Consider $\frac{y^2}{x} = 4a$ ... (1)

(If the parameter is on one side of the equation exclusively, then the same gets eliminated once we differentiate)

Now differentiating (1) *w.r.t* x we have

$$\frac{x \cdot 2y \frac{dy}{dx} - y^2 \cdot 1}{x^2} = 0 \quad \text{or} \quad 2xy \frac{dy}{dx} - y^2 = 0$$

ie., $2x \frac{dy}{dx} - y = 0$, is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$2x\left(-\frac{dx}{dy}\right) - y = 0 \quad \text{or} \quad 2x\,dx + y\,dy = 0$$

$\Rightarrow \int 2x\,dx + \int y\,dy = c$

ie., $x^2 + \frac{y^2}{2} = c$ or $2x^2 + y^2 = 2c = k$ *(say)*

Thus $\mathbf{2x^2 + y^2 = k}$, is the required O.T.

2. *Find the O.T of the family of astroids* $x^{2/3} + y^{2/3} = a^{2/3}$

>> Consider $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating *w.r.t* x, we have

$$\frac{2}{3} \cdot x^{-1/3} + \frac{2}{3} \cdot y^{-1/3} \frac{dy}{dx} = 0$$

ie., $x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$, is the D.E of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have,

$$x^{-1/3} + y^{-1/3}\left(-\frac{dx}{dy}\right) = 0 \quad \textit{ie.,} \quad x^{-1/3}\,dy = y^{-1/3}\,dx$$

ie., $y^{1/3}\,dy = x^{1/3}\,dx$ by separating the variables.

$\Rightarrow \int y^{1/3}\,dy - \int x^{1/3}\,dx = c$

ie., $\frac{y^{4/3}}{(4/3)} - \frac{x^{4/3}}{(4/3)} = c$ or $x^{4/3} - y^{4/3} = -\frac{4c}{3} = k$ *(say)*

Thus $\mathbf{x^{4/3} - y^{4/3} = k}$ is the required O.T.

3) Find the orthogonal trajectories of the family of curves

$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ 'λ' being the parameter **.** **(July 2015)**

Soln: $we\,have\,\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ ....(1)

Differentiating the (1) equation we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\cdot\frac{dy}{dx}=0$$

$$i.e\quad \frac{x}{a^2}=\frac{-y}{b^2+\lambda}\cdot\frac{dy}{dx}\text{........(2)}$$

$$Also\ \ from(1)\qquad \frac{x^2}{a^2}-1=\frac{-y^2}{b^2+\lambda}$$

$$\Rightarrow \frac{x^2-a^2}{a^2}=\frac{-y^2}{b^2+\lambda}\text{............(3)}$$

$Now, dividing (2)$ by(3) we get

$$\frac{x}{x^2-a^2}=\frac{y}{y^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{x}{x^2-a^2}=\frac{1}{y}\cdot\frac{dy}{dx}$$

$Now\ let\,us\,replace\ \frac{dy}{dx}\ by\,\frac{-dx}{dy}$

$$\therefore \frac{x}{x^2-a^2}=\frac{1}{y}\cdot\left(-\frac{dx}{dy}\right)$$

$or\ \ ydy=-\frac{x^2-a^2}{x}dx\ by\,separating\,the$ variables

$$\Rightarrow \int ydy=-\int xdx+a^2\int\frac{dx}{x}+c$$

$i.e\ \frac{y^2}{2}=\frac{-x^2}{2}+a^2\log x+c\ \ is\,the\,required\ orthogonal\,trajectories$

4) Find the orthogonal trajectory of the cardiods $r=a\ 1-\cos\theta$ **, using the differential equation method .** **(**
Jan 2015 ,Jan2014)

Soln:

Differentiating the given equation w.r.t θ,we get $\frac{dr}{d\theta} = a\sin\theta$.Substituting for a in the given equation,we get

$r = \left(\frac{1-\cos\theta}{\sin\theta}\right)\frac{dr}{d\theta}$.........DE of given equation

Changing $\frac{dr}{d\theta}$ *to* $-r^2\frac{d\theta}{dr}$ $\quad r = \left(\frac{1-\cos\theta}{\sin\theta}\right)\left(-r^2\frac{d\theta}{dr}\right)$

$\frac{dr}{r} + \left(\operatorname{cosec}\theta - \cot\theta\right)d\theta = 0$........DE of orthogonal trajectories

solving this equation,we get

$\log r + \log\left(\operatorname{cosec}\theta - \cot\theta\right) - \log\sin\theta = \log c$

$r\frac{\operatorname{cosec}\theta - \cot\theta}{\sin\theta} = c \Rightarrow r\frac{1-\cos\theta}{\sin^2\theta} = c$

$r = c\left(1+\cos\theta\right)$, this is requried orthogonal trajectories.

MODULE V

LINEAR ALGEBRA

CONTENTS:

Definition: A system of mn nos. arranged in a rectangular formation along m-rows & n-columns & bounded by the brackets or is called as m by n matrix or mxn matrix Matrix is denoted by a single capital letters A,B,C etc.

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}_{m\times n}$$

Elementary operations on Matrices:

The following 3 operations are said to be elementary operations

1. Interchange of any two rows or columns.
2. Multiplication of each element of a row or column by a non-Zero scalar or constant.
3. Addition of a scalar multiple of one row or column to anpother row or column.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \xrightarrow{R_3 \to kR_1 + R_3} B = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (ka_1 + c_1) & (ka_2 + c_1) & (ka_3 + c_3) \end{bmatrix}$$

If a matrix A gets transferred into another matrix B by any of these transformations then A is said to be equivalent to B written as $A \sim B$.

Echelon form or Row reduced Echelon form.

A matrix A of order mxn is said to be in a row reduced echelon form if

1. The leading element (the first non-Zero entry) of each row is unity.
2. All the entries below this leading entry is Zero.
3. The no of Zeros appearing before the leading entry in each row is greater than that appears in its previous row.
4. The Zero rows must appear below the non-zero rows.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Normal form of a matrix

The given matrix A is reduced to an echelon form first by applying a series of elementary row transformations.

Later column transformations row performed to reduce the matrix to one of the following four forms, called the normal form of A.

$i)\, Ir \quad ii)\, Ir, o \quad iii) \begin{bmatrix} Ir \\ \hline o \end{bmatrix} \quad iv) \begin{bmatrix} Ir & o \\ o & o \end{bmatrix}$ where Ir is the identity matrix of order r.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$I_3 \qquad I_3,0 \qquad \begin{bmatrix}I_2\\0\end{bmatrix}\quad \begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$

Rank of a matrix: The number of non-zero rows in echelon or normal form. It's in denoted by f(A)

1. Reduce the matrix to the row reduced echelon form

$$A=\begin{bmatrix}1&2&-2&3\\0&0&1&2\\-1&-3&2&-1\\2&4&-1&3\end{bmatrix}$$

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 + R_1 \quad R_4 \to R_4 - 2R_1$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&-1&0&2\\0&0&3&-3\end{bmatrix}\quad A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&8&3\\0&0&3&-3\end{bmatrix}$$

$R_3 \to R_3/8, \quad R_4 \to R_4/3 \quad R_4 \to R_4 - R_3$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&1&-3\end{bmatrix}\quad A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&0&-11/8\end{bmatrix}$$

$R_4 \to \dfrac{-8}{11}R_4$

$$A=\begin{bmatrix}1&2&-2&3\\0&1&8&1\\0&0&1&3/8\\0&0&0&1\end{bmatrix}$$

2) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$ find rank of a matrix

$Sol: R_3 \to R_3 - 3R_1, \quad R_3 \to R_3 - 13R_2$

$$A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 13 & -2 & -8 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -28 & -34 \end{bmatrix}$$

$\rho(A) = 3$

3) $\begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 5 & 1 & 1 & 4 \end{bmatrix}$ find the rank

$Sol: R_2 \to 4R_2 - R_1, \quad R_3 \to 8R_3 - 3R_1 \quad R_4 \to R_4 + R_2$

$$A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & -6 & 5 & 6 \\ 0 & -2 & 3 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

4) Using the elementary transformation reduce the matrix $A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}$ to echelon form

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\to A=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix} = \begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{bmatrix}$$

$R_3 \to R_3 - 2R_2, \quad R_2 \to {-R_2}/{7}$

$$=\begin{bmatrix}1 & 2 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{bmatrix} \quad A=\begin{bmatrix}1 & 2 & -2\\ 0 & 1 & -8/7\\ 0 & 0 & 0\end{bmatrix}$$

5) Applying elementary transformations reduce the following matrix to the normal form & hence find rank of matrix given $\begin{bmatrix}3 & 2 & 5 & 7 & 12\\ 1 & 1 & 2 & 3 & 5\\ 3 & 3 & 6 & 9 & 15\end{bmatrix}$

$Sol: R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}3 & 2 & 5 & 7 & 12\\ 1 & 1 & 2 & 3 & 5\\ 3 & 3 & 6 & 9 & 15\end{bmatrix} = \begin{bmatrix}1 & 1 & 2 & 3 & 5\\ 3 & 2 & 5 & 7 & 12\\ 3 & 3 & 6 & 9 & 15\end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 3R_1$

$$A=\begin{bmatrix}1 & 1 & 2 & 3 & 5\\ 0 & -1 & -1 & -2 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

This echelon form, now we have to perform column trans to reduce to the normal form.

$c_2 \to c_2 - c_1, \quad c_3 \to c_3 - 2c_1 \quad c_4 \to c_4 - 3c_1 \quad c_5 \to c_5 - 5c_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to c_3 - c_2, \quad c_4 \to c_4 - 2c_2 \quad c_3 \to c_5 - 3c_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -R_2} A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

6) By performing elementary row & column transformations, reduce the following matrix to the normal form $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$Sol:\ R_2 \leftrightarrow R_2$

$$A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_4 \to R_4 - 4R_2 \qquad R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$c_2 \to c_2 + 2c_1, \quad c_3 \to c_3 - c_1 \quad c_4 \to c_4 + 4c_1, \quad c_5 \to c_5 - 2c_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to c_3 + c_2, \quad c_4 \to c_4 - 3c_2, \quad c_5 \to c_5 - c_2 \quad c_4 \to c_4 - 9c_3 \quad c_5 \to c_5 - 4c_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

7) Reducing the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$ into normal form and find the rank

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1 \quad R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - 2R_2 \quad R_3 \to \frac{-1}{3} R_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$R_2 \to -R_2 \quad c_3 \to 2c_3 + 2c_2 \quad c_4 \to 2c_4 - c_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_3 \to {c_2}/{2} \quad c_4 \to c_4 - 3c_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$== \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 3.$$

8) Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

$Sol: R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1 \quad R_4 \to R_4 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_3 \leftrightarrow R_4 \quad R_3 = \frac{1}{3} R_3$

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c_2 \to c_2 - 2c_1, \quad c_3 \to c_3 - 2c_1 \quad c_4 \to c_4 - 3c_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rho(A) = 4$$

9) Find the rank of the matrix by reducing to the normal form

1)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$$

$Sol: R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - 3R_1$

2)
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - 7R_2 \quad R_4 \to R_4 - 6R_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$c_2 \to c_2 - c_1, \quad c_3 \to c_3 - c_1 \quad c_4 \to c_4 - c_1 \quad c_3 \to c_3 - 2c_2 \quad c_4 \to c_4 - 5c_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$c_4 \to c_4 - 2c_3 \quad c_4 \to \frac{1}{18}c_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4.$

10) Find the value of K such that the following matrix A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix}$ may have rank equal to a) 3 b) 2.

3) $Sol: R_2 \to R_2 - R_1 \quad R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & k \\ 1 & 4 & 10 & k^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 3 & 9 & k^2-1 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & k-1 \\ 0 & 0 & 0 & k^2-3k+2 \end{bmatrix}$$

a) Rank of A can be 3 if the equivalent form of A has 3 non-Zero rows.

This is possible if $k^2 - 3k + 2 \neq 0$

$i.e,\ (k-1)(k-2) \neq 0$

$\rho(A) = 3 \quad if \ k \neq 1 \ \& \ k \neq 2$

4) b) Rank of A can be 2 if the equivalent form of A has 2 non-zero rows.

5) This is possible if $k^2 - 3k + 2 = 0$

$i.e,\ (k-1)(k-2) = 0 \Rightarrow k = 1 \ or \ k - 2$

6) $\iota(A) = 2 \quad if \ k = 1 \ \& \ k = 2$

$$A:B = \begin{bmatrix} a_{11}a_{12}.........a_{1n} : b_1 \\ a_{21}a_{22}.........a_{2n} : b_2 \\ a_{m1}a_{m2}.........a_{mn} : b_m \end{bmatrix}$$

The given sys of equation is consistent & will have unique soln.

Let us convert $A:B$ into a set of equation as follows

x + y + z = 6		
-2y + z = 7	-2y+3=-1	x+y+z=6
-3z = -9	-2y=-4	x=6-2-3=1
$\Rightarrow$ z = 3	y=2	x=1

x=1, y=2, z=3is the unique soln.

7) Solve the system of equations: x+2y=3z=0
2x+3y+z=0
4x+5y+4z=0
x+y-2z=0

$$Sol:\ A:B = \begin{bmatrix} 1 & 2 & 3 & :0 \\ 2 & 3 & 1 & :0 \\ 4 & 5 & 4 & :0 \\ 1 & 1 & -2 & :0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 4R_1$ $R_4 \to R_4 - R_1$ $R_3 \to R_3 - 3R_2$ $R_4 \to R_4 - R_2$

$$= \begin{bmatrix} 1 & 2 & 3 & :0 \\ 0 & -1 & -5 & :0 \\ 0 & -3 & -8 & :0 \\ 0 & -1 & -5 & :0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & :0 \\ 0 & -1 & -5 & :0 \\ 0 & 0 & 7 & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

$\rho(A) = 3$

$\rho\ A:B = 3 \Rightarrow n = 3$

Hence the system is consistent & will have trivial soln x=0 y=0 z=0

8) Does the following system of homogenous equations possess a non-trivial solutions? If so find them

$x_1 + x_2 - x_3 + x_4 = 0$

$x_1 - x_2 + 2x_3 + x_4 = 0$

$3x_1 + x_2 + x_4 = 0$

$$A:B = \begin{bmatrix} 1 & 1 & -1 & 1 & :0 \\ 1 & -1 & 2 & -1 & :0 \\ 3 & 1 & 0 & 1 & :0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 3R_1$ $R_3 - R_2$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 & :0 \\ 0 & -2 & 3 & -2 & :0 \\ 0 & -2 & 3 & -2 & :0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 & :0 \\ 0 & -2 & 3 & -2 & :0 \\ 0 & 0 & 0 & 0 & :0 \end{bmatrix}$$

Gauss elimination method:

The simplest method of solving systems of the form (1) of section 5.2 is the elimination method.

The Working Rule for the method is as given below.

Working rule:

Step1: Reduce the augmented matrix (A:B) to the form where A is in echelon form or in upper triangular form, by employing appropriate elementary row operations.

Step2: Write the linear equations associated with the reduce form obtained in Step 1. Let the number of equations in this reduced system be equal to r, If r=n, then the reduced system yields the unique solution the given system. If r<n, then n-r unknowns in the reduce system can be chosen arbitrarily and the reduced system yields infinitely many solutions of the given system.

1. : Solve the following system of linear equations by the Gauss elimination method

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + 2x_3 = 2$$

Sol: For the given system, the coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

And the augmented matrix is $A:B = \begin{bmatrix} 1 & 1 & 1 : 4 \\ 2 & 1 & -1 : 1 \\ 1 & -1 & 2 : 2 \end{bmatrix}$

We reduce this matrix $A:B$ to the upper triangular form by using elementary operations

Using the row operation $R_2 \to R_2 - 2R_1$ *and* $R_3 \to R_3 - R_1$, *We get*

$$A:B \sim \begin{bmatrix} 1 & 1 & 1 : 4 \\ 0 & -1 & -3 : -7 \\ 0 & -2 & 1 : -2 \end{bmatrix}$$

Now, Using the row operation $R_3 \to R_3 - 2R_2$ in this, we get

$$A:B \sim \begin{bmatrix} 1 & 1 & 1 : 4 \\ 0 & -1 & -3 : -7 \\ 0 & 0 & 7 : 12 \end{bmatrix}$$

We note that A is now reduced to the upper triangular form. The linear equations which correspond to this reduced form of $A:B$ are

$x_1 + x_2 + x_3 = 4$

$-x_2 - 3x_3 = -7$

$7x_3 = 12$

From equation (iii), wew find that $x_3 = 12/7$.

$x_2 = 7 - 3x_3 = 7 - \frac{36}{7} = \frac{13}{7}$

Substituting for $x_3$and x_2 found above in (i), we get

$x_1 = 4 - x_2 - x_3 = 4 - \frac{13}{7} - \frac{12}{7} = \frac{3}{7}$.

Thus, x_1=3/7, x_2=13/7, x_3=12/7 constitute the solutionof the given system.

2. Solve the following system of equations by Gauss's elimination method:

$$4x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$

Sol: The augmented matrix is $A:B = \begin{bmatrix} 4 & 1 & 1 & : 4 \\ 1 & 4 & -2 & : 4 \\ 3 & 2 & -4 & : 6 \end{bmatrix}$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & : 4 \\ 0 & 15/4 & -9/4 & : 3 \\ 0 & -10 & 2 & : -6 \end{bmatrix}$$

Using $R_2 \to R_2 - (1/4)R_1, R_3 \to R_3 - 3R_2$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & : 4 \\ 0 & 5 & -3 & : 4 \\ 0 & -5 & 1 & : -3 \end{bmatrix}$$

Using $R_2 \to (4/3)R_2, R_3 \to (1/2)R_3$

$$A:B \sim \begin{bmatrix} 4 & 1 & 1 & : 4 \\ 0 & 5 & -3 & : 4 \\ 0 & 0 & -2 & : 1 \end{bmatrix}$$

Using $R_3 \to R_3 - R_2$,

We note that A is now reduced to the upper triangular form. The equations that correspond to (i) are

$4x_1 + x_2 + x_3 = 4$

$5x_2 - 3x_3 = 4$

$-2x_3 = 1.$

These yield.

$$x_3 = -\frac{1}{2},\ x_2 = \frac{1}{5}(4+3x_3) = \frac{1}{2},\ x_1 = \frac{1}{4}(4 - x_2 - x_3) = 1.$$

Thus, x_1=1, x_2=1/2, x_3=-1/2 constitute the solution of the given system.

3._ Solve the following system of equations by Gauss's elimination method:

$$x + 2y + 2y = 1$$
$$2x + y + z = 2$$
$$3x + 2y + 2z = 3$$
$$x + z = 0$$

Sol: The augmented matrix is $A:B = \begin{bmatrix} 1 & 2 & 2:1 \\ 2 & 1 & 1:2 \\ 3 & 2 & 2:3 \\ 0 & 1 & 1:0 \end{bmatrix}$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & -3 & -3:0 \\ 0 & -4 & -4:0 \\ 0 & 1 & 1:0 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & 1 & 1:0 \\ 0 & 1 & 1:0 \\ 0 & 1 & 1:0 \end{bmatrix}$$

Using $R_2 \to (-1/3)R_2, R_3 \to (-1/4)R_3$

$$A:B \sim \begin{bmatrix} 1 & 2 & 2:1 \\ 0 & 1 & 1:0 \\ 0 & 0 & 0:0 \\ 0 & 0 & 0:0 \end{bmatrix}$$

Using $R_3 \to R_3 - R_2, R_4 \to R_4 - R_2,$

We note that A is now reduced to the echelon form. The system correspond to (i) is

$x + 2y = 2z = 1$

$y + z = 0$

These are two equations for three unknowns. Therefore, we can choose one of the unknown arbitrarily. Taking z=k, we get y=-k and x=1

Thus x=1,y=-k, z=k, Where k is arbitrary, is a solution of the given system.

5) Solve the following system of equations by Gauss's elimination method:

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + x_4 = -6$$

Sol : Consider augmented matrix $A:B$ *by* $R_1 \leftrightarrow R_4$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 1 & 7 & 1 & 1 & : & 12 \\ 1 & 1 & 6 & 1 & : & -5 \\ 5 & 1 & 1 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - 5R_1$

$$A:B = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 6 & 0 & -3 & : & 18 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & -4 & -4 & -19 & : & 34 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & -4 & -21 & : & 46 \end{bmatrix}$$

$R_4 \to 5R_4 + 4R_3$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 4 & : & -6 \\ 0 & 2 & 0 & -1 & : & 6 \\ 0 & 0 & 5 & -3 & : & 1 \\ 0 & 0 & 0 & -117 & : & 234 \end{bmatrix}$$

Hence we have $x_1 + x_2 + x_3 + 4x_4 = -6$

$2x_2 - x_4 = 6$

$5x_3 - 3x_4 = 1$

$-117x_4 = 234$

$\therefore x_4 = -2, x_3 = -1, x_2 = 2$ *and* $x_1 = 1$ *is the reqd. soln.*

Gauss -Jordan method:

This method can be regarded as the modification of Gauss – elimination method. This method aims in reducing the coefficient matrix A to a diagonal matrix.

1) **Applying Gauss Jordan method solve** $2x+3y-z=5$, $4x+4y-3z=3$, $2x-3y+2z=2$

Soln: $A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 4 & 4 & -3 & : & 3 \\ 2 & -3 & 2 & : & 2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A:B = \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & -2 & -1 & : & -7 \\ 0 & -6 & 3 & : & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 & : & 5 \\ 0 & 2 & 1 & : & 7 \\ 0 & -2 & 1 & : & -1 \end{bmatrix}$$

$R_1 \to 2R_1 - 3R_2, R_3 \to R_3 + R_2$

$$A:B = \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 2 & : & 6 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -5 & : & -11 \\ 0 & 2 & 1 & : & 7 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3, R_2 \to R_2 - R_3$

$$A:B = \begin{bmatrix} 4 & 0 & 0 & : & 4 \\ 0 & 2 & 0 & : & 4 \\ 0 & 0 & 2 & : & 6 \end{bmatrix}$$

Hence $4x = 4, 2y = 4, 2z = 6$

$\therefore x = 1, y = 2, z = 3$

2. Apply Gauss - Jordan method to solve the system of equations

$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$
$$x + y + z = 9$$

>> As it is convenient to have the leading coefficient as 1 we shall interchange the first and third equations. The augmented matrix will be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2, \quad R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_1 \to R_2 + R_1, \quad R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$-1/4 \cdot R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \to 2R_3 + R_1, \quad R_2 \to 3R_3 + R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Hence we have $x = 1, \ -y = -3, \ z = 5$

Thus $\mathbf{x = 1, \ y = 3, \ z = 5}$ is the required solution.

Iterative methods of solution of a system of algebraic equations

In this article we discuss two numerical iterative methods for solving a system of algebraic equations.

These two methods cannot be applied to any system of equations. It is applicable only when the numerically large coefficients are along the leading / principal diagonal of the coefficient matrix A associated with the system of equations usually represented in the form $AX = B$. Such a system is called a *diagonally dominant system*.

The methods are illustrated for the following system of three independent equations in three unknowns.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of equations is said to be diagonally dominant if

$$|a_{11}| > |a_{12}| + |a_{13}|, \ |a_{22}| > |a_{21}| + |a_{23}|, \ |a_{33}| > |a_{31}| + |a_{32}|$$

Sometimes we may have to rearrange the given system of equations to meet this requirement. If this condition is satisfied, the solution exists as the iteration process will converge.

Gauss - Seidel iterative method

We write the system of equations in the form

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \quad \ldots (1)$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \quad \ldots (2)$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \quad \ldots (3)$$

We start with the trial solution (*initial approximation*)

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

The first approximation are as follows.

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12} \cdot 0 - a_{13} \cdot 0\right] = \frac{b_1}{a_{11}}$$

This approximation is immediately used in (2) so that we have

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\,x_1^{(1)} - a_{23}\cdot 0\right]$$

ie., $$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right]$$

Finally, we use both these approximations in (3), so that we have

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}\,x_1^{(1)} - a_{32}\,x_2^{(1)}\right]$$

This completes first iteration.

The process is continued till we get the solution to the desired degree of accuracy.

Problem 1

Solve the following system of equations by Gauss - Seidel method

$$10x + y + z = 12$$

$$x + 10y + z = 12$$

$$x + y + 10z = 12$$

The given system of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{10}\left[12 - y - z\right] \qquad \ldots (1)$$

$$y = \frac{1}{10}\left[12 - x - z\right] \qquad \ldots (2)$$

$$z = \frac{1}{10}\left[12 - x - y\right] \qquad \ldots (3)$$

Let us start with the trial solution $x = 0,\ y = 0,\ z = 0$.

First iteration :

$$x^{(1)} = \frac{1}{10}\left[12 - 0 - 0\right] = 1.2$$

$$y^{(1)} = \frac{1}{10}\left[12 - 1.2 - 0\right] = 1.08 \qquad z^{(1)} = \frac{1}{10}\left[12 - 1.2 - 1.08\right] = 0.972$$

Second iteration :

$$x^{(2)} = \frac{1}{10}\left[12 - 1.08 - 0.972\right] \approx 0.9948$$

$$y^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 0.972\right] \approx 1.00332$$

$$z^{(2)} = \frac{1}{10}\left[12 - 0.9948 - 1.00332\right] \approx 1.000188$$

Third iteration :

$$x^{(3)} = \frac{1}{10}\left[12 - 1.00332 - 1.000188\right] = 0.99965$$

$$y^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.000188\right] = 1.00002$$

$$z^{(3)} = \frac{1}{10}\left[12 - 0.99965 - 1.00002\right] = 1.00003$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}\left[12 - 1.00002 - 1.00003\right] = 0.999995 \approx 1$$

$$y^{(4)} = \frac{1}{10}\left[12 - 1 - 1.00003\right] = 0.999997 \approx 1$$

$$z^{(4)} = \frac{1}{10}\left[12 - 1 - 1\right] = 1$$

Thus $x = 1,\ y = 1,\ z = 1$

Problem 2

Solve the following system of equations by Gauss - Seidel method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

>> The equations are diagonally dominant and hence we first write them in the following form.

$$x = \frac{1}{20}\left[17 - y + 2z\right] \qquad y = \frac{1}{20}\left[-18 - 3x + z\right] \qquad z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

We start with the trial solution $x = 0,\ y = 0,\ z = 0$

First iteration :

$$x^{(1)} = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3(0.85)\right] = -1.0275$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2(0.85) + 3(-1.0275)\right] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}\left[17 - (-1.0275) + 2(1.0109)\right] = 1.0025$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3(1.0025) + 1.0109\right] = -0.9998$$

$$z^{(2)} = \frac{1}{20}\left[(25 - 2(1.0025) + 3(-0.9998)\right] = 0.9998$$

Third iteration :

$$x^{(3)} = \frac{1}{20}\left[17 - (-0.9998) + 2(0.9998)\right] = 0.99997$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3(0.99997) + 0.9998\right] = -1.0000055$$

$$z^{(3)} = \frac{1}{20}\left[(25 - 2(0.99997) + 3(-1.0000055)\right] = 1.0000022$$

Thus $x = 1$, $y = -1$, $z = 1$ is the required solution.

Problem 3

Employ Gauss - Seidel iteration method to solve

$$5x + 2y + z = 12$$

$$x + 4y + 2z = .15$$

$$x + 2y + 5z = 20$$

Carryout 4 iterations taking the initial approximation to the solution as (1 , 0 , 3)

Soln:

>> The given system of equations are diagonally dominant and we put them in the following form.

$$x = \frac{1}{5}[12 - 2y - z]$$

$$y = \frac{1}{4}[15 - x - 2z]$$

$$z = \frac{1}{5}[20 - x - 2y]$$

By data, $x^{(0)} = 1$, $y^{(0)} = 0$, $z^{(0)} = 3$

First iteration :

$$x^{(1)} = \frac{1}{5}[12 - 2(0) - 3] = 1.8$$

$$y^{(1)} = \frac{1}{4}[15 - 1.8 - 2(3)] = 1.8$$

$$z^{(1)} = \frac{1}{5}[20 - 1.8 - 2(1.8)] = 2.92$$

Second iteration :

$$x^{(2)} = \frac{1}{5}[12 - 2(1.8) - 2.92] = 1.096$$

$$y^{(2)} = \frac{1}{4}[15 - 1.096 - 2(2.92)] = 2.016$$

$$z^{(2)} = \frac{1}{5}[20 - 1.096 - 2(2.016)] = 2.9744$$

Third iteration :

$$x^{(3)} = \frac{1}{5}[12 - 2(2.016) - 2.9744] = 0.99872$$

$$y^{(3)} = \frac{1}{4}[15 - 0.99872 - 2(2.9744)] = 2.01312$$

$$z^{(3)} = \frac{1}{5}[20 - 0.99872 - 2(2.01312)] = 2.995$$

Fourth iteration :

$$x^{(4)} = \frac{1}{5}[12 - 2(2.01312) - 2.995] = 0.995752$$

$$y^{(4)} = \frac{1}{4}[15 - 0.995752 - 2(2.995)] = 2.003562$$

$$z^{(4)} = \frac{1}{5}[20 - 0.995752 - 2(2.003562)] = 2.9994248$$

Thus the solution after four iterations correct to four decimal places is given by

$$\boldsymbol{x = 0.9958, \quad y = 2.0036, \quad z = 2.9994}$$

LINEAR TRANSFORMATION:

A Linear transformation in two dimensions is given by

$y_1 = a_1x_1 + a_2x_2$

$y_2 = b_1x_1 + b_2x_2$

This can be represented in the matrix form as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =)\ Y = AX$$

Similarly a linear transformation in 3 dimensions along with its matrix form is as,

$y_1 = a_1x_1 + a_2x_2 + a_3x_3$

$y_2 = b_1x_1 + b_2x_2 + b_3x_3$

$y_3 = c_1x_1 + c_2x_2 + c_3x_3$

A is called transformation matrix

If $|A| \neq 0$ Then y=AX is called non-Singular transformation or regular transformation.

If $|A| = 0$ Then y=AX is called Singular transformation

X= $A^{-1}y$ is called the inverse transformation.

Let z=By =B(AX)=(BA)X=CX Where C=BA Z=CX is called a composite linear transformation .

1. Show that the transformation
   $y_1 = 2x_1 + x_2 + x_3;\quad y_2 = x_1 + x_2 + 2x_3 \quad y_3 = x_1 - 2x_3$ is regular. Write down the inverse transformation.

Sol: The given transformation may be written as

Y=AX

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} = 2(-2) - 1(-2-2) + 1(-1) = -4 + 4 - 1 = -1 \neq 0$$

$|A| \neq 0 \Rightarrow$ A is a non-singular matrix

The transformation is regular

The inverse transformation is X= $A^{-1}Y$

$$A^{-1} = \frac{adjA}{|A|} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$$X = A^{-1}Y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1 = 2y_1 + 2y_2 + y_3$

$x_2 = 4y_1 + 5y_2 + 3y_3$ is the inverse transformation.

$x_3 = y_1 - y_2 - y_3$

2. Prove that the following matrix is orthogonal

$$A\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$\text{Sol:Consider} \quad AA' = I = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. If A=1/3 $\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal. Find a,b,c & A^{-1} A is orthogonal $AA' = I$

$$Sol: \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \quad \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+4+a^2 & 2+2+ab & 2-4+ac \\ 2+2+ab & 4+1+b^2 & 4-2+bc \\ 2-4+ac & 4-2+bc & 4+4+c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$5+a^2 = 9$ $\quad$ $5+b^2 = 9$ $\quad$ $8+c^2 = 9$

$a^2 = 4$ $\quad$ $b^2 = 4$ $\quad$ $c^2 = 1$

$a = 2$ $\quad$ $b = 2$ $\quad$ $c = 1$

$$AA' = I \Rightarrow A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

4. Find the inverse transformation of the following linear transformation

$y_1 = x_1 + 2x_2 + 5x_3$

$y_2 = 2x_1 + 4x_2 + 11x_3$

$y_3 = -x_1 + 2x_3$

Sol: Y=AX

$$A^{-1}Y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 & -9 & 2 \\ -4 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1 = 19y_1 - 9y_2 + 2y_3$

$x_2 = -4y_1 + 2y_2 - y_3$ *is the inverse transformation*

$x_3 = -2y_1 + y_2$

5. Represents each of the transformation $y_1 = z_1 - 2z_2$ & $x_2 = -y_1 - 4y_2$, $y_2 = 3z$ by the use of matrix & find the composite transformation which express x_1, x_2 in terms of z_1, z_2

$Sol: x_1 = 3y_1 + 2y_2$

$x_2 = -y_1 + 4y_2$

$$\Rightarrow x = AY \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1 = z_1 + 2z_2$

$$y_2 = 3z_1 \quad \Rightarrow y = BZ \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$X = AY = A(BZ) = ABZ = AB\ Z$

$$AB = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$x_1 = 9z_1 + 6z_2$

$x_2 = 11z_1 + 2z_2$ is the required composite transformation

6.` Given the linear transformation

$y_1 = 5x_1 + 3x_2 + 3x_3$
$y_2 = 3x_1 + 2x_2 + 2x_3$
$y_3 = 2x_1 - x_2 + 2x_3$

$Sol: z_1 = 4x_1 + 2x_3$
$z_2 = x_2 + 4x_3$
$z_3 = 5y_3$

Express y_1, y_2, y_3
interms of z_1, z_2, z_3

Given : Y=AX

$$Z = BX \Rightarrow X = B^{-1}Z \qquad B^{-1}\begin{bmatrix} 1/4 & 0 & -1/10 \\ 0 & 1 & -4/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$Y = (AB^{-1})Z$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 5/4 & 3 & -23/10 \\ 3/4 & 1 & -23/10 \\ 1/2 & -1 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Eigen values and Eigen vectors of a square matrix:

Definition: Let A be a given square matrix of order n. suppose I a non-zero Column vector X of order n and real or complex no. λ Such that AX= λ x

Then X is called an Eigen vector of A.

λ is called the corresponding Eigen value of A.

Working Rule:

1. Given square matrix A write down the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation for Eigen values $\lambda_1, \lambda_2, \lambda_3, ...$
3. To find Eigen vector, write down the matrix equation as

$$A - \lambda I \quad X = 0 \quad where\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

4. We set $\lambda = \lambda_1$ in the matrix equation & solve it for Eigen vector x_1. Similarly we obtain Eigen vector $x_2, x_3.....$ for corresponding Eigen value $\lambda_2, \lambda_3.....$

1. Find the Eigen values & corresponding eigen vector of the following matrix

$$A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$$

$$Sol: |A - \lambda I| = \begin{vmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{vmatrix} = (-3-\lambda)(7-\lambda)+16$$

$$= \lambda^2 - 4\lambda - 5$$

$$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$$

The roots of this equation are λ_1=5 & λ_2=-1. These are the two Eigen value of the given matrix A. Let X= x, y^T, then the matrix equation $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = \lambda_1 = 5$

$$\begin{bmatrix} -8 & 8 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-8x+8y=0, -2x+2y=0. Both of these reduces to some equation x-y=0 =) x=y. If we choose x=a, then y=a These, when $\lambda = \lambda_1 = 5, x_1 = \begin{bmatrix} a \\ a \end{bmatrix}$ is the solution of (1)

$\lambda_0 = \lambda_2 = -1$ equation (1) becomes

$$\begin{bmatrix} -2 & 8 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $-2x + 8y = 0,\ x - 4y = 0$. Hence if we choose y=b, then x=4b

Thus when $\lambda = \lambda_2 = -1,\ x_2 = \begin{bmatrix} Ab \\ b \end{bmatrix}$ is the soln of (1)

2. Find the Eigen values & the Eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Sol: For the given matrix, the characteristic polynomial is

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$= (2-x)^3 - (2-\lambda)$$

$$= (\lambda - 3)(2-\lambda)(\lambda - 1)$$

The characteristic equation of the given matrix is

$(2-\lambda)(\lambda-3)(\lambda-1)=0$

The roots are $\lambda_1=1 \quad \lambda_2=2 \quad \& \ \lambda_3=3$ these are the Eigen value of the given matrix.

$[x, y, z]^T = X$, then the matrix equation (A-λI)x=0

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

$For\ \lambda=\lambda_1=1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x+z=0 \ \& \ y=0$

If we choose x=a, then z=-a; $x_1 = [a_1 \ 0_1 \ -a]^T$

$If\ a=1 \quad x_1 = [1\ 0\ -1]^T$

$$For\ \lambda=\lambda_2=2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x=0,\ z=0.$ we take y=b; $x_2 = [0\ \ b\ \ 0]^T$

$If\ b=2 \quad x_2 = [0\ \ 2\ \ 0]^T$

Let

$$For\ \lambda=\lambda_3=3 \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x+z=0,\ y=0.$ we take x=c; then z=c $x_3 = [c\ \ o\ \ c]^T$

$If\ c=3 \quad x_3 = [3\ \ 0\ \ 3]^T$

3. Find the matrix P which reduces the matrix $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form

Hence find A^4

$$|A-\lambda I| = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$$

$\lambda_1=-2 \quad \lambda_2=3 \quad \lambda_3=6$

$$|A-\lambda I| = \begin{bmatrix} 1\text{-}\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$$

$\lambda_1 = -2 \quad \lambda_2 = 3 \quad \lambda_3 = 6$

$$\text{Case(1):} \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x + y + 3z = 0$

$x + 7y + z = 0$

$$\frac{x}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x}{-20} = \frac{-y}{0} = \frac{z}{20}$$

$Case ii): \lambda_2 = 3$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 3z = 0$

$x + 2y + z = 0$

$3x + y - 2z = 0$

$$\frac{x}{\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5}$$

$Case iii): \lambda_3 = 6$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x + y + 3z = 0$

$x - y + z = 0$

$3x + y - 5z = 0$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 1 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{4} = \frac{y}{-8} = \frac{z}{4}$$

$$\frac{x}{4} = \frac{-y}{-8} = \frac{z}{4}$$

$$p = \; x_1\, x_2\, x_3 \; = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$p^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$p^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

$$A^4 = PD^4 p^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

5) Diagonalizable the matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ *and find* A^5

$$(A - \lambda I) = 0$$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda^3 - 2\lambda = 0$$

$$-\lambda\left[\lambda^2 - 3\lambda + 2 \right. = 0$$

$$x = 0 \quad \lambda = 1 \quad \lambda = 2$$

$Case i : \lambda = 0$

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$11x - 4y - 7z = 0$

$7x - 2y - 5z = 0$

$10x - 4y - 6z = 0$

$$\frac{x}{\begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & -5 \\ 10 & -6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 7 & -2 \\ 10 & -4 \end{vmatrix}}$$

$$\frac{x}{-8} = \frac{-y}{8} = \frac{z}{-8}$$

$Case ii : \lambda = 1$

$$\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$10x - 4y - 7z = 0$

$7x - 3y - 5z = 0$

$10x - 4y - 7z = 0$

$$\frac{x}{\begin{vmatrix} -4 & -7 \\ -3 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 10 & -7 \\ 7 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 10 & -4 \\ 7 & -3 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2}$$

$Case iii : \lambda = 2$

$$\begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$9x - 4y - 7z = 0$

$7x - 4y - 5z = 0$

$10x - 4y - 8z = 0$

$$\frac{x}{\begin{vmatrix} -4 & -5 \\ -4 & -8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & -5 \\ 10 & -8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 7 & -4 \\ 10 & -4 \end{vmatrix}}$$

$$\frac{x}{12} = \frac{-y}{-6} = \frac{z}{12}$$

$$p = \; x_1 \, x_2 \, x_3 \;= \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$p^{-1} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$p^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

$$A^5 = PD^5 p^{-1} = \begin{bmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{bmatrix}$$

Quadratic forms:

A homogeneous expression of the second degree in any number of variables is called a quadratic form (Q.F).

In general for two variables $x_1, x_2 \;\; i.e, n = 1, 2 \;\; a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ is called QF in two variables.

The matrix A of the above Q.F is $A = \begin{bmatrix} coeff \; of \; x_1^2 & \frac{1}{2} \; coeff \; of \; x_1x_2 \\ \frac{1}{2} \; coeff \; of \; x_1x_2 & coeff \; of \; x_1^2 \end{bmatrix}$

$Eg: x^2 + y^2 + xy \qquad A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

2) $2x^2 + 3y^2 + 6xy \qquad A = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$

3) $5x_1^2 + 7x_2^2 + 12\,x_1x_2 \qquad A = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$

Similarly Q.F in 3 variables is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_2 + 2a_{23}x_1x_2$$

$$A = \begin{bmatrix} coeff\ of\ x_1^2 & \frac{1}{2}coeff\quad x_1x_2 & \frac{1}{2}coeff\quad x_1x_3 \\ \frac{1}{2}Coeff\ of\quad x_1x_2 & coeff\ of\ x_1^2 & \frac{1}{2}coeff\quad x_2x_3 \\ \frac{1}{2}Coeff\ of\quad x_1x_2 & \frac{1}{2}coeff\quad x_2x_3 & coeff\ of\ x_3^2 \end{bmatrix}$$

$$Examples: 1)\ QF: x^2 + y^2 + z^2 + xy + 2yz = 4zx \qquad A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{1}{2} & 1 & \\ 2 & 1 & 1 \end{bmatrix}$$

$$2)\ 5x^2 + 2yz + 6y^2 + 9z^2 + 4xy \qquad A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 9 \end{bmatrix}$$

$$3)\ xy + yz + zx \quad A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Canonical Form (sum of squares):

$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + ... + \lambda_n y_n^2$ *where* $\lambda_1, \lambda_2, ...\lambda_n$ *are* Eigen values called canonical form.

Rank Index & Signature of canonical form

The number of non-zero terms present in a canonical form of Q is called rank of Q it, (r)

$Ex: 2y_1^2 + y_2^2 - y_3^2 \Rightarrow r = 3 \quad y_1^2 + y_3^2 \Rightarrow r = 2$

1. The number of the terms present in a canonical form is called index of Q. (p)

$Ex: 2y_1^2 + 3y_2^2 - 5y_3^2 \quad p = 2$

2. The difference between the negative terms in a canonical form is called signature of Q (s)

$Ex: y_1^2 - 3y_2^2 + y_3^2 \quad s = 2 - 1 = 1$

Nature of Quadratic Form:

r=rank, p=index n=number of variables

Condition	Meaning	Nature of Q.F	Eg;
r=n,p=n	All n Co efficient are positive	+ve definite	$2y_1^2 + y_2^2 + 8y_3^2 +$
r=n,p=0	All n Co efficient are -ve	-ve definite	$-y_1^2 - y_2^2 - 6y_3^2$
r=p,p<n for r=2=p 2<3	At least one of the Co-efficient Zero & all other Co-efficientp are +ve	+ve Semi definite	$y_2^2 + 5y_3^2$
r<n,p=0	At least one of the Co-efficient Zero & all other Co-efficient are -ve	-ve Semi definite	$-y_2^2 - 10y_3^2$

Note: Q.F is indefinite if some of the Co-efficient are +ve and some are -ve

Eg: $y_1^2 - y_2^2 + 3y_3^2$

Working rule to reduce Q.F to Canonical (sum of squares) form by orthogonal transformation.

1. Write down the matrix A to Q.F
2. Find the Eigen values & the corresponding eigen vectors of matrixA.
3. Normalise the Eigen vector x_1, x_2, x_3

$$i.e, x_1^1 = \frac{x_1}{\|x_1\|}$$

$$If \;\; x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \|x_{11}\| = \sqrt{a^2 + b^2 + c^2}$$

$$111^{iy} \; x_2^1 = \frac{x_2}{\|x_2\|} \qquad x_3^1 = \frac{x_3}{\|x_3\|}$$

4. Write down the associated orthogonal model matrix $Q = \begin{bmatrix} x_1^1 & x_2^1 & x_3^1 \end{bmatrix}$
5. Since $pp^1 = I \Rightarrow p^{-1} = p^1$

   Then $p^{-1}AP = p^1AP =$ diagonal matrix $\lambda_1, \lambda_2, \lambda_3$

6. The associated canonical form is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

7. x=py where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \& Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

   will give us the orthogonal linear transformation.

1) Obtain the canonical form of the quadratic form

   $Sol: 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$

   $$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

   $(A - \lambda I) = 0$

   $$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} = 0$$

   $(2-\lambda)\ \ (2-\lambda)^2 - 1\ \ +1\ \ -(2-\lambda)-1\ \ -1\ \ 1+(2-\lambda)\ \ = 0$

   $(2-\lambda)\ \ 4+\lambda^2 - 4\lambda - 1\ \ -\ \ 2-\lambda+1\ \ -\ \ 3-\lambda\ \ = 0$

   $2-\lambda\ \ \ \lambda^2 - 4\lambda + 3\ \ -\ \ 3-\lambda\ \ -3+\lambda = 0$

   $2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda - 3 + \lambda - 3 + \lambda = 0$

$-\lambda^3 + 6\lambda^2 - 9\lambda = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda = 0$

$\lambda(\lambda^2 - 6\lambda + 9) = 0 \quad \lambda(\lambda - 3)^3 = 0$

$\lambda = 0, 3, 3\ i.e, \lambda_1 = 0,\ \lambda_2 = 0,\ \ \lambda_3 = 0$ are roots and are the Eigen values of A.

2) The canonical form of the given Q.P that we get by an orthogonal transformation is $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 3y_2^2 + 3y_3^2$

   Sol: Since one Co- efficient in this canonical form is zero & the other two are +ve, the Q.F is +ve Semi-definite

   Rank, r=2 Index, p=2 & Signature, s=2.

3) Reduce the Q.F $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to the canonical form by an orthogonal transformation

   $$Sol: A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$(A-\lambda I)=0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}=0$$

$$(1-\lambda)\ \ (3-\lambda)^2-1\ \ -0=0$$

$$(1-\lambda)\ \ 9+\lambda^2-6\lambda-1\ \ =0$$

$$(1-\lambda)\ \ \lambda^2-6\lambda+8\ \ =0$$

$$\lambda^2-6\lambda+8-\lambda^3+6\lambda^2-8\lambda=0$$

$$-\lambda^3+7\lambda^2-14\lambda+8=0 \Rightarrow \lambda^3-7\lambda^2+14\lambda-8=0$$

$$(\lambda-1)\ \ (\lambda-2)\ \ (\lambda-4)=0$$

The eigen values of A are $\lambda_1=1\quad \lambda_2=2\quad \lambda_3=4$

$Case i: For\ \lambda_1=1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}}=\frac{z}{\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{3}=\frac{-y}{0}=\frac{z}{0}$$

$$x_1=\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$11x_1 11=\sqrt{1^2+0+0}=1$$

$$x_1^1=\frac{x_1}{\|x_1\|}=\frac{100^{\ T}}{1}=\ 1\quad 0\quad 0^{\ T}$$

$$\lambda_2=2$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}=\frac{z}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{0}=\frac{-y}{-1}=\frac{z}{+1}$$

$$x_2=\begin{bmatrix}3\\1\\+1\end{bmatrix}$$

$$11x_2 11=\sqrt{0+1+1}=\sqrt{2}$$

$$x_2' =\frac{x_2}{\|x_2\|}=\frac{1}{\sqrt{2}}=\ 0\quad 1\quad +1\ ^T=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}^T$$

$\lambda_3=4$

$$\begin{bmatrix}-3 & 0 & 0\\0 & -1 & -1\\0 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x}{\begin{vmatrix}-1 & -1\\-1 & -1\end{vmatrix}}=\frac{-y}{\begin{vmatrix}-3 & 0\\0 & -1\end{vmatrix}}=\frac{z}{\begin{vmatrix}-3 & 0\\0 & -1\end{vmatrix}}$$

$$\frac{x}{0}=\frac{-y}{3}=\frac{z}{3}$$

$$x_1=\begin{bmatrix}0\\-3\\3\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$11x_3 11=\sqrt{0+1+1}=\sqrt{2}$$

$$x_3^1=\frac{x_3}{\|x_3\|}=\frac{0\ -1\ \ 1\ ^T}{\sqrt{2}}=\begin{bmatrix}1 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^T$$

The orthogonal model matrix for A is

$$Q=\begin{bmatrix}x_1^1 & x_2^1 & x_3^1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

X=Q.F is the orthogonal transformation that reduces the given Q.F to the canonical form.
The canonical form is

$$\lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2=y_1^2+2y_2^2+4y_3^2$$

Rayleigh's power method

Rayleigh's power method is an iterative method to determine the numerically largest eigen value and the corresponding eigen vector of a square matrix.

Working procedure:

- Suppose A is the given square matrix, we assume initially an eigen vector X_0 in a simple for like $[1,0,0]'$ *or* $[0,1,0]'$ *or* $[0,0,1]'$ *or* $[1,1,1]'$ and find the matrix product AX_0 which will also be a column matrix.
- We take out the largest element as the common factor to obtain $AX_0 = \lambda^1 X^1$.
- We then find AX^1 and again put in the form $AX^1 = \lambda^2 X^2$ by normalization.
- The iterative process is continued till two consecutive iterative values of λ and X are same upto a desired degree of accuracy.
- The values so obtained are respectively the largest eigen value and the corresponding eigen vector of the given square matrix A.

Problems:

1) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ given } [\text{redacted}]\ 0\ \ 0]^T$$

$$Solution: \ AX^0 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^1 X^1$$

$$AX^1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^2 X^2$$

$$AX^2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99\begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.994 \end{bmatrix} = 2.997\begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^7 X^7$$

Thus the **largest eigen value** is approximately **3** and the corresponding **eigen vector** is [1 ,0, 1]'

2) Using the Power method find the largest eigen value and the corresponding eigen vector starting with the given initial vector.

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \textit{given} \left[1 \quad 0.8 \quad -0.8\right]^T$$

$$Solution: \ AX^0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^1 X^1$$

$$AX^1 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^2 X^2$$

$$AX^2 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \begin{bmatrix} 5.994 \\ 5.988 \\ -5.988 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.999 \\ -0.999 \end{bmatrix} = \lambda^5 X^5$$

Thus after five iterations the numerically largest eigen value is **5.994** and corresponding eigen vector is **[1, 0.999, -0,999]'**

6) Using Rayleigh's power method to find the largest Eigen value and the corresponding Eigen vector of the matrix.
(Dec 2012)

Sol: $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad X^{(0)} = [1, 0, 0]'$

$$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0.3333 \end{bmatrix} = \begin{bmatrix} 7.3332 \\ -3.3332 \\ 3.3332 \end{bmatrix} = 7.3332\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.4545 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 7.818 \\ -3.818 \\ 3.818 \end{bmatrix} = 7.818\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.488 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 7.952 \\ -3.952 \\ 3.952 \end{bmatrix} = 7.952\begin{bmatrix} 1 \\ -0.4969 \\ 0.4969 \end{bmatrix}$$

The largest Eigen value is $\lambda = 7.952$ and the corresponding Eigen vector is

$$\begin{bmatrix} 1 & -0.4969 & 0.4969 \end{bmatrix}'$$