STRENGTH OF MATERIALS
(COMMON TO CV/TR/EV/CTM)

Sub Code : 10 CV 33 **IA Marks : 25**
Hrs/ Week : 04 **Exam Hours : 03**
Total Hrs. : 52 **Exam Marks : 100**

PART – A

UNIT 1:

Simple Stress and Strain

1.1 Introduction, 1.2 Properties of Materials, 1.3 Stress, Strain, Hook's law, 1.4 Poisson's Ratio, 1.5 Stress – Strain Diagram for structural steel and non ferrous materials, 1.6 Principles of superposition, 1.7 Total elongation of tapering bars of circular and rectangular cross sections. Elongation due to self – weight

7 Hours

UNIT 2:

Simple Stress and Strain (continued…)

2.1 Composite section, 2.2 Volumetric strain, expression for volumetric strain, 2.3 Elastic constants, relationship among elastic constants, 2.4 Thermal stresses (including thermal stresses in compound bars).

6 Hours

UNIT 3:

Compound Stresses

3.1 Introduction, 3.2 Stress components on inclined planes, 3.3 General two dimensional stress system, 3.4 Principal planes and stresses, 3.5 Mohr's circle of stresses.

8 Hours

UNIT 4:
Bending Moment and Shear Force in Beams

4.1 Introduction, 4.2 Types of beams loadings and supports, 4.3 Shearing force in beam, 4.4 Bending moment, 4.5 Sign convention, 4.6 Relationship between loading, shear force and bending moment, 4.7 Shear force and bending moment equations, SFD and BMD with salient values for cantilever beams, simply supported beams and overhanging beams considering point loads, UDL, UVL and Couple.

7 Hours

PART B

UNIT 5:
Bending Stress, Shear Stress in Beams

5.1 Introduction – Bending stress in beam, 5.2 Assumptions in simple bending theory, 5.3 Pure bending derivation of Bernoulli's equation, 5.4 Modulus of rupture, section modulus, 5.5 Flexural rigidity, 5.6 Expression for horizontal shear stress in beam, 5.7 Shear stress diagram for rectangular, symmetrical 'I' and 'T' section (Flitched beams not included).

6 Hours

UNIT 6:
Deflection of Beams

6.1 Introduction – Definitions of slope, deflection, 6.2 Elastic curve derivation of differential equation of flexture, 6.3 Sign convention 6.4 Slope and deflection for standard loading classes using Macaulay's method for prismatic beams and overhanging beams subjected to point loads, UDL and Couple.

6 Hours

UNIT 7:
Torsion of Circular Shafts

7.1 Introduction – Pure torsion-torsion equation of circular shafts, 7.2 Strength and stiffness, 7.3 Torsional rigidity and polar modulus, 7.4 Power transmitted by shaft of solid and hollow circular sections.

6 Hours

UNIT 8:
Elastic Stability of Columns
8.1 Introduction – Short and long columns, 8.2 Euler's theory on columns, 8.3 Effective length slenderness ration, 8.4 radius of gyration, buckling load, 8.5 Assumptions, derivations of Euler's Buckling load for different end conditions, 8.6 Limitations of Euler's theory, 8.7 Rankine's formula and problems.

6 Hours

TEXT BOOKS:
1. **Strength of Materials,** Subramanyam, Oxford University Press, Edition 2008
2. **Mechanics of Materials**, B.C Punmia Ashok Jain, Arun Jain, LakshmiPublications, New Delhi.
3. **Strength of Materials**, Basavarajaiah and Mahadevappa Universities Press (2009).

REFERENCE BOOKS:
1. **Strength of Materials**, Singer Harper and Row Publications.
2. **Elements of Strength of Materials**, Timoshenko and Young Affiliated East west Press
3. **Mechanics of Materials,** James M. Gere (5th Edition), Thomson Learning.

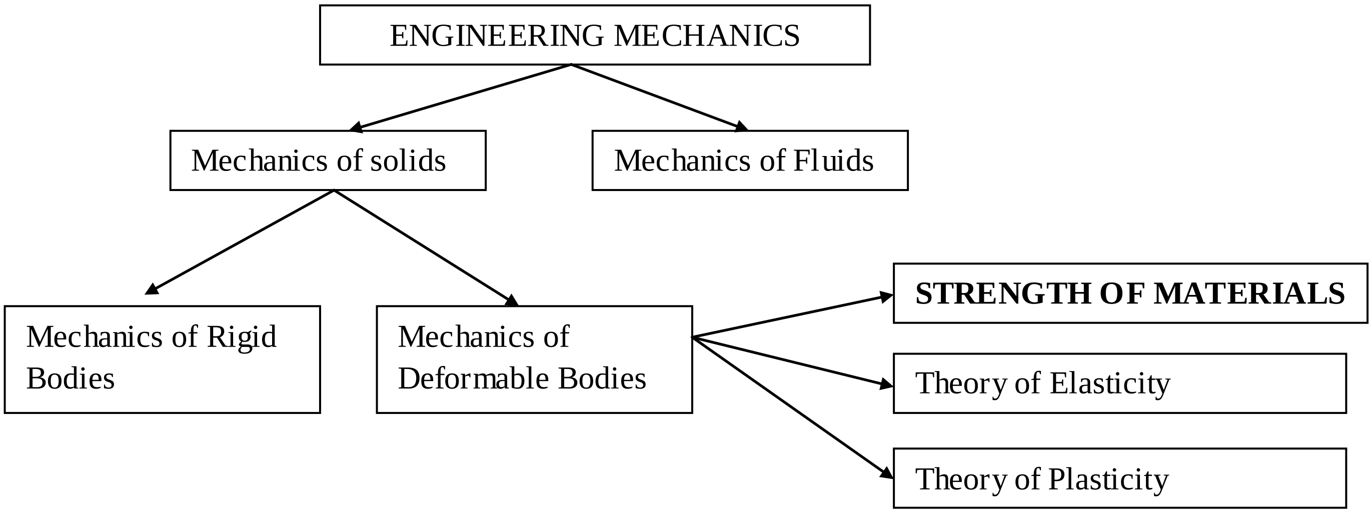
www.rejinpaul.com

CONTENTS

UNIT -I

SIMPLE STRESSES AND STRAINS

1. CLASSIFICATION OF ENGINEERING MECHANICS



2. INTRODUCTION

2.1. Definition

Strength of materials is a branch of engineering mechanics which deals with the effects of forces applied on the bodies or structures or materials which are deformable in nature.

It deals with the relations between the externally applied loads or forces and the internal effects in the body. In day to day work, we come across bodies or members such as beams, columns Shafts etc which are made up of Steel, Concrete, Timber, Aluminum etc

When materials are loaded they first deform before actual failure takes place. Hence before selecting any material for engineering purpose, it is important to know the behavior of the material under the action of loads and also the strength of the material.

The assessment of the strength and behavior of the materials can be done by knowing the various properties of the materials such as rigidity, Plasticity, Elasticity Etc

2.2 Assumptions in Strength of Materials

1. All bodies are **deformable**.
2. Materials are perfectly **elastic**.
3. Materials are **isotropic** and **homogenous**.
4. **Principle of superposition** is valid

www.rejinpaul.com

5. **St. Venant's Principle** is valid. ("At points away from the loading points, the behaviour of material will be independent of gripping forces or type of application of load or local effects.")

2. SIMPLE STRESSES AND STRAINS

When a material is subjected to external forces or loads, it tends to deform. However due to cohesion between the particles in the body, the material offers resistance to deformation. (If the force is increased the resistance as well as the deformation also increases).

(The material will offer necessary resistance to deformation when the load is within a certain limit). When material is not capable of offering necessary resistance against external forces, permanent deformation will set in and failure of the member may occur, if the load is increased.

3.1. Stress (p, f, σ)

Whenever some external forces acts on a body it sets up a deformation and the body inturn offers some resistance against deformation . this resistance per unit area to deformation is known as “stress”.

Mathematically stress is defined as the force per unit area.

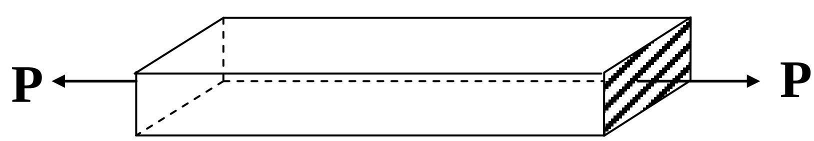


Fig.1

In the above case the stress action on the body is given as

p = P/A, Where,

p - Intensity of stress

A - cross sectional area

P - Applied load

Unit – N/mm^2, kN/m^2, N/cm^2 etc

www.rejinpaul.com

3.2. Types of Stress:-

3.2.1. Tensile Stress (p_t)

When a load applied on a body tends to pull the material away from each other, causing extension of the body in the direction of the applied load, the load is called a Tensile load and the corresponding stress induced is called Tensile stress.

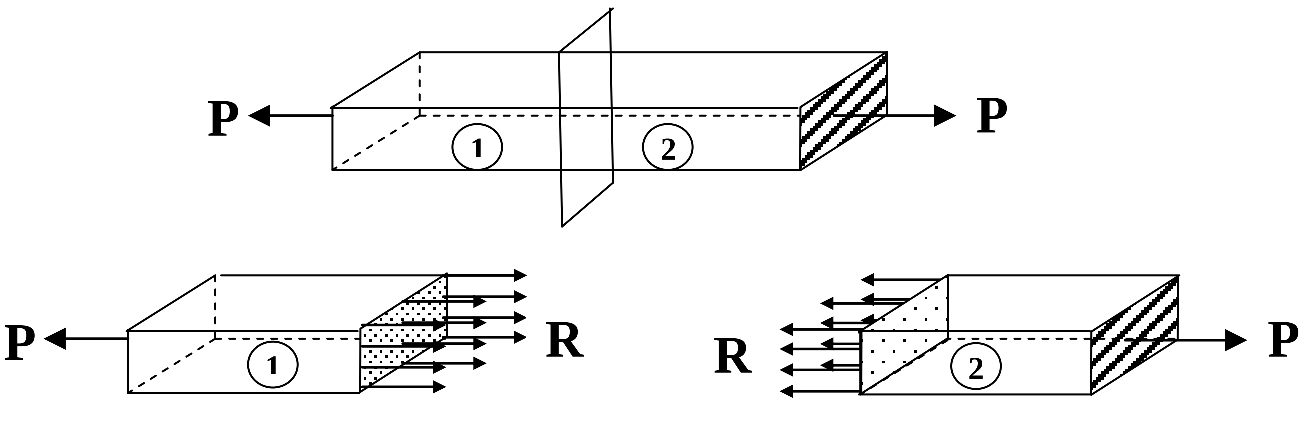


Fig 2

Let us consider a uniform bar of cross sectional area 'A' and subjected to an axial pull 'P' at both ends. Let us consider a section to divide the bar into two parts. For equilibrium the two portions of the bar at the sectional plane, a resisting force 'R' is developed due to equilibrium R= P.

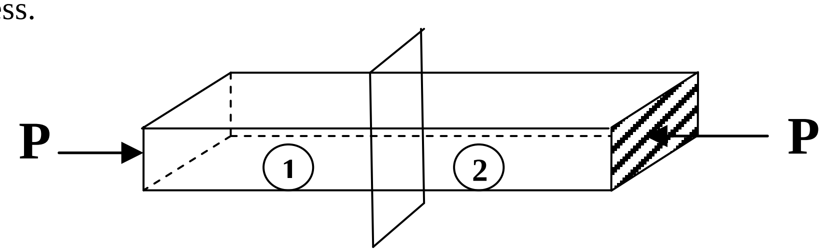
Therefore tensile stress, σ_t =R/A=P/A

Units of stress –

In SI system the units are expressed in N/mm^2, kN/m^2, N/cm^2 (it is also expressed in Pascal 1 pascal = 1N/m^2)

3.1.2. Compressive Stress:- (p_c or f_c or σ_c)

When a load applied on a body tends to push the particles of the material closer to each other, causing shortening of the body in the direction of the applied forces, the applied force is known as compressive force and the corresponding stress is known as compressive stress.



www.rejinpaul.com

Fig-3

The compressive stress at any section along the length of the load is given as

$$\sigma_c = \frac{Resisting\ force}{Cross\ sectional\ area}$$

$$\sigma_c = \frac{R}{A} = \frac{P}{A}$$

Units – N/mm^2, kN/m^2, N/cm^2 etc

3.1.3. Shear stress:- (τ or q)

When a load applied on a body causes one portion of the body to slide over the adjoining portion, such a force is known as Shear Force and the corresponding stress is known as Shear Stress.

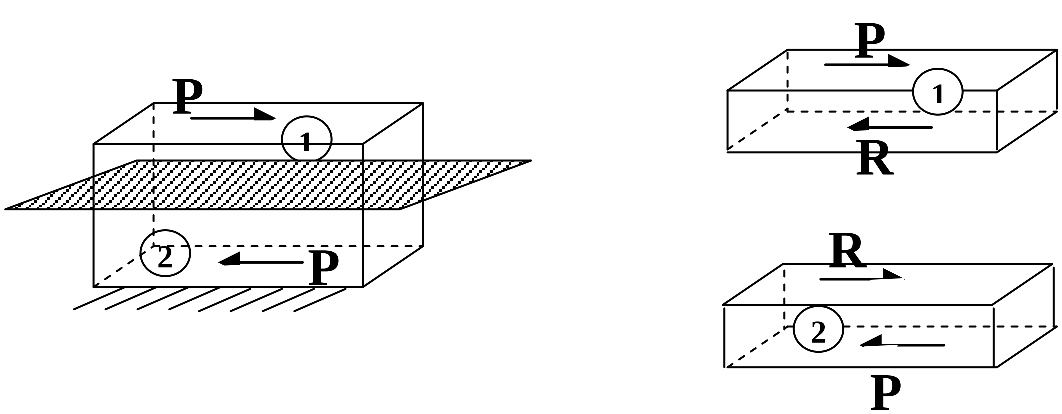


Fig-4

As shown in the fig, the body might separate into two portions causing one portion to slide over the another. At the plane of separation a resisting force 'R' is developed. Thus shear stress is given as

$$\tau = \frac{Resisting\ force}{Applied\ Area}$$

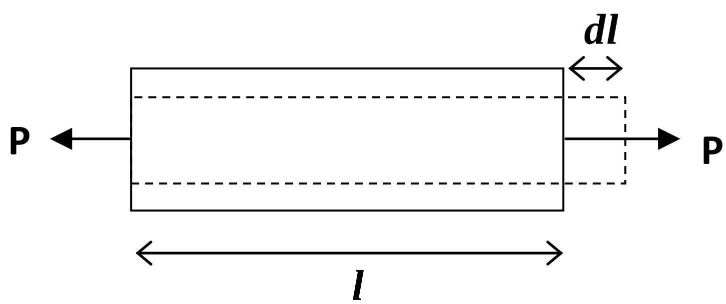
$$\tau = \frac{R}{A} = \frac{P}{A}$$

In the above, Tensile stresses and compressive stresses are known as Direct stresses. Whereas Shear stresses is known as Tangential stresses.

3.2. Strain:- (e or ϵ)

When a force is applied on a body, the body changes its dimension, The measure of deformation is known as strain.

Mathematically, strain is defined as the ratio of change in length to the original length. It is a dimensionless quantity

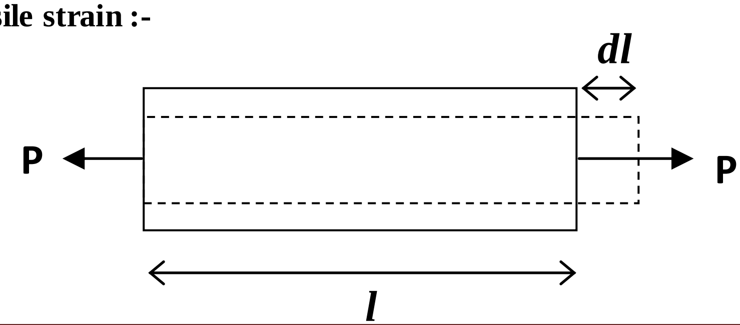


Consider, a bar subjected to an axial force P. Let l be the original length of the bar. Let (l+dl) be the new length of the bar, such that dl represents change in length. Therefore, Strain

$$\epsilon = \frac{dl}{l}$$

The strain depends on the nature of the load acting on a body stresses are induced in the body and we can observe, the following types of strains.

3.2.1 Tensile strain :-

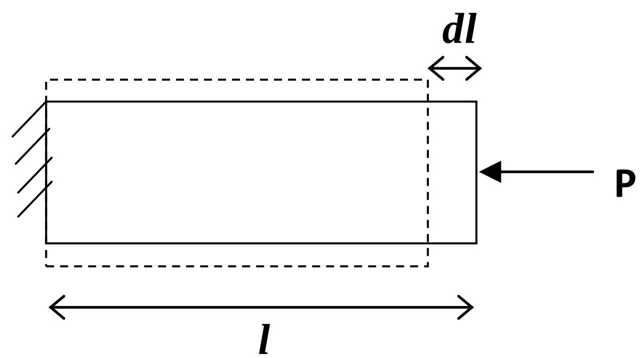


When a tensile force acts on a body, it elongates in the direction of force P by an amount of dl and then the strain is called Tensile Strain.

Thus, tensile strain is given as

$$\epsilon = +\frac{dl}{l}$$

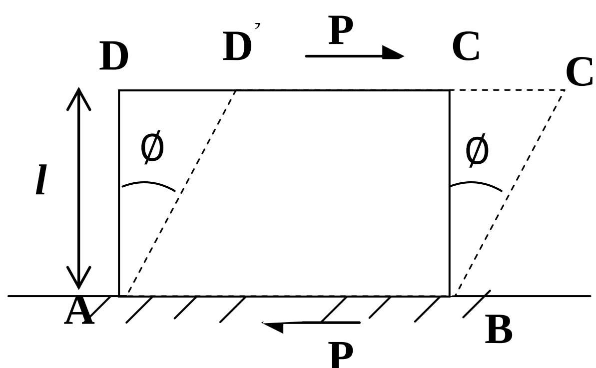
3.2.2. Compressive strain:



When an axial compressive force acts on a body causing shortening of the body by an amount 'dl' in the direction of the applied force, the strain observed is called compressive strain and it is given as

$$\epsilon = -\frac{dl}{l}$$

3.2.3. Shear strain :-



Fig

Consider an element of unit thickness subjected to a shear force as shown in fig. The body deforms as shown (Point D shifts to D' and Point C shifts to C')

Let Φ represent the angular rotation of the vertical faces. Let dl represent the horizontal on transverse displacement of the upper face with respect to the lower face. This displacement occurs over a length 'l'. In such a case shear strain is defined as,

$$\text{Shear Strain} = \frac{Transverse\ displacement}{Distance\ from\ lower\ face(perpendicular\ height)}$$

From the triangle ADD', $\tan\phi = \frac{dl}{l}$

For very small values of Φ we have $\tan\phi = \phi$

$$\phi = \frac{dl}{l}$$

This implies that, shear strain can be measures directly by measuring the angular rotation Φ (Φ is measured in radians)

4. Elasticity

When an external force acts on a body, the body tends to deform and deformation continues till full resistance to external forces is set up. Once the load causing deformation is removed, the body returns to its original shape. This property by virtue of which a body regains its original shape after the external forces are removed is called Elasticity.

A material is said to be perfectly elastic if it regains its original shape completely.

Steel, Copper, Aluminum, Brass, Concrete and wood are considered to be perfectly elastic within certain limits.

Note:-

1. A homogeneous material is one which is made up of same kind if material throughout.
2. An isotropic material is one which has the same elastic properties in all direction.

4.1. Elastic Limit

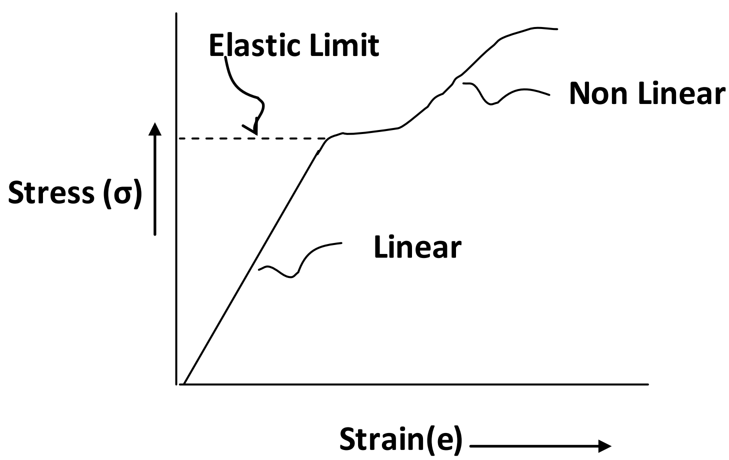


Fig 5

When any material is loaded or stressed, it deforms, when the load is removed, the body regains its original shape. However if the stress is increased beyond a certain value or limit, the material fails to regain its original shape even when the load is removed. This limiting stress upto which the material behaves as an elastic material is called elastic limit.

5. Hook's Law

It states that "Within the elastic limit the stress is directly proportional to strain"

i,e. Stress α strain

Stress = Strain x constant

$$\frac{Stress}{strain} = constant$$

This constant is called Elastic modulus or Modulus of Elasticity or Young's Modulus and it is denoted as E

$$\therefore \mathbf{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\epsilon}}$$

i,e Young's modulus has the same unit as the stress

$$\text{Consider } E = \frac{\sigma}{\epsilon}$$

$$\text{But } \sigma = \frac{P}{A} \text{ and } \epsilon = \frac{dl}{l}$$

$$\therefore \mathbf{E} = \frac{\frac{P}{A}}{\frac{dl}{l}}$$

$$\mathbf{E} = \frac{\boldsymbol{P.l}}{\boldsymbol{A.dl}}$$

$$\boldsymbol{dl} = \frac{\boldsymbol{P.l}}{\boldsymbol{A.E}} \quad \textbf{or} \quad \boldsymbol{dl} = \frac{\boldsymbol{\sigma.l}}{\boldsymbol{E}}$$

Whenever a force is acting on a body whose cross sectional area is A and length l and E is its elastic modulus, the change in length is given by

$$\boldsymbol{dl} = \frac{\boldsymbol{P.l}}{\boldsymbol{A.E}} = \frac{\boldsymbol{\sigma\, l}}{\boldsymbol{E}}$$

Values of Young's modulus E for different materials

Material	Value of E
1) Steel	210 x 10^3 N/mm^2
2) Wrought Iron	190 x 10^3 N/mm^2
3) Cast Iron	120 x 10^3 N/mm^2
4) Copper	90 x 10^3 N/mm^2
5) Brass	80 x 10^3 N/mm^2
6) Aluminum	70 x 10^3 N/mm^2
7) Wood	10 x 10^3 N/mm^2

6. Principle of superposition

When a number of loads are acting on a body the resulting strain according to principle of superposition will be "The algebraic sum of the strains caused by the individual forces"

If an elastic body is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body , deformations the individual sections can be calculated if the free body diagrams of the individual sections are considered separately. The net (total) deformation is sum of the individual deformations.

7. Mechanical properties of materials

The following are considered as the most important properties of engineering materials

1) Elasticity
2) Plasticity
3) Ductility
4) Malleability
5) Brittleness
6) Toughness
7) Hardness

Any material cannot posses all the above properties because the different properties oppose each oter. Hence the engineering Materials can be classified as follows depending upon their Mechanical properties

a) Elastic Materials:- These are materials which undergo deformation due to application of forces and once the forces are removed the material regains its original shape.
b) Plastic materials :- These are materials which do not regain their original slope even after the external loads acting on the mare removed

c) Ductility :- these are material that can undergo considerable deformation without much increase in he load od\r in simple terms, these are materials that can be drawn into wires
d) Malleable materials:- These are Materials which can be extended in two directions easily or in simple terms, materials which can be beaten into thin sheets.
e) E) Brittle materials:- these are materials which do not undergo any deformation before failure when external forces act on them.
f) Tough materials :- These are materials which can resist sudden loads or shock loads without showing any fracture on failure
g) Hard material:- These are materials that have the ability to resist surface abrasion or indentation (Markings)

Various tests are carried out on engineering materials to assess their mechanical properties in a material testing laboratory. They are

1) Tension test
2) Compression test
3) Impact test
4) Shear test
5) Torsion test
6) Bending test
7) Fatigue test
8) Hardness test

8. St Venant's Principle:-

It states that, "In a bar carrying direct or normal loads, except in the extreme end regions of the bar, the stress distribution over the cross section is uniform".

If we consider a bar of uniform cross section (b x b) and subjected to direct axial load P, we can consider three different sections at different distances from the extreme as in fig. The stress distribution at different distances is different and is represented in each case as shown above.

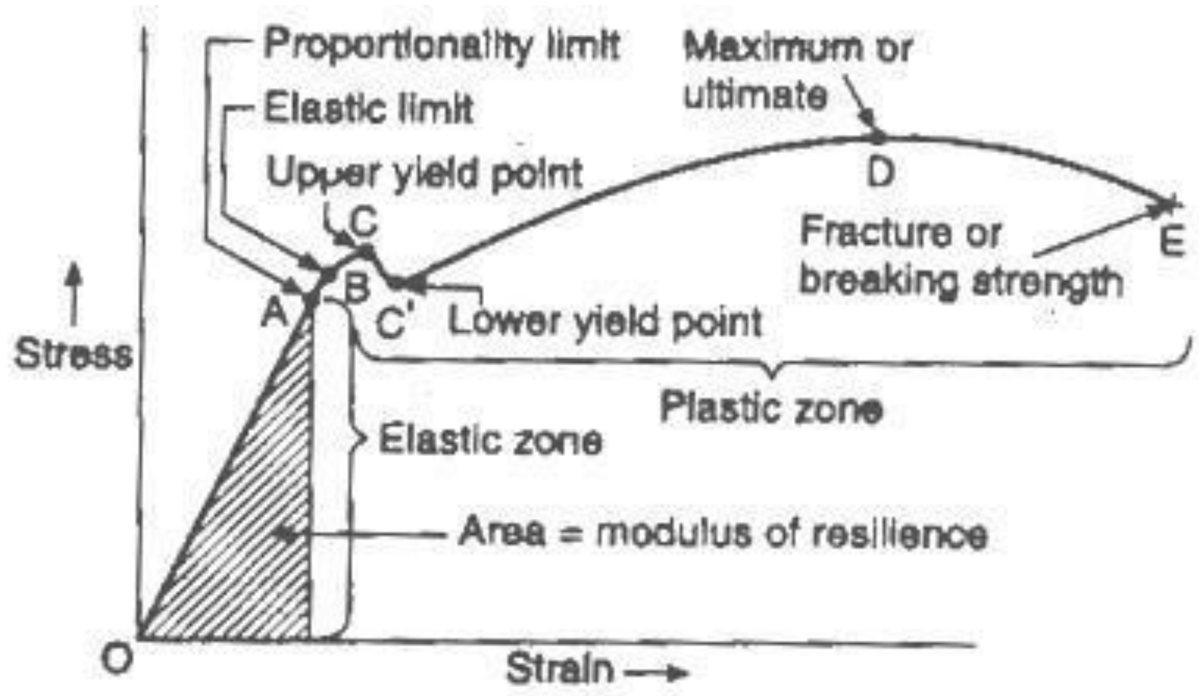
We see that the stress distribution is uniform at section like 3. This stress distribution is possible if the load P acts through the centroid of the cross section or axially.

9. Tension Test on Mild Steel specimen:-

A typical tensile test specimen on mild steel is as shown above. The ends of the specimen are gripped into a universal testing machine.

The specimen has lager diameter at the ends to see that the specimen at the ends to see that the specimen does not fail in the end regions. The specimen should fail in the gauged portion. The deformation is recorded as the load is applied(increases).

The elongation is recorded with the help of strain gauges. The loading is done till the specimen fails. A graph of stress verses strain is plotted and a typical stress strain curves for mild steel is as follows.



Fig

Point A -Proportionality Limit
Point B- Elastic Limit
Point C- Lower Yield point
Point C'-Upper yield Point
Point D- Ultimate stress
Point E- Failure or Rupture or Breaking stress

From the graph the plot from O-A is linear i,e the stress is proportional to the strain within this limit as such A is called proportionality limit. "Hooks Law" is valid in this region. On further increase in load, the curve takes a fall as represented from A to C. Between A and C there exists a point B. The material behaves like an elastic material till this limit and as such B is called Elastic Limit. Loading the material beyond C causes permanent deformation of the specimen i,e the cross sectional area reduces and length increases till

www.rejinpaul.com

the upper yield point C is reached. Loading the specimen beyond C, the material regains some strength and this continues till ultimate stress (D) is reached. On further increase in load the specimen finally fails at failure stress(E).

1) Proportional Limit / Limit of proportionality :-
   It is the limiting or maximum stress value upto which the stress is proportional to the strain.
2) Elastic Limit :-This is the maximum or limiting stress value such that there is mo permanent deformation in the material.
3) Yield Point /Yield Strength:- These are the lower stresses at which the extension of the specimen is rapid without much increase in the strain.
4) Ultimate stress or yield strength:- this is the maximum stress, the material can resist.
5) Breaking stress/ Breaking strength:- this is stress at which the specimen fails or breaks or ruptures.

9.1 Working stress and factor safety :-

In designing any engineering components stressing the material upto its ultimate strength is not advisable for following reasons

1) Stressing the material till ultimate strength causes the deformation or the failure of the member.
2) The material may not be 100% reliable.
3) The material may contain minor defects.

Hence the material is stressed to a point much lesser than ultimate strength. Such a stress is called working stress which is normally equal to stress at proportional limit.

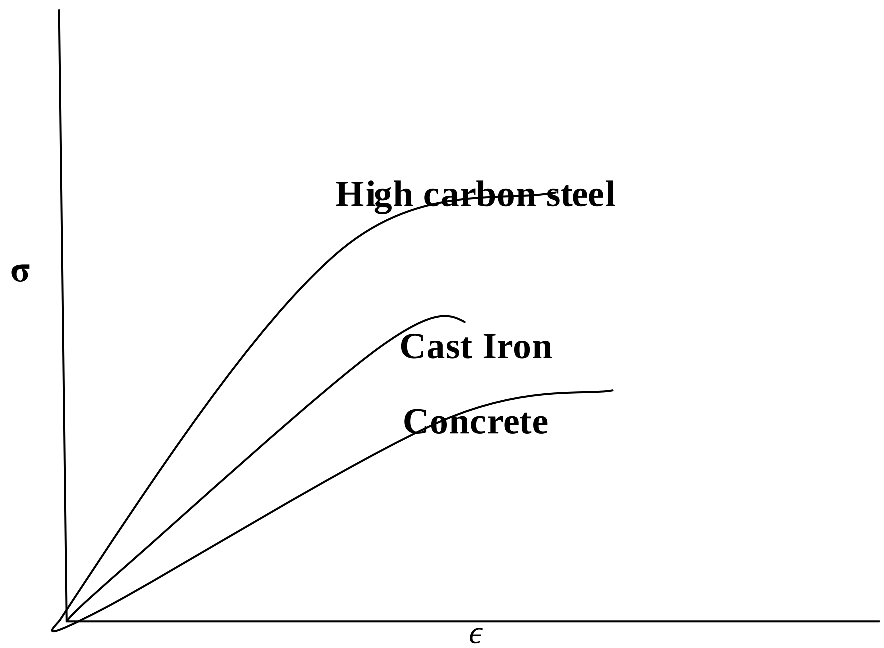
The ratio of the ultimate stress to the working stress is known as “Factor of Safety”

Different materials have different strength and also different reliability and hence the factor of safety for these materials will also vary and there are presented in the tabular column

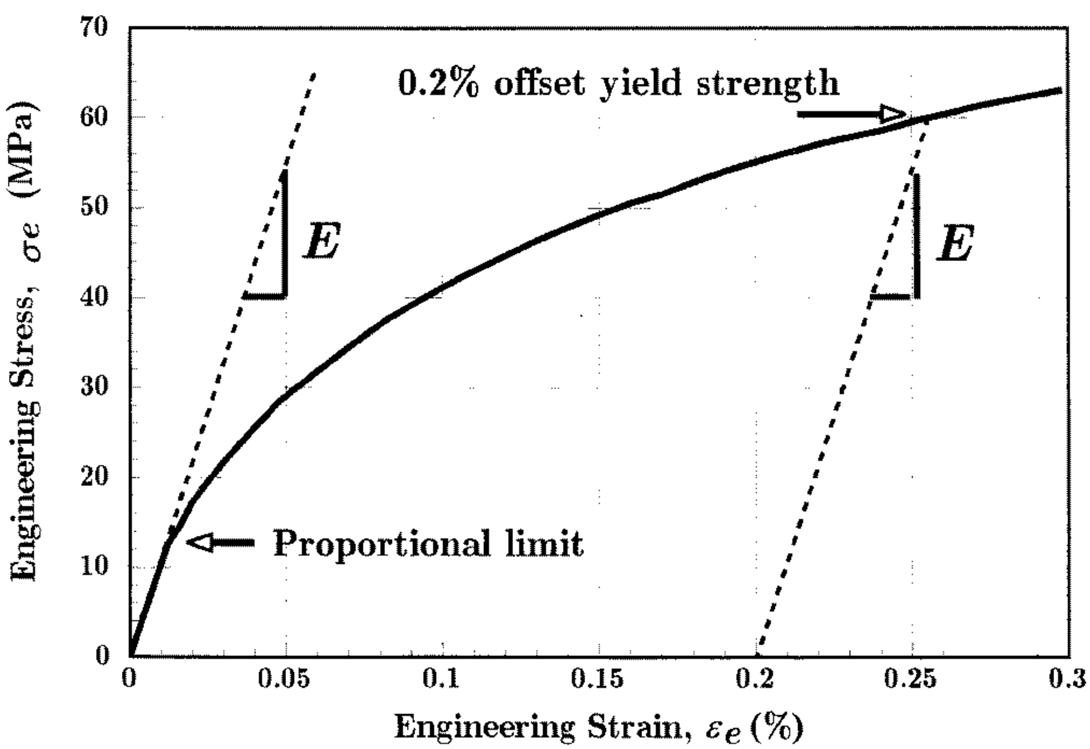
Material	Steel	Concrete	Timber
Factor of safety	1.85	3	4to 6

Stress- strain behavior of Non-Elastic materials

The behavior of certain Non-Elastic materials such as Concrete, Aluminum, Cast Iron and Carbon Steel is represented as shown above. These materials do not show specific yield points and their failure is sudden. In such cases to determine the approximate yield point or stress, the stress corresponding to 0.2% strain is calculated and such a stress obtained is called proof stress.


High carbon steel
σ
Cast Iron
Concrete
ϵ

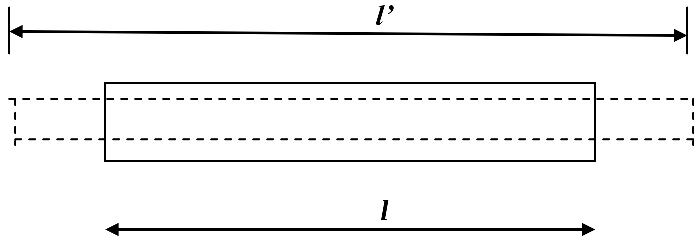
Fig


0.2% offset yield strength
E
E
Proportional limit
Engineering Stress, σe (MPa)
Engineering Strain, εe (%)
70
60
50
40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3

.

Fig

Percentage Elongation



Fig

It is defined as the ratio of the final extension at failure to the original length expressed as a percentage.

If l represents the original length and l' represents the final length at failure, percentage elongation is given as

$$\%\ \text{elongation} = \frac{(l'-l)}{l} \times 100$$

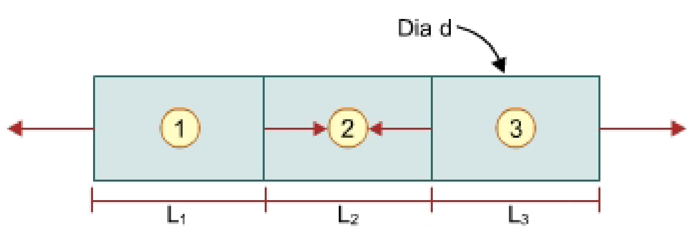
Percentage reduction in area:-

It is defined as the ratio of maximum change in cross sectional area to the original cross sectional area.

If 'A' represents original cross sectional area at failure

$$\text{Percentage reduction in area} = \frac{(A'-A)}{A} \times 100$$

9. Deformation of Uniform Bars



Procedure:

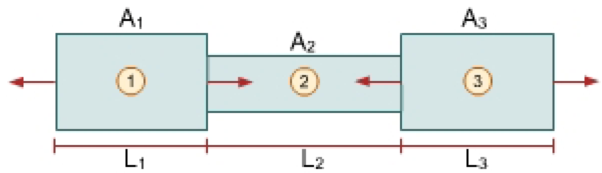
STEP 1:
Check whether the system of forces is in equilibrium or not. If there is any unknown force in the system, determine it using equilibrium condition.

STEP 2:
Separate each part and find the force acting on each part.

STEP 3:
The total deformation is given by the algebraic sum of deformation of each part.

$$dl = (dl_1 \pm dl_2 \pm dl_3) = \frac{P}{AE}(L_1 \pm L_2 \pm L_3)$$

Deformation of Varying Cross section Bars



Procedure:

STEP 1:
Check whether the system of forces is in equilibrium or not. If there is any unknown force, determine it using equilibrium condition.

STEP 2:
Separate each part and find the force acting on each part.

STEP 3:
The total deformation is given by the algebraic sum of deformation of each part.

$$dl = (dl_1 \pm dl_2 \pm dl_3) = \frac{P}{E}\left(\frac{L_1}{A_1} \pm \frac{L_2}{A_2} \pm \frac{L_3}{A_3}\right)$$

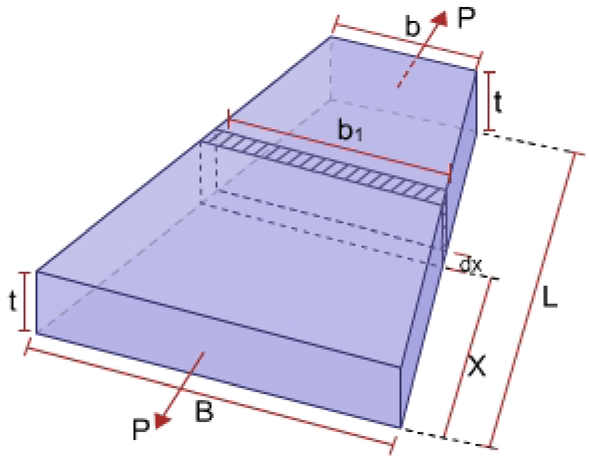
If the Young's Modulus of the material is different, then the change in length is given by

$$dl = (dl_1 \pm dl_2 \pm dl_3) = P\left(\frac{L_1}{A_1E_1} \pm \frac{L_2}{A_2E_2} \pm \frac{L_3}{A_3E_3}\right)$$

Deformation of Tapering Bars of Uniform Thickness

Let us consider a tapering bar of const thickness 't' and width varying from 'B' to 'b' over a length 'L' subjected to direct load 'P'. Let us consider an elemental strip of length 'dx' at a distance 'x' as shown in fig. The elemental strip can be considered to be uniform.

We have deformation of elemental strip $= \frac{PL}{AE} = \frac{P\,dx}{(b_1 t)E}$



Deformation of entire bar = $\int_0^l \frac{P}{t\,Eb_1}dx$

To express 'b_1' in terms of 'x' or as a function of 'x':-

b_1 = B – decrease in width over the length 'x'
Decrease in width over the length 'L' = B –b
∴ Decrease in width over the length 'x' = $\frac{B-b}{L}$x = Kx
∴ b_1 = B - K x
Elongation of the smallest element is δl = $\frac{P\,L}{A\,E} = \frac{P}{E} \times \frac{dx}{(B-Kx)t}$

∴ Total Elongation = $dl = \int_0^l \frac{P}{E} \times \frac{dx}{(B-Kx)t} = \frac{P}{t\,E}\int_0^l \frac{dx}{(B-Kx)}$

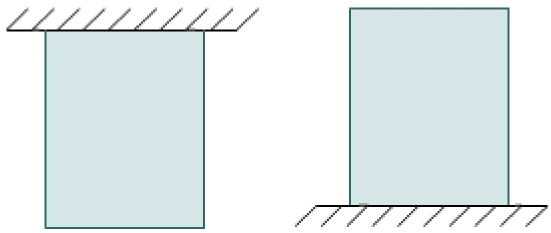
$$= \frac{P}{tE}\left[-\frac{1}{K}log_e(B-Kx)\right]_0^l$$

$$= -\frac{P}{t\,E}\frac{l}{B-b}(log_e b - log_e B)$$

$$dl = \frac{Pl}{tE(B-b)}log_e\frac{B}{b} = \frac{2.303Pl}{tE(B-b)}log_e\frac{B}{b}$$

Deformation due to Self Weight

When a bar is suspended at its top, it will undergo tensile deformation due to self weight. The top most section is subjected to maximum force equal to the total weight of the bar and bottom most section is subjected to no force due to self weight.



When a bar is supported at its bottom, it will undergo compressive deformation due to self weight. In this case bottom most section is subjected to maximum load due to self weight and top most section is subjected to no force due to self weight.

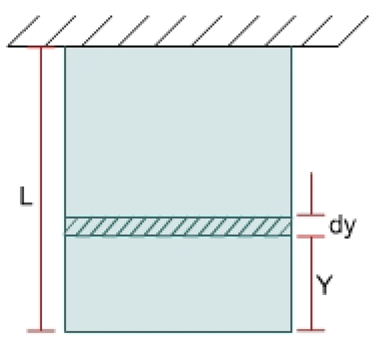
Note:

Specified weight or weight density of a material is the weight per unit volume of the material.

$$\gamma = \frac{weight}{Volume} = \frac{W}{V} \text{ N/mm}^2$$

$$\text{i.e. } W = \gamma \text{ X } V$$

Let us consider uniform bar or area of cross section 'A' and length 'L' supported as shown in figure. Let us consider an elemental strip of length 'dy' at a distance 'y' as shown in figure.



Force on elemental strip = weight of material below the elemental strip.

$$= \gamma \text{ x Volume}$$

$$= \gamma \text{ x } (AY)$$

$$\text{Deformation of elemental strip} = \frac{(Load)(Length)}{(Area)(Young's\ Modulus)}$$

$$= \frac{(\gamma AY)(dy)}{(A)(E)}$$

$$= \frac{\gamma Y\ x\ dy}{E}$$

$$\text{Therefore deformation of entire bar} = \int_0^L \frac{\gamma}{E}\, y.dy$$

$$= \frac{\gamma}{E}\left|\frac{y^2}{2}\right|_0^L$$

$$\text{Therefore deformation of entire bar} = \frac{\gamma L^2}{2E}$$

$$= \frac{\gamma.L.LA}{2EA}$$

$$\frac{(\gamma V)L}{2AE}$$

$$\text{Deformation of entire bar} = \frac{WL}{2AE}$$

W = total weight of the bar.

If an external load 'W' is applied on the uniform bar, then its deformation will be = $\frac{WL}{AE}$
Therefore deformation due to self weight of uniform bar is equal to one-half the deformation of the same bar under an external load equal to the total weight of the bar

Problems:

1. **A mild steel rod 2.5 m long having a cross sectional area of 50 mm2 is subjected to a tensile force of 1.5 kN. Determine the stress, strain, and the elongation of the rod. Take E = 2 × 105 N/mm^2**

Solution:

Data Given
Length of the rod 'L' = 2.5 m = 250 mm
Area of cross-section 'A' = 50 mm^2
Tensile force 'P' = 1.5 kN = 1.5 × 10^3 N
Young's Modulus 'E' = 2 × 105 N/mm^2

www.rejinpaul.com

Stress σ = P/A= 1.5×10^3 / 50 =30N /mm^2

Since, E = Stress / Strain Strain = Stress / E = 30 / 2 × 105 = 0.0015

Also, Elongation = Strain x Original length = 0.0015 × 2500 = 0.375 mm.

2. **A load of 100 kN is to be lifted with the help of a steel wire of 5 m length. The permissible limit of stress for wire is 150 N/mm^2. Find the minimum diameter of the steel wire and the elongation at the permissible limit. Take E = 2 X 10^5 N/mm^2**

Solution:

Given Data

Tensile load 'P' = 100 kN

Length of wire 'L' = 5 m = 5000 mm

Stress 'σ' = 150 N/mm^2

E = 2 × 10^5 N/mm^2

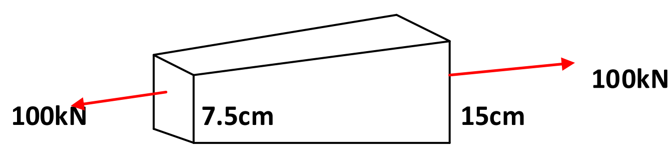
$$\sigma = \frac{P}{A} = \frac{4\,P}{\pi\ d^2} = \frac{4 \times 100x1000}{\pi d^2} = 150$$

d = 29.13 mm

δL = (100 × 10^3) × 5000 / 666.67 × 2 × 10^5 = 3.75 mm.

3.**A steel bar AB of uniform thickness 2 cm, tapers uniformly from 15 cm to 7.5 cm in a length of 50 cm. Determine the elongation of the plate; if an axial tensile force of 100 kN is applied on it and E = 2 × 10^{11} N/m^2.**

Solution:



$$dl = \frac{Pl}{tE(B-b)} log_e \frac{B}{b} = \frac{100\times10^3\times500}{25\times2\times10^5\times(150-75)} log_e \frac{150}{75} = 0.115mm$$

Unit 2

COMPOSITE SECTION

A section made up of more than one material designed to resist the applied load is called 'COMPOSITE SECTION'.

In a composite section, materials are placed in parallel and under the load the section behaves like a single material. Under the load, there will not be separation of materials. Materials of composite sections undergo equal amount of deformation under load.

Ex: A section of Reinforced Cement Concrete - it is made up of 2 materials namely, Concrete and Steel.

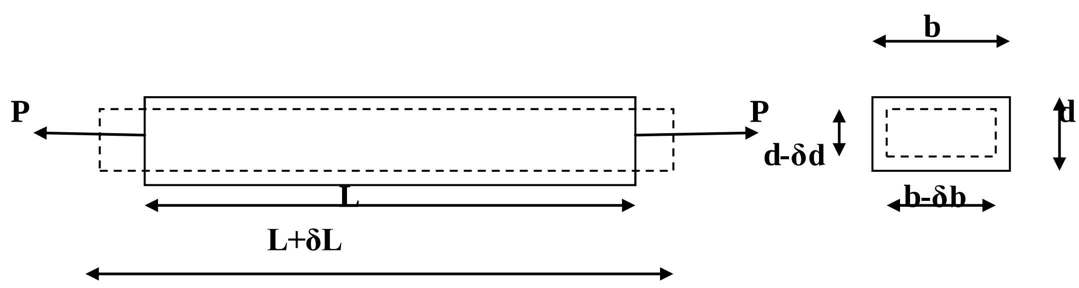
1. Total load applied on the composite section is shared among different materials of composite section, i.e., $P = P_1 + P_2 + P_3 + \dots$
2. Different materials of composite section should undergo equal amount of deformation i.e., $\delta L_1 = \delta L_2 = \delta L_3 = \dots$
3. The material can also have same strain if the materials are of same length

Modular ratio: Modular ratio between two materials is defined as the ratio of Young's Modulus of Elasticity of two materials.

Eg: Modular ratio between steel and concrete is 15, i.e., $E_s / E_c = 15$

Poisson's Ratio:

In any engineering problems the elongation along the tensile force 'P' direction is accomapnied with the contraction in the traverse direction. Consider a bar AB of length 'L' as shown in the fig is subjected to a load of P. Due to the tensile length the bar elongates by 'δL' and compressed in the other (mutually perpendicular) direction by ' δb'.



The change in length is = δL and the change in breadth = - δb

www.rejinpaul.com

Mathematically Poisson's Ratio is defined as the ratio between lateral strain to the longitudinal strain and denoted by µ or 1/m.

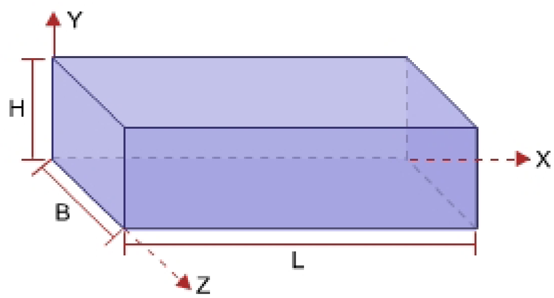
$$\text{`}\mu \text{ or } 1/m = \frac{\epsilon_b}{\epsilon_l} = \frac{\delta b/_b}{\delta l/_l}$$

Volumetric Strain: It is defined as the ratio between change in volume to the actual volume

$$\epsilon_V = \frac{\delta V}{V} = \frac{dV}{V}$$

Volumetric strain is a dimensionless quantity and has no unit.

To show that volumetric Strain of a rectangular block of material is the algebraic sum of strain along 3 mutually perpendicular dimensions or directions:



Let us consider, a rectangular block of material of dimensions (L×B×H) as shown in fig. The volume of rectangular block is,

Initial Volume V = LBH

Final Volume = (v+dv) = (L+dl)(B+dB)(H+dH)

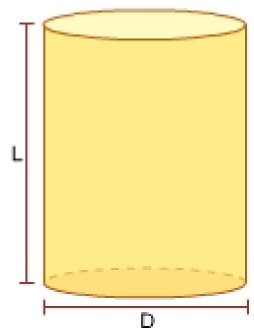
Change in Volume = dv = (V+dV)-V = LB dH + BH dL +LH dB
(After neglecting the higher order)

$$\text{Volumetric Strain } \epsilon_V = \frac{dV}{V} = \frac{LB\ dH+BH\ dL+LH\ dB}{LBH} = \frac{dL}{L} + \frac{dB}{B} + \frac{dH}{H}$$

$$\epsilon_V = \epsilon_l + \epsilon_B + \epsilon_h$$

www.rejinpaul.com

To show that volumetric strain of a cylinder is given by the algebraic sum of strain along the length and twice the strain along diameter:



Let us consider a cylinder of length 'L' and diameter 'D'. The volume of a cylinder is given by,

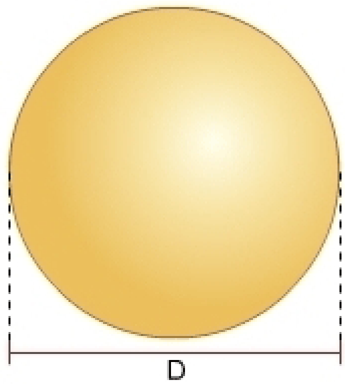
$$V = \frac{\pi D^2}{4} L$$

Differentiating $$dV = \frac{\pi}{4}[D^2 \times dL + L \times 2D \times dD]$$

$$\epsilon_V = \frac{dV}{V} = \frac{D^2 \times dL + L \times 2D \times dD}{D^2 L}$$

$$\epsilon_V = \frac{dL}{L} + \frac{2\ dD}{D} = \epsilon_l + 2\ \epsilon_d$$

To show that Volumetric Strain of a sphere is thrice the strain along diameter:



Let us consider a sphere of diameter 'D' Volume of a sphere is given by

$$V = \frac{4}{3}\pi R^3 = \frac{\pi}{6} D^3$$

$$dV = \frac{\pi}{6} D^2 dD$$

$$\text{Volumetric Strain} = \frac{dV}{V} = 3 \times \frac{dD}{D} = 3\epsilon_d$$

Note:

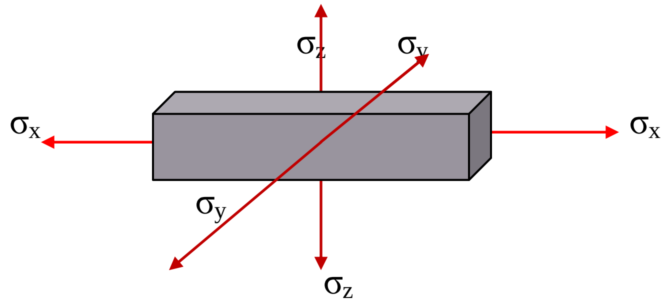
1. A direct stress applied on a material causes direct strain along its line of action and lateral strain along directions perpendicular to its line of action.
   If tensile stress is applied, there will be positive direct strain and negative lateral strain.
   If comp. stress is applied, there will be negative direct strain and positive lateral strain.
2. 'σ_x' produces direct strain along X-direction and lateral strain along 'Y' and 'Z' directions.
   'σ_y' produces direct strain along Y-direction and lateral strain along 'X' and 'Z' directions.
   'σ_Z' produces direct strain along Z-direction and lateral strain along 'X' and 'Y' directions
3. If a material is subjected to 3-D direct stress system, then,
   a. Total strain along X- direction is given by the algebraic sum of direct strain due to 'σ_x 'and lateral strain due to 'σ_y' and 'σ_z'
   b. Total strain along Y - direction is given by the algebraic sum of direct strain due to 'σ_Y' and lateral strain due to 'σ_Y' and 'σ_Z'
   c. Total strain along Z - direction is given by the algebraic sum of direct strain due to 'σ_Z' and lateral strain due to 'σ_X' and 'σ_y'
   d.

   We have,

$$E = \frac{\sigma}{\epsilon} \; or \; \epsilon = \frac{\sigma}{E}$$

And $$\mu = \frac{\epsilon_{lat}}{\epsilon_{long}}$$

$$\epsilon_{lat} = \mu \times \epsilon_{long} = \frac{\mu\,\sigma}{E}$$



Stress	Direct Stress	Lateral Strain
σ_x	$+\frac{\sigma_x}{E}$	$-\mu\frac{\sigma_x}{E}$
σ_y	$+\frac{\sigma_y}{E}$	$-\mu\frac{\sigma_y}{E}$
σ_z	$+\frac{\sigma_z}{E}$	$-\mu\frac{\sigma_z}{E}$

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E} \text{ ------- } (1)$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_z}{E} \text{ ------- } (2)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_x}{E} \text{ -------} (3)$$

Equations (1), (2) and (3) relate 3 - dimensional stresses and corresponding strains. They are called 'Generalized Hook's Law Statements'

Thermal Stress

When a body is subjected to change in temperature its dimensions will also be changed (since the bodies are subjected to thermal expansion or contraction). For metals when the temperature of a body is increased there is a corresponding increase in its dimensions.

When the body is allowed to expand (without restraining) no stress develops. But, in case the body is restrained prevents the expansion, then the stresses in the body will develop. These stresses are called as thermal stress. It may be tensile or compressive depending upon whether the contraction is prevented or extension is prevented.

Mathematically it is defined as , let us consider a bar of length L placed between two supports to prevent the extension in its length. If the temperature of bar is increased through Δt°C, the bar will be increased in length by an amount

$\Delta L = L.\alpha.\Delta t$

Where α is the coefficient of thermal expansion.

The thermal stress is $= \sigma_t = E\,\epsilon = E\,L\alpha\,\Delta t/L = E\,\alpha\,\Delta t$

Note: If the supports used to prevent the expansion yield by an amount δ, the total amount prevented will become (ΔL –δ).

The thermal stress is $= \sigma_t = E\,\epsilon = E\,(\Delta L - \delta)\alpha\,\Delta t/L$

Problems:

1. **The principal stresses at a point in an elastic material are 70 N/mm^2 tensile, 30 N/mm^2 tensile, and 50 N/mm^2 compressive.**
   Calculate the volumetric strain. Given E = 2 × 10^{11} N/m^2; μ = 0.30

   Solution:

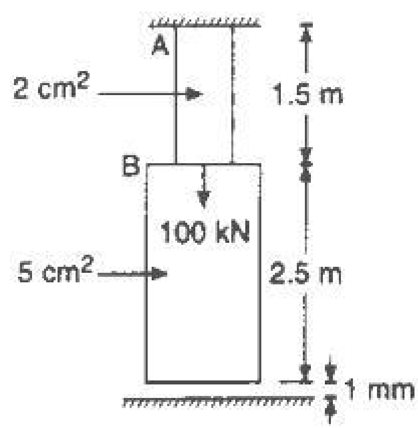
$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E} = \frac{1}{2\times10^5}(70 - 0.3\times30 + 0.3\times50) = 3.8\times10^{-4}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_z}{E} = \frac{1}{2\times10^5}(30 - 0.3\times70 + 0.3\times50) = 1.2\times10^{-4}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_x}{E} = \frac{1}{2\times10^5}(-50 + 0.3\times70 + 0.3\times50) = -0.7\times10^{-4}$$

$$\epsilon_V = \epsilon_x + \epsilon_y + \epsilon_z = 4.3\times10^{-4}$$

2. A composite bar ABC, rigidly fixed at A and 1 mm above the lower support, is loaded as shown in figure. If the cross-sectional area of section AB is 2 cm^2 and that of the section BC is 5 cm^2, determine the reactions at the ends, and the stresses in two sections. Take E = 2 × 10^7 N/cm^2.



Solution: Length: AB = L_1 = 1.5 m = 150 cm, Area of AB = A_1 = 2 cm^2

Length: BC = L_2 = 2.5 m = 250 cm, Area of BC = A_2 = 5 cm^2

Distance between C and lower support = 0.1 cm

Load on the bar 'P' = 100 kN, Young's modulus= E = 2×10^7 N/cm^2

Assume that, reaction at end A = R_A, Reaction at end C = R_C It is notable that the bar is rigidly fixed at A and loaded at B, not at C. Thus, only AB is under tension while BC under no stress until C touches the rigid support

Let δL = Increase in length of AB when subjected to a load to 100 kN.

From fundamentals δL = PL/AE

$$\delta L = \frac{10\times10^3\times150}{2\times2\times10^7} = 0.375cm$$

Since, the increase in length AB is more than 0.1 cm therefore some part of load will be required to increase AB by 0.1 cm and remaining will be shared by the portions AB and BC of the bar.

Thus using; δL = PL / AE

$0.1 = P_1 \times 150 / 2.0 \times 2 \times 10^7$ $P_1 = 26.67$ kN $\Rightarrow P - P_1 = P_2 = 73.33$ kN

The load P_2 will be shared by AB and BC. Let the reaction at A (beyond 0.1 cm) = R_{A1} And the reaction at C (beyond 0.1 cm) = R_C

$R_{A1} + R_C = 73.33$ kN …………..(1)

Let δL_1 = Increase in length of AB (beyond 0.1 cm)

δL_2 = Decrease in length of BC (beyond 0.1 cm)

$\delta_{L1} = \frac{R_{A1}\times150}{2\times2\times10^7}$ and $\delta_{L2} = \frac{R_c\times250}{5\times2\times10^7}$

$\delta_{L1} = \delta_{L2}$

$\frac{R_{A1}\times150}{2\times2\times10^7} = \frac{R_c\times250}{5\times2\times10^7} \Rightarrow R_{A1} = 2/3\ R_c$

$\therefore$ R_C = 44 kN and R_{A1} = 29.33 kN

Total reaction at A = 26.67 + 29.3 = 56 kN and total reaction at B = 44 kN Therefore, stresses in two sections are $\sigma_{AB} = R_A/A_1 = 56 \text{ X } 10^3/2 = 28 kN/cm^2$

$$\sigma_{BC} = R_c/A_2 = 44 \text{ X } 10^3/5 = 8.8 kN/cm^2$$

Unit 3

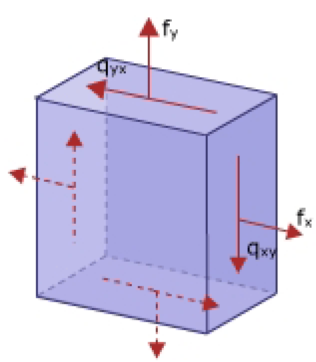
Compound Stresses:

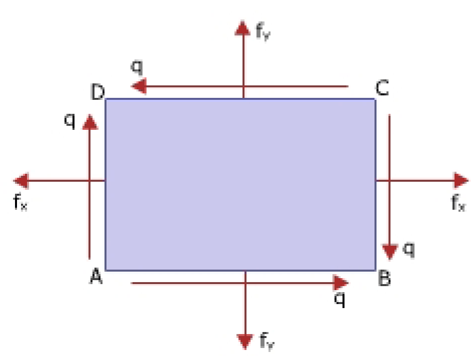
GENERAL

Structural members are subjected to various kinds of loads. This results in combination of different stresses which changes from point to point. When an element (considered at any point) in a body is subjected to a combination of normal stresses (tensile and/or compressive) and shear stresses over its various planes, the stress system is known as compound stress system. In a compound stress system, the magnitude of normal stress may be maximum on some plane and minimum on some plane, when compared with those acting on the element. Similarly, the magnitude of shear stresses may also be maximum on two planes when compared with those acting on the element. Hence, for the considered compound stress system it is important to find the magnitudes of maximum and minimum normal stresses, maximum shear stresses and the inclination of planes on which they act.

PLANE STRESS OR 2–D STRESS SYSTEM OR BIAXIAL STRESS SYSTEM

Generally a body is subjected to 3–D state of stress system with both normal and shear stresses acting in all the three directions. However, for convenience, in most problems, variation of stresses along a particular direction can be neglected and the remaining stresses are assumed to act in a plane. Such a system is called 2–D stress system and the body is called plane stress body.





In a general two dimensional stress system, a body consists of two normal stresses (fx and fy), which are mutually perpendicular to each other, with a state of shear (q) as shown in figure. Further, since planes AD and BC carry normal stress fx they are called planes of fx. These planes are parallel to Y–axis. Similarly, planes AB and CD represent planes of fy, which are parallel to X–axis.

PRINCIPAL STRESSES AND PRINCIPAL PLANES

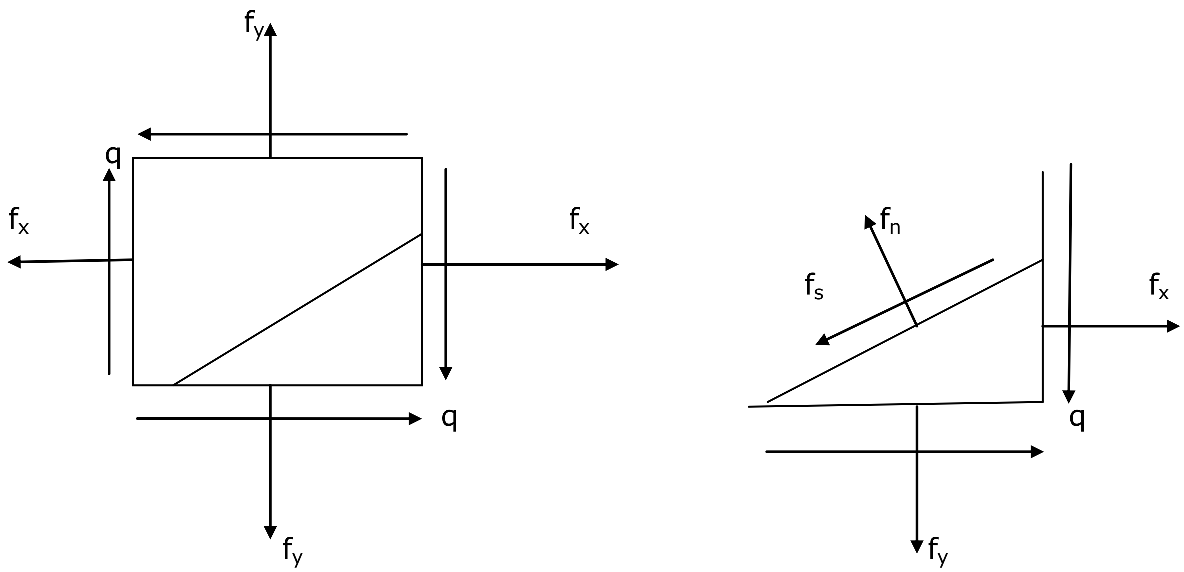
For a given compound stress system, there exists a maximum normal stress and a minimum normal stress which are called the Principal stresses. The planes on which these Principal stresses act are called Principal planes. In a general 2-D stress system, there are two Principal planes which are always mutually perpendicular to each other. Principal planes are free from shear stresses. In other words Principal planes carry only normal stresses.

MAXIMUM SHEAR STRESSES AND ITS PLANES

For a given 2–D stress system, there will be two maximum shear stresses (of equal magnitude) which act on two planes. These planes are called planes of maximum shear. These planes are mutually perpendicular. Further, these planes may or may not carry normal stress. The planes of maximum shear are always inclined at 450 with Principal planes.

EXPRESSIONS FOR NORMAL AND TANGENTIAL COMPONENTS OF STRESS ON A GIVEN PLANE

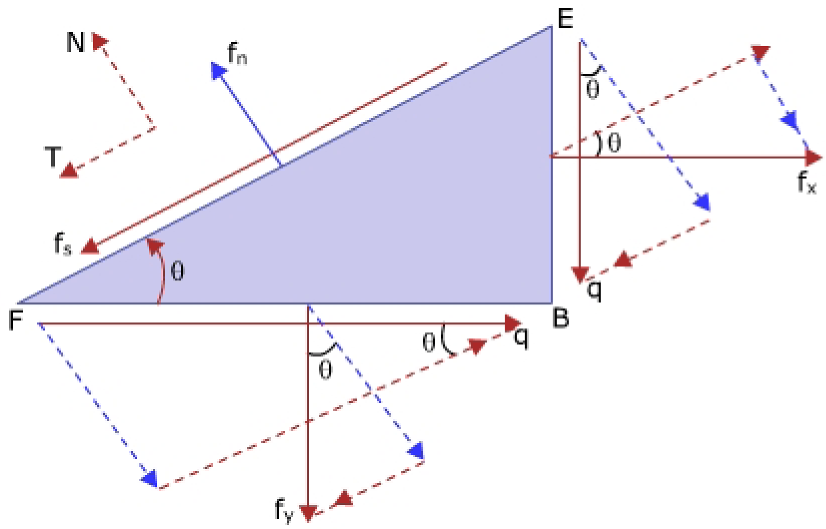
Consider a rectangular element ABCD of unit thickness subjected to a general 2-D stress system as shown in figure. Let f n and f s represent the normal and tangential components of resultant stress 'R' on any plane EF which is inclined at an angle '?' measured counter clockwise with respect to the plane of f y or X–axis.



www.rejinpaul.com

To derive expression for f_n

Consider the Free Body Diagram of portion FBE as shown in figure.



Applying equilibrium along N-direction, we have

$$\Sigma F_N = 0 \quad [\ \nwarrow\ +ve]$$

$$f_n\,(EF.1) - f_x\,(BE.1)\sin\theta - q\,(BE.1)\cos\theta - f_y\,(BF.1)\cos\theta - q\,(BF.1)\sin\theta = 0$$

$$f_n = f_x \frac{BE}{EF}\sin\theta + q\frac{BE}{EF}\cos\theta + f_y\frac{BF}{EF}\cos\theta + q\frac{BF}{EF}\sin\theta$$

Since $\frac{BE}{EF} = \sin\theta$ and $\frac{BF}{EF} = \cos\theta$

$$\therefore f_n = f_x \sin^2\theta + 2q\sin\theta\cos\theta + f_y\cos^2\theta$$

But $\cos 2\theta = 2\cos^2\theta - 1$ Hence $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

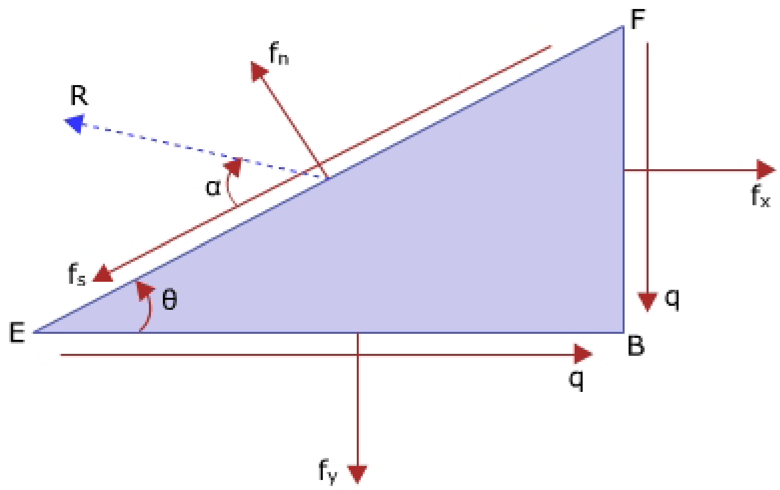
Also $\cos 2\theta = 1 - 2\sin^2\theta$ Hence $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$f_x = f_x\,\frac{1}{2}(1-\cos 2\theta) + f_y\,\frac{1}{2}(1+\cos 2\theta) + q\sin 2\theta$$

$$f_n = \left(\frac{f_x + f_y}{2}\right) - \left(\frac{f_x - f_y}{2}\right)\cos 2\theta + q\sin 2\theta \qquad \text{-------- (1)}$$

Equation (1) is the desired expression for normal component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of f_Y or X–axis

To derive expression for f_s
Consider the Free Body Diagram of portion FBE shown in figure above. For equilibrium along T direction, we have



$\Sigma F_T = 0$ [↙ + ve]

$$f_s(EF.1) - f_x(BE.1)\cos\theta + q(BE.1)\sin\theta + f_y(BF.1)\sin\theta - q(BF.1)\cos\theta = 0$$

$$f_s = f_x \frac{BE}{EF}\cos\theta - q\frac{BE}{EF}\sin\theta - f_y\frac{BF}{EF}\sin\theta + q\frac{BF}{EF}\cos\theta$$

Since $\frac{BE}{EF} = \sin\theta \qquad \frac{BF}{EF} = \cos\theta$

$$\therefore f_s = f_x \sin\theta\cos\theta - q\sin^2\theta - f_y\cos\theta\sin\theta + q\cos^2\theta$$

$$\therefore f_s = (f_x - f_y)\sin\theta\cos\theta + q(\cos^2\theta - \sin^2\theta)$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$f_s = \left(\frac{fx - fy}{2}\right)\sin 2\theta + q\cos 2\theta \qquad \text{-------- (2)}$$

Equation (2) is the desired expression for tangential component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X–axis.

www.rejinpaul.com

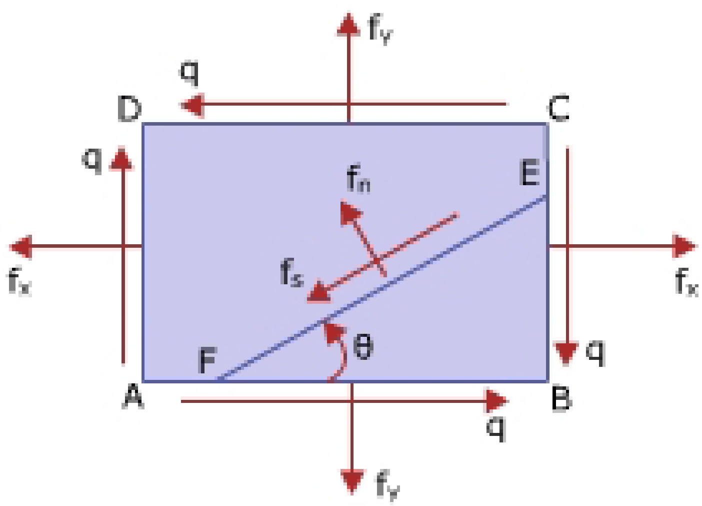
Note:
The resultant stress 'R', and its inclination ' α ' on the given plane EF which is inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X–axis, can be determined from the normal (fn) and tangential (fs) components obtained from eqns. (1) and (2).

$$R = \sqrt{f_n^2 + f_s^2}$$

$$\alpha = \tan^{-1}\left(\frac{f_n}{f_s}\right)$$

Expresions for Principal stresses and Principal planes

Consider a rectangular element ABCD of unit thickness subjected to general 2-D stress system as shown in figure. Let f_n and f_s represent the normal and tangential components of stress on any plane EF which is inclined at an angle ' θ' measured counter clockwise with respect to the plane of f_y or X–axis



The expression for normal component of stress $f_n$on any given plane EF is given by

$$f_n = \frac{f_x + f_y}{2} - \frac{f_x - f_y}{2}\cos 2\theta + q \sin 2\theta \qquad \text{-------- (1)}$$

To find values of θ at which f n is maximum or minimum, the necessary condition is

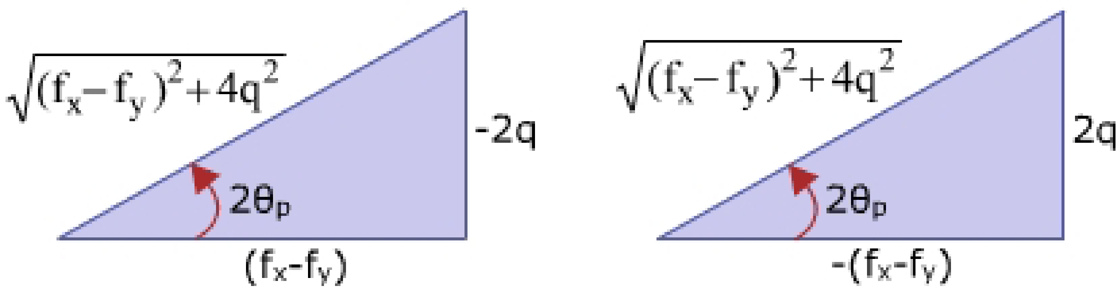
$$\frac{df_n}{d\theta} = 0$$

From eqn. (1) $$-\left(\frac{f_x - f_y}{2}\right)(-2\sin 2\theta) + 2q\cos 2\theta = 0$$

$$\therefore \tan 2\theta_p = -\frac{2q}{f_x - f_y} \qquad \text{------- (2)}$$

Inclination of principal planes can be obtained from eqn. (2). It gives two values of θ differing by 90^o. Hence, Principal planes are mutually perpendicular. Here, the two principal planes are designated as θ_{p1} and θ_{p2}.

Graphical representation of eqn. (2) leads to the following



From the above figures,

$$\sin 2\theta_p = \pm \frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}} \qquad \cos 2\theta_p = \pm \frac{(f_x - f_y)}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

Substituting in eqn.(1)

$$f_n = \frac{f_x + f_y}{2} \pm \frac{f_x - f_y}{2} \frac{fx - fy}{\sqrt{(fx - fy)^2 + 4q^2}} \pm q \frac{2q}{\sqrt{(fx - fy)^2 + 4q^2}}$$

On simplification,

$$f_{n1,2} = \frac{fx + fy}{2} \pm \frac{1}{2}\sqrt{(fx - fy)^2 + 4q^2} \qquad \text{--------- (3)}$$

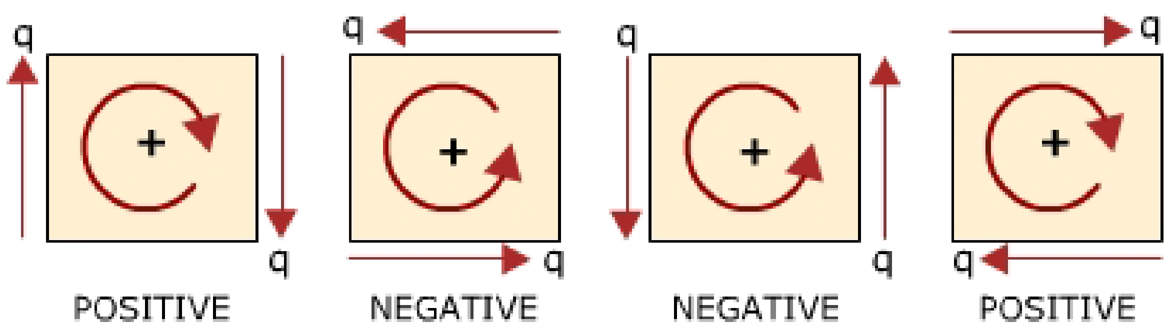
Equation (3) is the desired expression for Principal stresses. Here, the Principal stresses are represented by f_{n1} and f_{n2}.

Mohr's Circle

The formulae developed so far (to find f_n, f_s, f_{n-max} , f_{n-min} , θ_{p1} , θ_{p2} , $f_{s\,max}$, θ_{s1} , θ_{s2}) may be used for any case of plane stress. A visual interpretation of these relations, devised by the German Engineer Christian Otto Mohr in 1882, eliminates the necessity of remembering them. In this interpretation a circle is used; accordingly, the construction is called **Mohr's Circle.** If this construction is plotted to scale the results can be obtained graphically; usually, however, only a rough sketch is drawn and results are obtained from it analytically.

Rules for applying Mohr's Circle to compound stresses

1. The normal stresses f_x and f_y are plotted along X-axis. Tensile stresses are treated as positive and compressive stresses are treated as negative.
2. The shear stress q is plotted along Y-axis. It is consider positive when its moment about the center of the element is clockwise and negative when its moment about the center of the element is anti-clockwise.

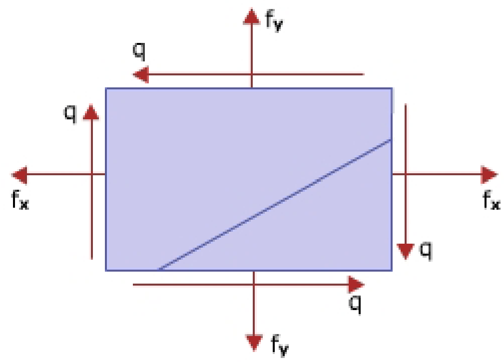


3. Positive angles in the circle are obtained when measured in counter clockwise sense. Further, an angle of '2θ' in the circle corresponds to an angle θ in the element.

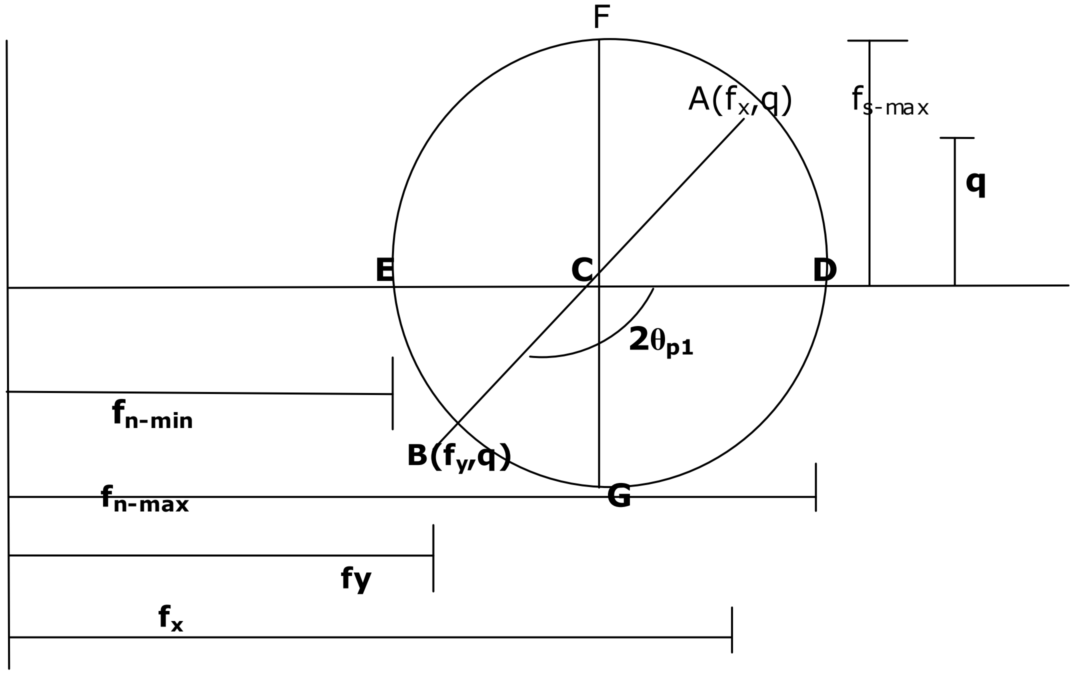
4. A plane in the given element corresponds to a point on the Mohr's circle. Further, the coordinates of the point on the Mohr's circle represent the stresses acting on the plane

Procedure to construct Mohr's circle

Consider an element subjected to normal stresses f_x and f_y accompanied by shear stress q as shown in figure. Let f_x be greater than f_y.



1. In the rectangular coordinate system, locate point A which will be should be a point on the circle representing the stress condition on the plane f_x of the element. The coordinates of point A are (f_x , q).
2. Similarly locate point B, representing stress conditions on plane fy of the element. The coordinates of point B are (f_y – q).
3. Join AB to cut X-axis at point C. Point C corresponds to the center of Mohr's circle.
4. With C as center and CA as radius, draw a circle.



Fig

From figure, it can be seen that OD and OE represent maximum and minimum normal stresses which are nothing but principal stresses. The coordinates of points D and E give the stress condition on principal planes. It can be seen that the value of shear stress is '0' on these two planes. Further, angles BCD = 2 θ_{p1} and BCE =2 θ_{p2} (measured counter clockwise) give inclinations of the principal planes with respect to plane of f_y or X-axis. It is seen that $2\theta_{p1} \sim 2\theta_{p2} = 180^0$.

Hence, $\theta_{p1} \sim \theta_{p2} = 90^0$.

It can be observed that shear stress reach maximum values on planes corresponding two points F and G on the Mohr's circle. The coordinates of points F and G represents the stress conditions on the planes carrying maximum shear stress. The ordinate CF and CG represent the maximum shear stresses. The angles BCG = $2\theta_{s1}$ and BCF $=2\theta_{s2}$ (measured counter clockwise) give inclinations of planes carrying maximum shear stress with respect to plane of f_y or X-axis. It is seen that $2\theta_{s1} \sim 2\theta_{s2} = 180^0$.

Hence, $\theta_{s1} \sim \theta_{s2} = 90^0$.

Also it is seen that $2\theta_{p1} \sim 2\theta_{s1} \sim 2\theta_{p2} \sim 2\theta_{s2} = 90^0$. Hence, $\theta_{p1} \sim \theta_{s1} \sim \theta_{p2} \sim \theta_{s2} = 45^0$.

To find the normal and tangential stresses on a plane inclined at θ to the plane of f_y , first locate point M on the circle such that angle BCM = 2θ(measured counter clockwise) as shown in figure. The coordinates of point M represents normal and shear stresses on that plane. From figure, ON is the normal stress and MN is the shear stress.

Problems:

1. **In a 2-D stress system compressive stresses of magnitudes 100 MPa and 150 MPa act in two perpendicular directions. Shear stresses on these planes have magnitude of 80 MPa. Use Mohr's circle to find,**

(i) Principal stresses and their planes
(ii) Maximum shears stress and their planes and
(iii) Normal and shear stresses on a plane inclined at 45^0 to 150 MPa stress.

Given, $f_x = -150$ MPa
$f_y = -100$ MPa
$q = 80$ MPa

If Mohr's circle is drawn to scale, all the quantities can be obtained graphically. However, the present example has been solved analytically using Mohr's circle.

Construct Mohr's circle with earlier fig

From figure

$$OC = \frac{f_x + f_y}{2} = -125\,MPa$$

To find Radius of Circle

$$CH = \frac{f_x - f_y}{2} = 25\,MPa$$

$$CA = \sqrt{CH^2 + HA^2} = 83.82$$

$$\therefore \text{ Radius} = CD = CE = CF = CG = CA = 83.82 \text{ Units}$$

To find Principal Stress and Principal Planes

$f_{n\,-max}$ = OC + CD

= – 125 – 83.82

= – 208.82 MPa

$f_{n\,min}$ = OC – CE

= – 125 – (–83.82)

= – 41.18 MPa

From figure

$$\alpha = \tan^{-1}\left(\frac{AH}{MC}\right) = 72^0.65$$

$$\text{But } 2\theta_{P1} = \angle ACH = \alpha = 72^0.65$$

$$\text{Hence, } \theta_{P1} = 36^0.32$$

$$\text{Further, } 2\theta_{P2} = \angle ACE = 180 + \alpha = 252^0.65$$

$$\text{Hence, } \theta_{P2} = 126^0.32$$

Unit 4

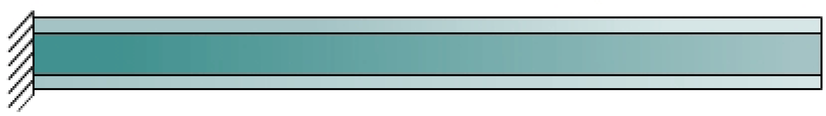
Bending Moment and Shear Force

TYPES OF BEAMS

a) Simple Beam

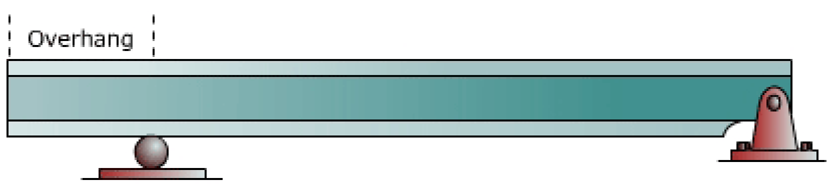
A simple beam is supported by a hinged support at one end and a roller support at the other end.

b) Cantilever beam



A cantilever beam is supported at one end only by a fixed support.

c) Overhanging beam.



An overhanging beam is supported by a hinge and a roller support with either or both ends extending beyond the supports.

Note: All the beams shown above are the statically determinate beams.

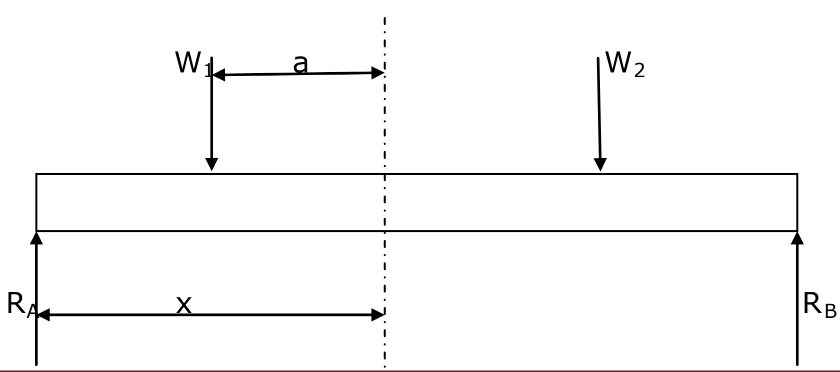


Fig 1

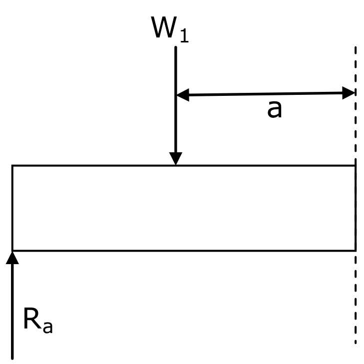


Fig 2 : Shear Force

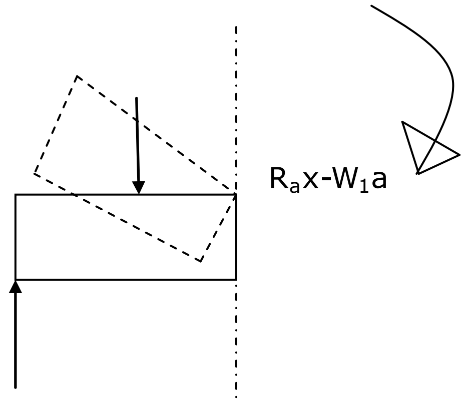


Fig 3 : Bending Moment

Consider a simply supported beam subjected to loads W_1 and W_2. Let R_A and R_B be the reactions at supports. To determine the internal forces at C pass a section at C. The effects of R_A and W_1 to the left of section are shown in Fig (b) and (c). In each case the effect of applied load has been transferred to the section by adding a pair of equal and opposite forces at that section. Thus at the section, moment $M = (W_1a-R_ax)$ and shear force $F = (R_A-W_1)$, exists. The moment M which tend to bends the beam is called bending moment and F which tends to shear the beam is called shear force.

Thus the resultant effect of the forces at one side of the section reduces to a single force and a couple which are respectively the vertical shear and the bending moment at that section. Similarly, if the equilibrium of the right hand side portion is considered, the loading is reduced to a vertical force and a couple acting in the opposite direction. Applying these forces to a free body diagram of a beam segment, the segments to the left and right of section are held in equilibrium by the shear and moment at section.

Thus the shear force at any section can be obtained by considering the algebraic sum of all the vertical forces acting on any one side of the section

Bending moment at any section can be obtained by considering the algebraic sum of all the moments of vertical forces acting on any one side of the section.

Shear Force

It is a single vertical force developed internally at any point on the beam to balance the external vertical forces and keep the point in equilibrium. It is therefore equal to algebraic sum of all external forces acting to either left or right of the section.

Bending Moment

It is a moment developed internally at each point in a beam that balances the external moments due to forces and keeps the point in equilibrium. It is the algebraic sum of moments to section of all forces either on left or on right of the section.

www.rejinpaul.com

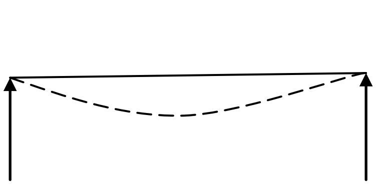
Types of Bending Moment

1) Sagging bending moment

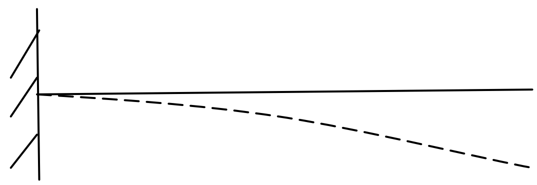
The top fibers are in compression and bottom fibers are in tension.

2) Hogging bending moment

The top fibers are in tension and bottom fibers are in compression.



Sagging Bending Moment



Hogging Bending Moment

Shear Force Diagram and Bending Moment Diagram

Shear Forces Diagram (SFD)
The SFD is one which shows the variation of shear force from section to section along the length of the beam. Thus the ordinate of the diagram at any section gives the Shear Force at that section.

Bending Moment Diagram (BMD)
The BMD is one which shows the variation of Bending Moment from section to section along the length of the beam. The ordinate of the diagram at any section gives the Bending Moment at that section.

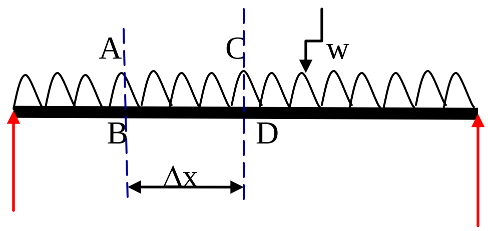
Importance of SFD and BMD
From the diagrams, one can easily determine the locations of maximum Shear Force or maximum Bending Moment. These locations most likely represent the zones of failure.

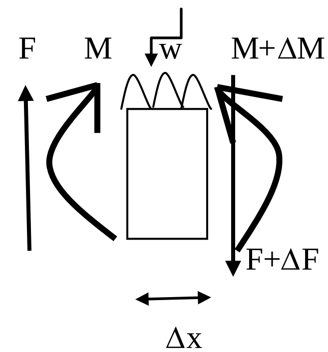
Point of Contraflexure
When there is an overhang portion, the beam is subjected to a combination of Sagging and Hogging moment. The point on the BMD where the nature of bending moment changes from hogging to sagging or sagging to hogging is known as point of contraflexure. Hence, at point

of contraflexure BM is zero. The point corresponding to point of contraflexure on the beam is called as point of inflection.

RELATIONS BETWEEN LOAD, SHEAR AND MOMENT

Consider a simply supported beam subjected to a Uniformly Distributed Load w/m. Let us assume that a portion PQRS of length Δx is cut and taken out. Consider the equilibrium of this portion





$\sum V=0$

$$F - (F+\Delta F) - w\,\Delta x = 0$$

$$\frac{\Delta F}{\Delta x} = -w$$

Limit $\Delta x \to 0$, then $\frac{dF}{dx} = -w$ or $F = \int w\,dx$

Taking moments about section CD for equilibrium

$$M-(M+\Delta M)+F\,\Delta x-(w(\Delta x)^2/2) = 0$$

$$F = \frac{\Delta M}{\Delta x}$$

Limit $\Delta x \to 0$ then $F = \frac{dM}{dx}$ or $M = \int F\,dx$

Rate of change of Shear Force or slope of SFD at any point on the beam is equal to the intensity of load at that point.

Properties of BMD and SFD

1) when the load intensity in the region is zero, Shear Force remains constant and Bending Moment varies linearly.

2) When there is Uniformly Distributed Load (UDL), Shear Force varies linearly and BM varies parabolically.

3) When there is Uniformly Varying Load (UVL), Shear Force varies parabolically and Bending Moment varies cubically.

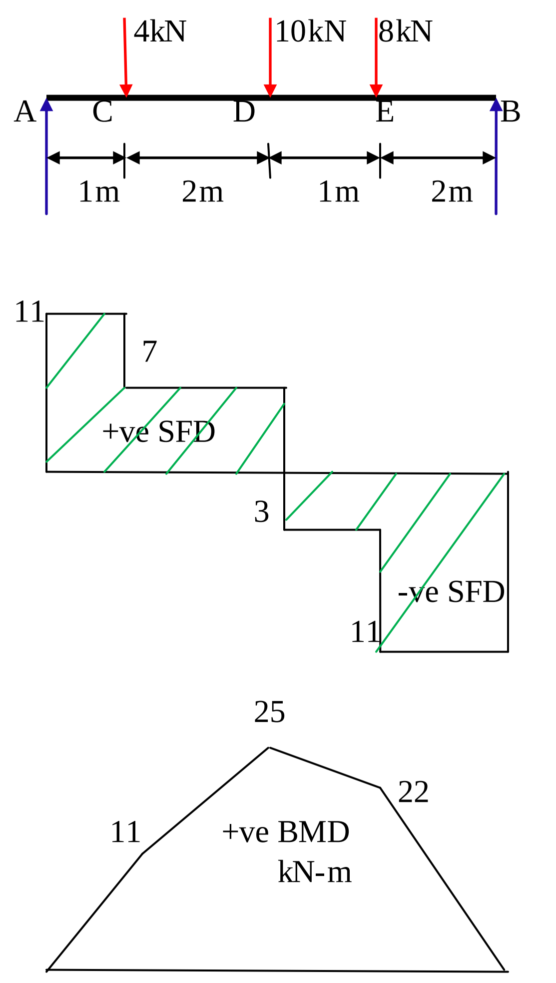
4) The ordinate of SFD changes abruptly at point load. Similarly BMD changes abruptly at

points where couple is acting.

5) The maximum or minimum Bending Moment occurs at a point where shear force is zero.

Problems:

1. **A simply supported beam is carrying point loads, as shown in figure. Draw the SFD and BMD for the beam.**



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 6 - (8 \times 4) - (10 \times 3) - (4 \times 1) = 0$

$R_B = 11$ kN

From condition of static equilibrium:

$R_A + 11 - 4 - 10 - 8 = 0$

$R_A = 11$ kN

Shear Force Calculations

SF at A $F_A = + R_A = + 11$ kN

SF left of C $F_C = + R_A = +11$ kN

SF right of C $F_C = + 11 - 4 = + 7$ kN

SF left of D $F_D = + 11 - 4 - 10 = 7$ kN

SF right of D $F_D = + 11 - 4 - 10 = - 3$ kN

SF left of E $F_E = + 11 - 4 - 10 = - 3$ kN

SF left of E $F_E = + 11 - 4 - 10 - 8 = - 11$ kN

SF left of B $F_B = + 11 - 4 - 10 - 8 = - 11$ kN

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1 m; $M_C = + R_A.1 = 11 \times 1 = 11$ kN m

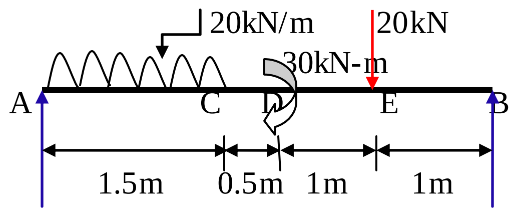
At x = 3 m; $M_D = 11 \times 3 - 4\,(3 - 1) = 25$ kN m

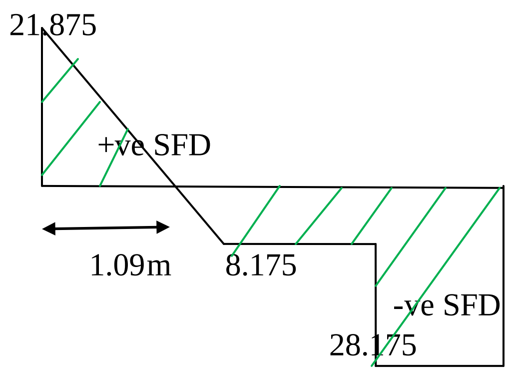
At x= 4m

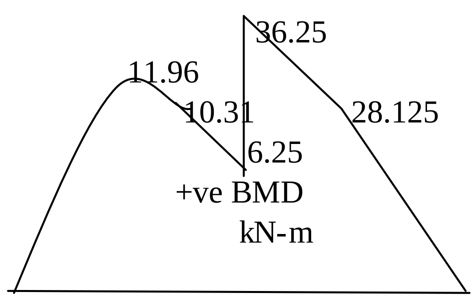
$M_E = 11 \times 4 - 4\,(4 - 1) - 10\,(4 - 3) = 22$ kN m

At X= 6m $M_B = 0$

2. Draw the SF and BM diagram for the simply supported beam loaded as shown in fig.







To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at A $F_A = + R_A = + 21.875$ kN

SF left of C $F_C = +21.875 - 30 = -8.175$ kN

SF right of C $F_C = +21.875 - 30 = -8.175$ kN

SF left of D $F_D = +21.875 - 30 = -8.175$ kN

SF right of D $F_D = +21.875 - 30 = -8.175$ kN

SF left of E $F_E = +21.875 - 30 = -8.175$ kN

SF left of E $F_E = +21.875\text{-}30\text{-}20 = -28.125$ kN

SF left of B $F_B = +21.875\text{-}30\text{-}20 = -28.125$ kN

SF is zero, between A & C at x = 1.09m

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1.5 m;

$M_C = 21.875 \times 1.5 - 20 \times 1.5^2/2 = 10.31$ kN m

At x = 1.09 m;

$M_{max} = 21.875 \times 1.09 - 20 \times 1.09^2/2 = 11.96$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5) = 6.25$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5)+30$

$= 36.25$ kN m

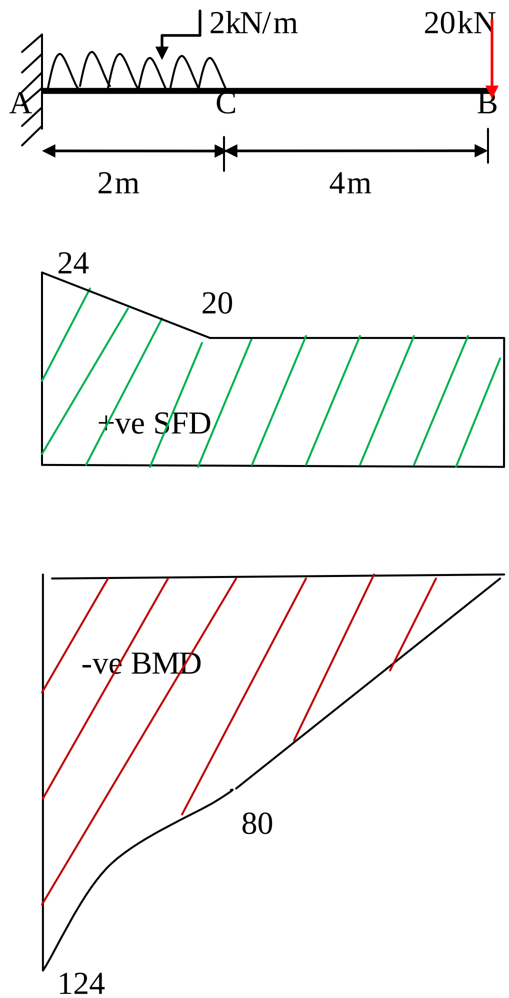
At x= 3m

$M_E = 21.875 \times 3 - 20 \times 1.5(0.75+1.5)+30$

$= 28.125$ kN m

At X= 4m $M_B = 0$

3. A cantilever is shown in fig. Draw the BMD and SFD. What is the reaction at supports?



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at B $F_B = +20$ kN

SF right of C $F_C = +20$ kN

SF left of C $F_C = +20$ kN

SF right of A $F_A = +20 + 2 \times 2 = 24$ kN

Since the moment is considered from right to left the anticlockwise moments are considered to be +ve and clockwise moments are considered to be -ve

Bending moment Calculations

At x = 0, $M_B = 0$

At x = 4 m; $M_c = -20 \times 4 = -80$ kN-m

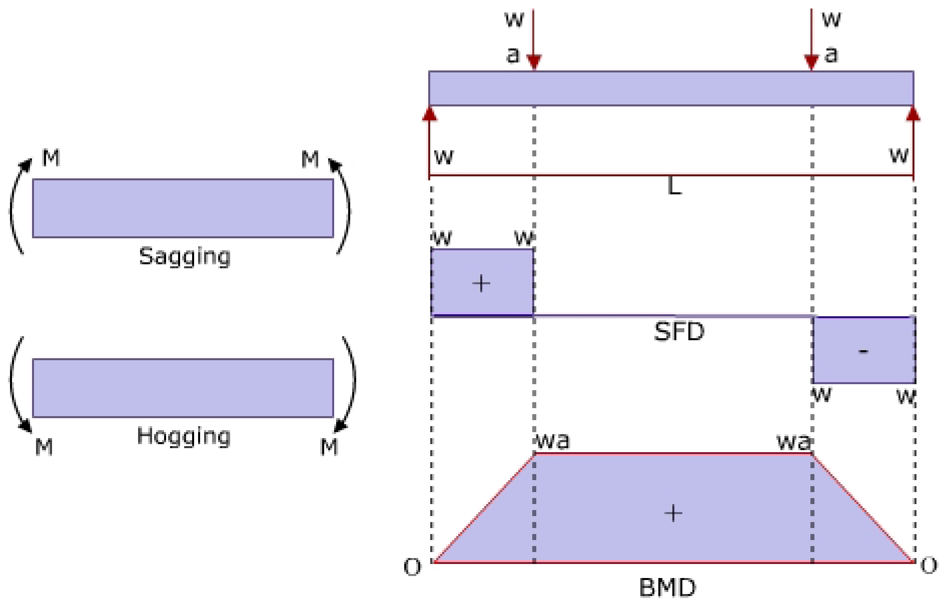
At x = 6 m;

$M_A = -(20 \times 6 + 2 \times 2^2/2) = 124$ kN m

Unit 5

Stresses in Beams

Pure Bending



A beam or a part of a beam is said to be under pure bending if it is subjected to only Bending Moment and no Shear Force.

Effect of Bending in Beams

The figure shows a beam subjected to sagging Bending Movement. The topmost layer is under maximum compressive stress and bottom most layer is under maximum tensile stress. In between there should be a layer, which is neither subjected to tension nor to compression. Such a layer is called "Neutral Layer". The projection of Neutral Layer over the cross section of the beam is called "Neutral Axis".

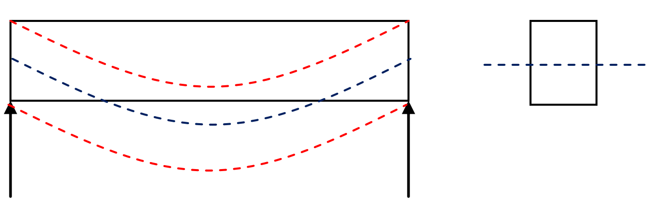


Fig-1

When the beam is subjected to sagging, all layers below the neutral layer will be under tension and all layers above neutral layer will be under compression. When the beam is subjected to hogging, all layers above the neutral layer will be under tension and all the layers below neutral layer will be under compression and vice versa if it is hogging bending moment

Assumptions made in simple bending theory

- The material is isotropic and homogenous.
- The material is perfectly elastic and obeys Hooke's Law i.e., the stresses are within the limit of proportionality.
- Initially the beam is straight and stress free.
- Beam is made up of number of layers and they undergo bending independently.
- Bending takes place over an arc of a circle and the radius of curvature is very large when compared to the dimensions of the beam.
- Normal plane sections before bending remain normal and plane even after bending.
- Young's Modulus of Elasticity is same under tension and compression.

Euler- Bernoulli bending Equation (Flexure Formula)

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

where,

M = Resisting moment developed inside the material against applied bending movement and is numerically equal to bending moment applied (Nmm)

I = Moment of Inertia of cross section of beam about the Neutral Angle. (mm^4)

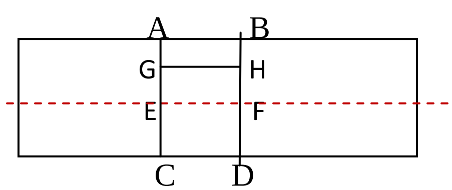
F = Direct Stress (Tensile or Compression) developed in any layer of the beam (N/mm^2)

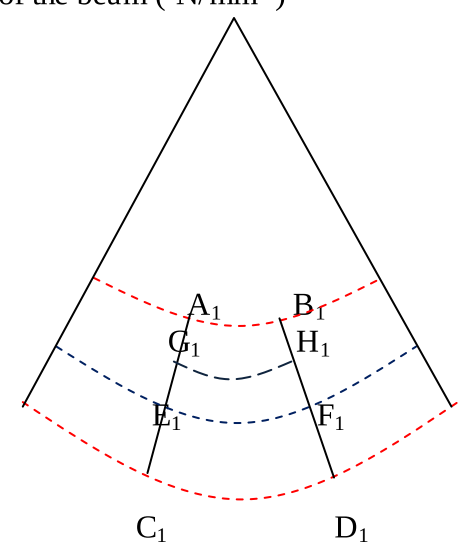
Y = Distance of the layer from the neutral axis (mm)

E = Young's Modulus of Elasticity of the material of the beam (N/mm^2)

R = Radius of curvature of neutral layer (mm)

Euler- Bernoulli's Equation





Consider two section very close together (AB and CD). After bending the sections will be at A_1 B_1 and C_1 D_1 and are no longer parallel. AC will have extended to A_1 C1 and B_1 D_1 will have compressed to B_1D_1. The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z-Z is called the neutral axis.

The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of $E_1F_1 = R$.

Let y be the distance(E'G') of any layer H_1G_1 originally parallel to EF.

Then $H_1G_1/ E_1F_1 =(R+y)\theta /R \theta = (R+y)/R$

and the strain ϵ at layer $H_1G_1= \epsilon = (H_1G_1'- HG) / HG = (H_1G_1- HG) / EF$

$$= [(R+y)\theta - R \theta] /R \theta$$

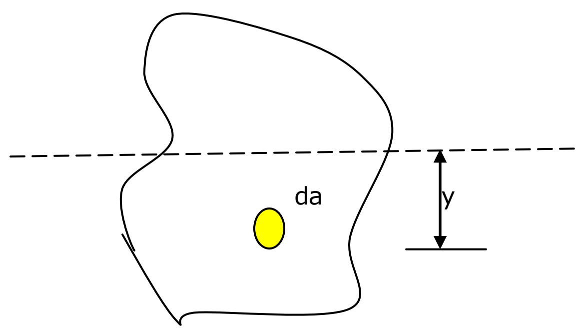
$$\epsilon = y /R.$$

The relation between stress and strain is $\sigma= E. \epsilon$ Therefore

$$\sigma = E. \epsilon = E. y /R$$

$$\sigma / E = y / R$$

Let us consider an elemental area 'da 'at a distance y, from the Neutral Axis.



stress developed over elemental area is $f = \frac{E}{R} y$

Force developed over elemental area = stress × area = $\frac{E}{R} y\ da$

Moment developed over elemental area about NA= Force × distance = $\frac{E}{R} y^2 da$

Total Moment developed from all the elements about the NA = $\int \frac{E}{R} da\, y^2$

$$M = \frac{E}{R}\int da\, y^2 = \frac{E}{R}\, I$$

$$\frac{M}{I} = \frac{E}{R} \ldots\ldots\ldots\ldots\ldots(2)$$

From Eqn (1) and (2), we get

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \quad \ldots\ldots\ldots\ldots\ldots(A)$$

Section Modulus(Z)

$$F = \frac{M}{I}.y$$

$$\Rightarrow f_{max} = \frac{M_{max}}{I}.y_{max}$$

$$i,e.,\ M_{max} = f_{max}.\frac{I}{y_{max}}$$

$$\text{Therefore,} \quad M_{max} = f_{max}.Z$$

Section modulus of a beam is the ratio of moment of inertia of the cross section of the beam about the neutral axis to the distance of the farthest fiber from neutral axis.

$$\text{Therefore,} \quad Z = \frac{I}{y_{max}} \qquad unit = mm^3$$

More the section modulus more will be the moment of resistive (or) moment carrying capacity of the beam. For the strongest beam, the section modulus must be maximum.

Problems:

1. **A beam made of C.I. having a section of 50 mm external diameter and 25 mm internal diameter is supported at two points 4 m apart. The beam carries a concentrated load of 100 N at its centre. Find the maximum bending stress induced in the beam**

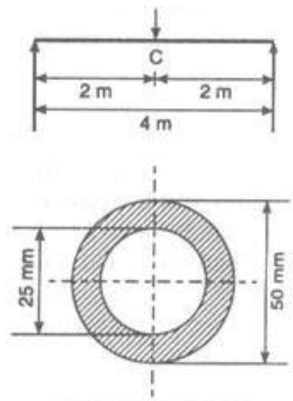
Solution: Outer diameter of cross-section D_0 = 50 mm

Inner diameter of choss-section D_i = 25 mm

$$A = \frac{\pi}{4}(D_0^2 - D_1^2) = 1472.62mm^2$$

$$I = \frac{\pi}{64}(D_0^4 - D_1^4) = 287.62 \times 10^3 mm^4$$

Length of span 'L' = 4 m
Load W = 100 N



Cross section of beam

For a simply supported beam with point load at center the maximum bending moment is $= \frac{WL}{4}$

$$= \frac{100 \times 4}{4} = 100\ Nm = 10000N - mm$$

From Bending equation

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \quad => f = \frac{M\,y}{I} = \frac{10000 \times 25}{287.62 \times 10^3} = 8.692\ N/mm^2$$

2. **A steel bar 10 cm wide and 8 mm thick is subjected to bending moment. The radius of neutral surface is 100 cm. Determine maximum and minimum bending stress in the beam.**

Solution : Assume for steel bar $E = 2 \times 10^5$ N/mm^2

y_{max} = 4mm
R = 1000mm
$f_{max} = E.y_{max}/R$ = (2 × 105 × 4)/1000

We get maximum bending moment at lower most fiber, Because for a simply supported beam tensile stress (+ve value) is at lower most fiber, while compressive stress is at top most fiber (–ve value).

f_{max} = 800 N/mm^2
f_{min} occurs at a distance of –4mm
R = 1000mm
$f_{min} = E.y_{min}/R$ = (2 × 105 x – 4)/1000
f_{min}= –800 N/mm^2

3. **A simply supported rectangular beam with symmetrical section 200mm in depth has moment of inertia of 2.26 x 10^{-5} m^4 about its neutral axis. Determine the longest span over which the beam would carry a uniformly distributed load of 4kN/m run such that the stress due to bending does not exceed 125 MN/m^2.**

Solution: Given data:

Depth d = 200mm = 0.2m
I = Moment of inertia = 2.26 × 10-5 m4
UDL = 4kN/m
Bending stress s = 125 MN/m^2 = 125 × 10^6 N/m^2
Span = ?

Since we know that Maximum bending moment for a simply supported beam with UDL on its entire span is given by = $WL^2/8$

i.e; $M = WL^2/8$ -------------(A)

From bending equation $M/I = f/y_{max}$

$y_{max} = d/2 = 0.2/2 = 0.1$m

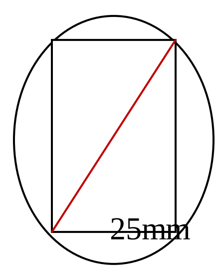
M = f.I/$y$max = [(125 × 106) × (2.26 × 10^{-5})]/ 0.1 = 28250 Nm

Substituting this value in equation (A); we get

28250 = (4 × 103)L^2/8

***L* = 7.52m**

4. **Find the dimension of the strongest rectangular beam that can be cut out of a log of 25 mm diameter.**



Solution: $b^2 + d^2 = 25^2$

$d^2 = 25^2 - b^2$

we Know $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$;

M = f (I/y) = f.Z

M will be maximum when Z will be maximum

$Z = I/y = (bd^3/12)/(d/2) = bd^2/6 = b.(25^2 - b^2)/6$

The value of Z maximum at dZ/db = 0;

i.e.; $d/db[25^2b/6 - b^3/6] = 0$

$25^2/6 - 3b^2/6 = 0$

$b^2 = 25^2/3$

b = 14.43 mm

d = 20.41 mm

Unit 6

Deflection of Beams

INTRODUCTION

Under the action of external loads, the beam is subjected to stresses and deformation at various points along the length. The deformation is caused due to bending moment and shear force. Since the deformation caused due to shear force in shallow beams is very small, it is generally neglected.

Elastic Line:

It is a line which represents the deformed shape of the beam. Hence, it is the line along which the longitudinal axis of the beam bends.

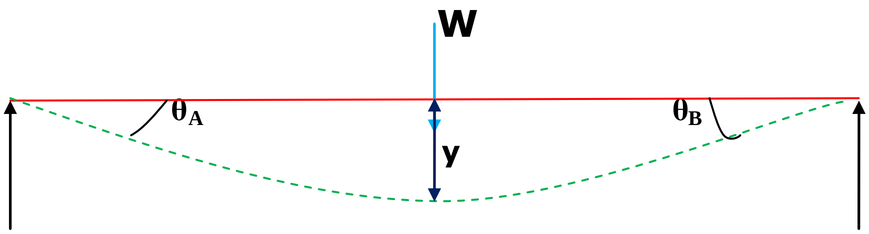
Deflection:

Vertical displacement measured from original neutral surface (refer to earlier chapter) to the neutral surface of the deformed beam.

Slope:

Angle made by the tangent to the elastic curve with respect to horizontal

The designers have to decide the dimensions of beam not only based on strength requirement but also based on considering deflection. In mechanical components excessive deflection causes mis-alignment and non performance of machine. In building it give rise to psychological unrest and sometimes cracks in roofing materials. Deflection calculations are required to impose consistency conditions in the analysis of indeterminate structures.



Strength:

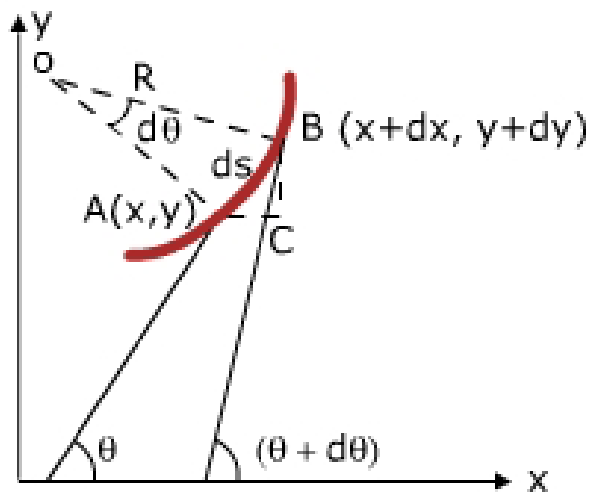
It is a measure of the resistance offered by the beam to load

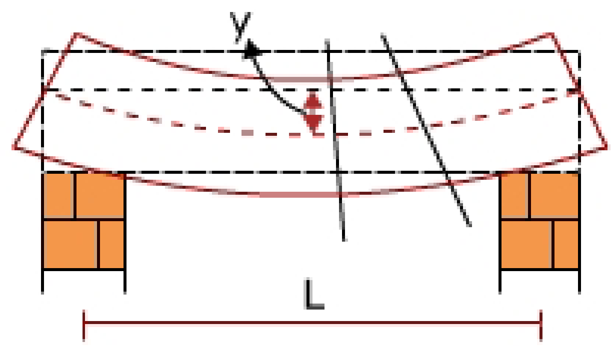
Stiffness:

It is a measure at the resistance offered by the beam to deformation. Usually span / deflection is used to denote the stiffness. Greater the stiffness, smaller will be the deflection. The term (EI) called "flexural rigidity" and is used to denote the stiffness.

Flexural Rigidity

The product of Young's modulus and moment of inertia (EI) is used to denote the flexural rigidity.





Let AB be the part of the beam which is bent into an arc of the circle. Let (x,y) be co-ordinates of A and (x + dx, y + dy) be the co-ordinates of B. Let the length of arc AB = ds. Let the tangents at A and B make angles q and (q + d q) with respect to x-axis.

We have $tan\theta = \frac{dy}{dx}$

Differentiating both sides with respect of x;

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\theta \quad \frac{d\theta}{ds} \quad \frac{ds}{dx} = \frac{d^2y}{dx^2} \qquad \text{----- (1)}$$

we have from figure $ds = Rd\theta$; $\dfrac{d\theta}{ds} = \dfrac{1}{R}$

again in Δ^{le} ABC, $\dfrac{ds}{dx} = \sec\theta$

From eq. 1; $\dfrac{d^2y}{dx^2} = \sec^2\theta \, \dfrac{1}{R} \, \sec\theta$

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\sec^2\theta \sec\theta} = \frac{\dfrac{d^2y}{dx^2}}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

Since dy/dx is small, its square is still small, neglecting $(dy/dx)^2$; we have

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

From bending theory $\dfrac{M}{I} = \dfrac{E}{R}$

$$\frac{M}{EI} = \frac{1}{R} \qquad \text{or}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$M = EI \, \frac{d^2y}{dx^2}$$

This is also known as Euler - Bernoulli's equation.

NOTE:

- While deriving Y-axis is taken upwards
- Curvature is concave towards the positive y axis.
- This occurs for sagging BM, which is positive.

Sign Convention

Bending moment 

Sagging + vc

If Y is $+^{ve}$ - Deflection is upwards

Y is $-^{ve}$ - Deflection is downwards

If θ is $+^{ve}$ – Slope is Anticlockwise

θ is $-^{ve}$ – Slope is clockwise

Methods of Calculating Deflection and Slope

- Double Integration method
- Macaulay's method
- Strain energy method
- Moment area method
- Conjugate Beam method

Each method has certain advantages and disadvantages.

Relationship between Loading, S.F, BM, Slope and Deflection

If Y - deflection

Differentiating $\frac{dy}{dx}$ - Slope (θ)

Differentiating $\frac{d^2y}{dx^2}$ - M. Bending moment

Differentiating $\frac{dM}{dx} = \frac{d^3y}{dx^3}$ = Shear force (F)

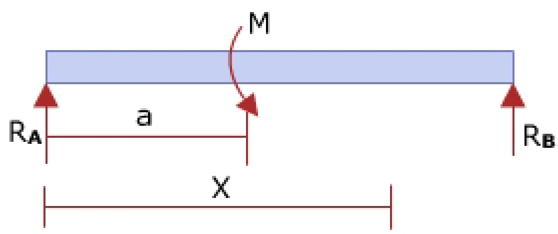
Differentiating $\frac{dF}{dx} = \frac{d^4y}{dx^4}$ = Loading (W)

Macaulay's Method

1. Take the origin on the extreme left.
2. Take a section in the last segment of the beam and calculate BM by considering left portion.
3. Integrate (x-a) using the formula

$$\int (x\text{-}a)\, dx = \frac{(x-a)^2}{2}$$

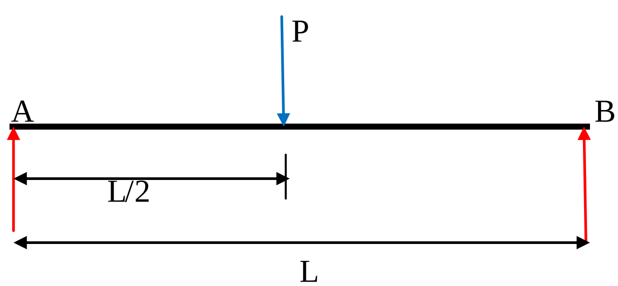
4. If the expression $(x\text{-}a)^n$ becomes negative on substituting the value of x, neglect the terms containing the factor $(x\text{-}a)^n$
5. If the beam carries UDL and if the section doesn't cuts the UDL, extend the UDL upto the section and impose a UDL in the opposite direction to counteract it.
6. If a couple is acting, the BM equation is modified as; $M = R_A x + M (x\text{-}a)^0$.



7. The constant C_1 and C_2 all determined using boundary conditions.
a) S.S. Beam – Deflection is zero at supports
b) Cantilever – Deflection and slope are zero at support.

Problems:

1. **Determine the maximum deflection in a simply supported beam of length L carrying a concentrated load P at its midspan**.



$$EI\,y'' = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle$$

$EI\,y' = \frac{1}{4}Px^2 - \frac{1}{2}P\langle x - \frac{1}{2}L\rangle^2 + C_1$ ....(1)

$EI\,y = \frac{1}{12}Px^3 - \frac{1}{6}P\langle x - \frac{1}{2}L\rangle^3 + C_1x + C_2$ ....(2)

At x =0; y =0 $\therefore$ C_2 =0
At x = L y =0

$0 = \frac{1}{12}PL^3 - \frac{1}{48}PL^3 + C_1L$
$C_1 = -\frac{1}{16}PL^2$
Maximum deflection occurs at x = L/2
Substituting the values of x and C_1 in equation…. (2)

$$EI\,y_{max} = \frac{1}{12}P(\frac{1}{2}L)^3 - \frac{1}{6}P(\frac{1}{2}L - \frac{1}{2}L)^3 - \frac{1}{16}PL^2(\frac{1}{2}L)$$

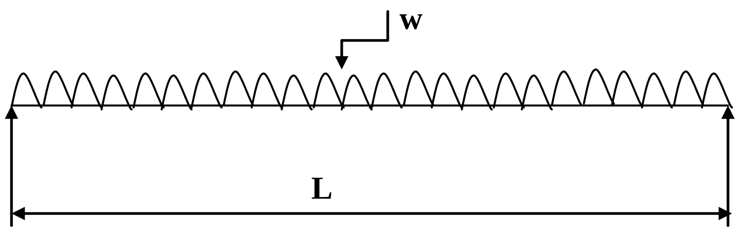
$$y_{max} = -\frac{PL^3}{48EI}$$

The negative sign indicates that the deflection is below the undeformed neural axis

$$\delta_{max} = \frac{PL^3}{48EI}$$

3. **Determine the maximum deflection in a simply supported beam of length L carrying a uniformly distributed load 'w' for the entire length of the beam.**

Solution : From the following fig



$EI\,y'' = \frac{1}{2}w_oLx - \frac{1}{2}w_ox^2$
$EI\,y' = \frac{1}{4}w_oLx^2 - \frac{1}{6}w_ox^3 + C_1$ **....................(1)**
$EI\,y = \frac{1}{12}w_oLx^3 - \frac{1}{24}w_ox^4 + C_1x + C_2$**............(2)**

At x =0 y=0 and C_2 =0

At x =L y =0

$0 = \frac{1}{12} w_o L^4 - \frac{1}{24} w_o L^4 + C_1 L$

$C_1 = -\frac{1}{24} w_o L^3$

Substituting the C_1 values in equation 2 we get

$$EI\,y = \frac{1}{12} w_o L x^3 - \frac{1}{24} w_o x^4 - \frac{1}{24} w_o L^3 x$$

x = L/2, y is maximum due to symmetric loading

$$EI\,y_{max} = \frac{1}{12} w_o L (\tfrac{1}{2} L)^3 - \frac{1}{24} w_o (\tfrac{1}{2} L)^4 - \frac{1}{24} w_o L^3 (\tfrac{1}{2} L)$$

$$EI\,y_{max} = -\frac{5}{384} w_o L^4$$

$$\delta_{max} = \frac{5 w_o L^4}{384 EI}$$

UNIT 7

TORSION OF SHAFTS

Bending Moment

The moment applied in a vertical plane containing the longitudinal axis is resisted by longitudinal tensile and compressive stresses of varying intensities across the depth of beam and are called as bending stresses. The moment applied is called Bending Moment.

Torsional Moment

The moment applied in a vertical plane perpendicular to the longitudinal axis i.e., in the plane of the cross section of the member, it causes twisting of layers which will be resisted by the shear stresses. The moment applied is called Torsion Moment or Torsional Moment. Torsion is useful form of transmitting power and its application is seen in screws and shafts.

PURE TORSION

A circular member is said to be in a state of pure torsion when it is subjected to a twisting moment which coincides with the axis of the shaft and not accompanied by bending and axial force.

ASSUMPTIONS IN TORSION THEORY

1. Material is homogenous and isotropic
2. Plane section remain plane before and after twisting i.e., no warpage of planes.
3. Twist along the shaft is uniform.
4. Radii which are straight before twisting remain straight after twisting.
5. Stresses are within the proportional limit.

DERIVATION OF TORSIONAL EQUATION:

Torsional Rigidity

$$\text{We have} \qquad \theta = \frac{TL}{CI_p}$$

As product (CI_P) is increased deformation q reduces. This product gives the strength of the section to resist torque and is called Torsional rigidity.

www.rejinpaul.com

Polar Modulus : (Z_P)

We have $\frac{T}{I_p} = \frac{f}{r}$

Maximum shear stress occurs at surface

$$T = f_s \cdot \frac{I_p}{R}$$

$$T = f_s \cdot Z_p$$

Where Z_P is called polar modulus $Z_p = \frac{I_p}{R}$

POWER TRANSMITTED BY SHAFT

Power transmitted = Torsional moment x Angle through which the torsional moment rotates / unit tank

If the shaft rotates with 'N' rpm

$$= T\left(\frac{N \cdot 2\pi}{60'}\right)$$

$$\text{Power transmitted} = \frac{2\pi NT}{60} \text{ N.m / sec}$$

$$\text{Power transmitted in kw} = \frac{2\pi NT}{60 \times 1000} = \frac{\pi NT}{30{,}000}$$

Note:

N is in rpm and T is in N-m

Problems:

1. **Find the maximum shear stress induced in a solid circular shaft of diameter 200 mm when the shaft transmits 190 kW power at 200 rpm**

Given data: Power transmitted, P = 190 kW, $I_p = \frac{\pi D^4}{32} = 1.57 \text{ X } 10^8 \text{ mm}^4$

speed N = 200 rpm and diameter of shaft = 200 mm.

$$P = \frac{2\pi NT}{60000}$$

$$T = \frac{60000 \times 190}{2\pi \times 200} = 9076.4N - m = 9.08 \times 10^6 \text{N-mm}$$

From the formulae $\frac{T}{I_p} = \frac{f_s}{r}$

Substituting all the values $f_s = 5.78 \text{N/mm}^2$.

2. **A solid shaft of mild steel 200 mm in diameter is to be replaced by hollow shaft of allowable shear stress is 22% greater. If the power to be transmitted is to be increased by 20% and the speed of rotation increased by 6%, determine the maximum internal diameter of the hollow shaft. The external diameter of the hollow shaft is to be 200 mm.**

Solution: Given that:

Diameter of solid shaft $d = 200$ mm

For hollow shaft diameter, $d_0 = 200$ mm

Shear stress; $t_H = 1.22\ t_s$

Power transmitted; $P_H = 1.20\ P_s$

Speed $N_H = 1.06\ N_s$

As the power transmitted by hollow shaft

$P_H = 1.20\ P_s$

$(2\pi.N_H.T_H)/60 = (2\pi.N_s.T_s)/60 \times 1.20$

$N_H.T_H = 1.20\ N_s.T_s$

$1.06\ N_s.T_H = 1.20\ N_sT_s$

$1.06/1.20\ T_H = T_s$

$1.06/1.20 \times \pi/16\ t_H\ [(d_0)^4 - (d_i)^4/d_0] = \pi/16\ t_s.[d]^3$

$1.06/1.20 \times 1.22\ t_s\ [(200)^4 - (d_i)^4/200] = t_s \times [200]^3$

$d_i = 104$ mm

3. **A solid shaft is subjected to a maximum torque of 1.5 MN.cm Estimate the diameter for the shaft, if the allowable shearing stress and the twist are limited to 1 kN/cm^2 and 1o respectively for 200 cm length of shaft. Take $G = 80 \times 10^5$ N/cm^2**

Solution: Since we have

$T/I_p = f_s/r = C.\theta/L$

$f_s = T.I_p\ r = 1.5 \times 10^6 / \theta/32.d^4 . d/2$

$1 \times 10^3 * 2\pi /1.5 \times 10^6 * 32 = 1/d^3$

$d = 19.69$ cm

$\theta = T.L / C.I_p$

$1.5 \times 10^6 * 2\pi / 1.5 \times 10^6 * 32 = 1 / d^3$

$d = 19.69$ cm

$\theta = T.L / C.Ip$

$1.5 \times 10^6 * 200\ d/80 * 10^5 * \pi/32\ d^4 = \pi/180$

$d^3 = 1.5 \times 10^6 * 180 * 200 * 32 / (80 * 10^5 * \pi * \pi)$

$d = 27.97$ cm

4. **A hollow circular shaft of 20 mm thickness transmits 300 kW power at 200 r.p.m. Determine the external diameter of the shaft if the shear strain due to torsion is not to exceed 0.00086. Take modulus of rigidity = 0.8 × 10^5 N/mm^2.**

Solution: Let d_i = inner diameter of circular shaft

d_0 = outer diameter of circular shaft

Then $d_0 = d_i + 2t$ where t = thickness

$d_0 = d_i + 2 * 20$

$d_0 = d_i + 40$

$d_i = d_0 - 40$

Since we have

Power transmitted = 2π NT/60

$300{,}000 = 2\pi * 200 * T / 60$

$\rightarrow$ T = 14323900 N mm

Also, we have $C = f_s/y$

$\rightarrow$ $0.8 * 10^5 = f_s / 0.00086$

$\rightarrow$ $f_s = 68.8$ N/mm^2

Now $T = \pi/16.\ f_s.(d_0^4 - d_i^4 / d_0)$

$14323900 = f_s /16 * 68.8\ (d_0^4 - (d_0 - 40)^4 / d_0)$

$1060334.6\ d_0 = d_0^4 - (d_0 - 40)^4$

$= (d_0^2 - d_0^2 + 80d_0 - 1600)*(d_0^2 + d_0^2 - 80d_0 + 1600)$

$= (80d_0 - 1600)\ (2d_0^2 - 80d_0 + 1600)$

$= 80\ (d_0 - 2_0) * 2 * (d_0^2 - 40\ d_0 + 800)$

$= 160\ (d_0^3 - 40d_0^2 + 800\ d_0 - 20\ d_0^2 + 800\ d_0 - 16000)$

$\rightarrow$ $1060334.6\ d_0 / 160 = d_0^3 - 60d_0^2 + 1600d_0 - 16000$

$\rightarrow$ $6627\ d_0 = d_0^3 - 60d_0^2 + 1600\ d_0 - 1600$

$\rightarrow$ $d_0^3 - 60d_0^2 + 1600d_0 - 6627\ d_0 - 16000 = 0$

$\rightarrow$ $d_0^3 - 60d_0^2 - 5027\ d_0 - 16000 = 0$

Using trial and error method to solve the above equation for d_0, we get d_0 = 107.5 mm.

Unit 8

Elastic Stability of Columns

Columns and Struts:

Columns and struts are structural members subjected to compressive forces. Theses members are often subjected to axial forces, although they may be loaded eccentrically. The lengths of these members are large compared to their lateral dimensions. In general vertical compressive members called columns and inclined compressive members are called struts.

CLASSIFICATION OF COLUMNS:

Columns are generally classified in to three general types. The distinction between types of columns is not well, but a generally accepted measure is based on the slenderness ratio (l_e/r_{min}).

Short Column :

A short column essentially fails by crushing and not by buckling. A column is said to be short, if $l_e/b \leq 15$ or $l_e/r_{min} \leq 50$, where l_e = effective length, b = least lateral dimension and r_{min} = minimum radius of gyration.

Long Column :

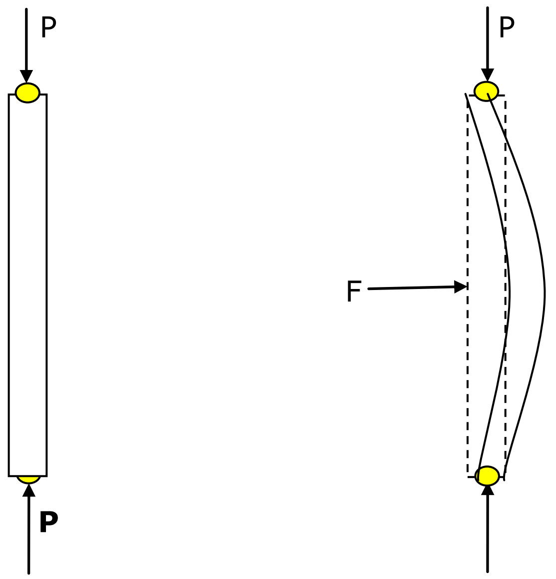
A long column essentially fails by buckling and not by crushing. In long columns, the stress at failure is less than the yield stress. A column is said to be long $l_e/b > 15$ or $l_e/r_{min} > 50$.

Intermediate Column :

An intermediate column is one which fails by a combination of crushing and buckling.

Elastic Stability of Column

Consider a long column subjected to an axial load P as shown in figure. The column deflects laterally when a small test load F is applied in lateral direction. If the axial load is small, the column regains its stable position when the test load is removed. At a certain value of the axial load, the column fails to regain its stable position even after the removal of the test load. The column is then said to have failed by buckling and the corresponding axial load is called Critical Load or failure Load or Crippling Load



SLENDERNESS RATIO (λ)

Slenderness ratio is defined as the ratio of effective length (l_e) of the column to the minimum radius of gyration (r_{min}) of the cross section.

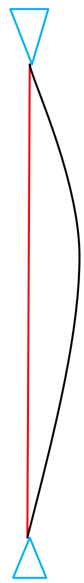
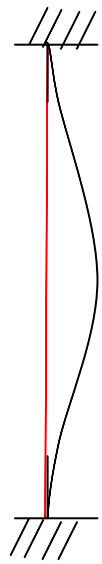
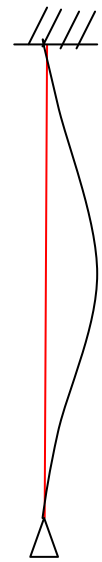
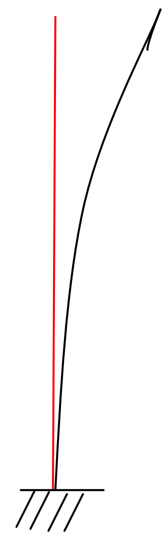
$$\lambda = \frac{l_e}{r_{min}}$$

Since an axially loaded column tends to buckle about the axis of minimum moment of inertia (I_{min}), the minimum radius of gyration is used to calculate slenderness ratio.

Further, $r_{min} = \sqrt{\frac{I_{min}}{A}}$, where A is the cross sectional area of column.

EFFECTIVE LENGTH OF COLUMN (*l*e)

Effective length is the length of an imaginary column with both ends hinged and whose critical load is the same as the column with given end conditions. It should be noted that the material and geometric properties should be the same in the above columns. The effective length of a column depends on its end condition. Following are the effective lengths for some standard cases.

Both ends are hinged	Both ends are fixed	One end fixed and other end hinged	One end fixed and other end is free
			
Effective Length $L_e = L$	Effective Length $L_e = \frac{L}{2}$	Effective Length $L_e = \frac{L}{\sqrt{2}}$	Effective Length $L_e = 2L$

Euler's Theorem

Theoretical analysis of the critical load for long columns was made by the great Swiss mathematician Leonard Euler (pronounced as Oiler). The assumptions made in the analysis are as follows:

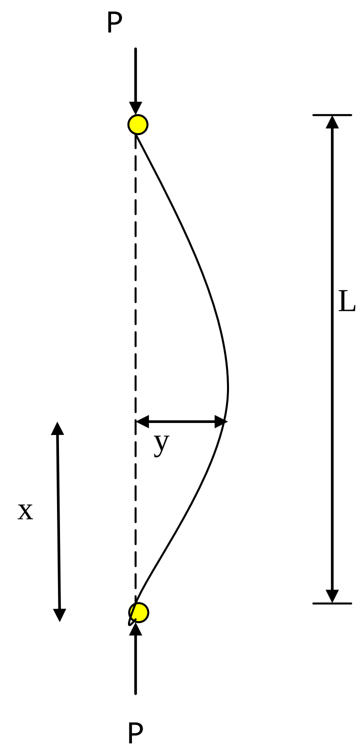
- The column is long and fails by buckling.
- The column is axially loaded.
- The column is perfectly straight and the cross sections are uniform (prismatic).
- The column is initially free from stress.
- The column is perfectly elastic, homogeneous and isotropic.

Eulers Critical Load for Long Columns

Case (1) Both ends hinged

Consider a long column with both ends hinged subjected to critical load P as shown.

www.rejinpaul.com



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P and deflection y is given by

$$M = -P\,y \qquad \text{------------ (1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \quad \text{or } M = EI\,\frac{d^2y}{dx^2} \text{ ....(2)}$$

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y = EI\,\frac{d^2y}{dx^2}$$

$$\text{or} \qquad \frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right) y = 0$$

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) \qquad \text{------(3)}$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at each end since it is pinned at each end. Applying these conditions (putting these values into the eq. (3)) gives us the following results: For y to be zero at x =0, the value of C_2 must be zero (since cos (0) = 1).

(i) Substituting y = 0 at x = L in eq. (3) lead to the following.

$$0 = C_1 \sin\left(L\sqrt{\frac{P}{EI}}\right)$$

While for y to be zero at x = L, then either C_1 must be zero (which leaves us with no equation at all, if C_1 and C_2 are both zero), or

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

which results in the fact that

$$\left(L\sqrt{\frac{P}{EI}}\right) = n\pi$$

$$\text{or} \qquad L\sqrt{\frac{P}{EI}} = n\pi \qquad \text{where } n = 0,1,2,2\ldots$$

$$\text{or} \qquad P = \frac{n^2\pi^2 EI}{L^2}$$

Taking least significant value of n, i.e. n = 1

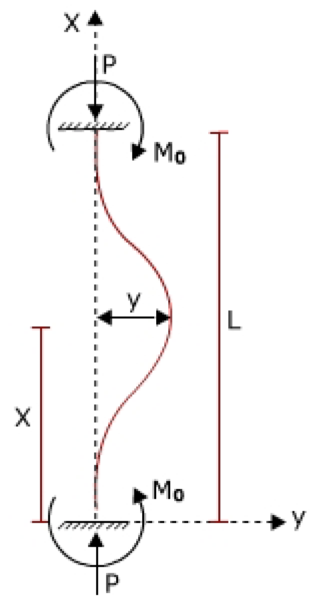
We have $$P = \frac{\pi^2 EI}{L^2}$$

or $$P_E = \frac{\pi^2 EI}{l_e^2}$$

where $l_e = L$.

Case (2) Both ends fixed

Consider a long column with both ends fixed subjected to critical load P as shown.



Consider a section at a distance x from the origin. Let y be the deflection of the column at this section. Bending moment in terms of load P, fixed end moment M 0 and deflection y is given by

$$M = -P\,y + M_0 \qquad \text{-------------(1)}$$

We can also write that for beams/columns the bending moment is proportional to the curvature of the beam, which, for small deflection can be expressed as

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

or $$M = EI\frac{d^2y}{dx^2} \qquad \text{------------(2)}$$

where E is the Young's modulus and I is the moment of Inertia.

Substituting eq.(1) in eq.(2)

$$-P\,y + M_0 = E\,I\,\frac{d^2y}{dx^2}$$

or
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = \frac{M_0}{EI}$$

This is a second order differential equation, which has a general solution form of

$$y = C_1 \sin\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad \text{---------} \ (3)$$

where C_1 and C_2 are constants. The values of constants can be obtained by applying the boundary conditions:

(i) y = 0 at x = 0. That is, the deflection of the column must be zero at near end since it is fixed. Applying this condition (putting these values into the eq. (3)) gives us the following result:

$$C_2 = -\frac{M_0}{P}$$

ii) At X = 0 $\frac{dy}{dx}$ =0, that is, the slope of the column must be zero, since it is fixed.

$$\frac{dy}{dx} = C_1\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) - C_2\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{-------}(4)$$

Substituting the boundary condition in eq. (4)

$$0 = C_1\sqrt{\frac{P}{EI}}$$

Hence, $C_1 = 0$

Substituting the constants C_1 and C_2 in eq. (3) leads to the following

$$y = -\frac{M_0}{P}\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \qquad \text{---------(5)}$$

To find Euler's critical load P E apply the third boundary condition.

when x = L , y = 0. That is, the deflection of the fixed end of the column must be zero. Applying these conditions (substituting these values into the eq. (5)) the following results were obtained

$$0 = -\frac{M_0}{P}\cos\left(L\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

$$1 = \cos\left(L\sqrt{\frac{P}{EI}}\right)$$

$$\left(L\sqrt{\frac{P}{EI}}\right) = n\pi$$

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad \text{where n=0,2,4,6,……}$$

Limitations of Euler's Theory

1. We have, Euler's critical load, $P_E = \frac{\pi^2 EI}{l_e^2}$ It can be seen that critical load depends only on the modulus and dimensions of column, and not on strength of the material. For instance, according to Euler's formula the critical load remains the same whether the columns is made of mild steel or high strength steel. However, one should note that the above columns will have different strengths.

2. We have, Euler's critical load,

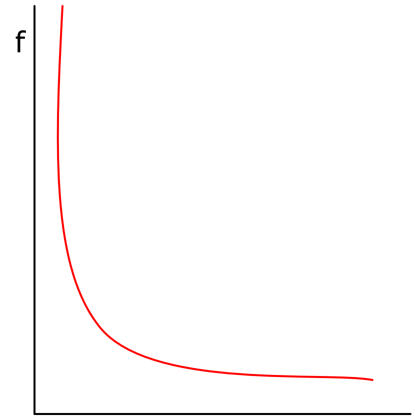
Therefore corresponding limiting stress at failure is given by

$$P_E = \frac{\pi^2 EI}{l_e^2}$$

$$f = \frac{P_E}{A} = \frac{\pi^2 EI}{l_e^2 A} = \frac{\pi^2 E}{\left(\frac{l_e}{r_{min}}\right)^2}$$

Since $\frac{I_{min}}{A} = r_{min}$

The variation of limiting stress 'f' versus slenderness ratio $\frac{l_e}{r_{min}}$ in the above equation is shown below.



The above plot shows that the limiting stress 'f' decreases as increases. In fact, when very small, limiting stress is is close to infinity, which is not rational. Limiting stress cannot be greater than the yield stress of the material.

3. Eulers formula determines the critical load, not the working load. Suitable factor of safety (which is about 1.7 to 2.5) should be considered to obtain the allowable load.

Rankine's critical Load

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} \qquad \text{............} \quad (1)$$

Where, P_R = Rankine's critical load

$P_C = f_C A$ = Crushing load for short columns

$P_E = \frac{\pi^2 EI}{l_e^2}$ = Euler's critical load for long columns

Rankine Gordon Load is given by the following empirical formula,

This relationship is assumed to be valid for short, medium and long columns. This relation can be used to find the load carrying capacity of a column subjected to crushing and/or buckling.

From eq. (1) $$P_R = \frac{P_E\, P_C}{P_E + P_C} = \frac{P_C}{1+\frac{P_C}{P_E}}$$

Substituting P_C and P_E in the above relation

$$P_R = \frac{f_c\ A}{1 + \left[\frac{f_c\ A}{\frac{\pi^2 E I}{l_e^2}}\right]} = \frac{f_c\ A}{1 + \left(\frac{f_c}{\pi^2 E}\right)\left[\frac{l_e^2\ A}{I}\right]}$$

Since $$\frac{I_{min}}{A} = (r_{min})^2$$

$$P_R = \frac{f_c\ A}{1 + a\left[\frac{l_e}{r_{min}}\right]^2}$$

Here, $a = \frac{f_c}{\pi^2 E} =$ Rankine Constant, and f_c = Crushing or yield stress

The Rankine formula is applicable to both short and long columns

The Rankine's formula for critical load is given by

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} \qquad \text{------------------(1)}$$

where, P_C = Critical load for short column that fails in crushing

P_E = Euler's critical load for long column that fails by buckling

In short columns, P_E is very large when compared with P_C , i.e., $P_E >> P_C$

Hence $\frac{1}{P_E} \approx 0$

Substituting in eq (1) $\frac{1}{P_R} = \frac{1}{P_c} \implies P_R = P_c$

Hence it can be concluded that Rankine's formula is valid for short columns.

In long columns, P_C is very large when compared with P_E , i.e., $P_C >> P_E$

Hence $\frac{1}{P_E} \approx 0$

Substituting in eq (1) $\frac{1}{P_R} = \frac{1}{P_c} \implies P_R = P_c$

Hence it can be concluded that Rankine's formula is valid for long columns.

Problems:

1. **A solid round bar 4.5 m long and 60 mm in diameter was found to extend 4.6 mm under a tensile load of 60 kN. This bar is used as a strut with both ends fixed. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.**

Given data: Diameter of the bar = 60 mm, length = 4.5 m, tensile load = 60 kN, FS = 4 and deflection = 4.6 mm. Effective length = $2l$ = 9000mm

Area of the bar = 2827mm^2, I = 10.18 X 10^6 mm^4

$$E = \frac{P/A}{dl/l} = 2.076 \times 10^4 N/mm^2$$

$$P_E = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 10.18 \times 10^6 \times 2.076 \times 10^4}{9000^2} = 25747\ N$$

Safe Load = 25747/4= 6436.75 N

2. **Find the Euler's critical load for a hollow cylindrical cast iron column 150 mm external diameter, 20 mm wall thickness if it is 6 m long with hinges at both ends. Assume Young's modulus of cast iron as 80 kN/mm^2. Compare this load with given by Rankine's formula using Rankine's constant a = 1/1,600 and f_c = 567 N/mm^2.**

Given Data: External diameter = 150 mm, thickness = 20 mm, length = 6 m, E = 80 kN/mm^2, a = 1/1,600 and f_c = 567 N/mm^2.

A_0 = 8167.41 mm^2 , I = 17.66 X 10^6 mm^4

From Euler's Condition $l_e = l$ $P_E = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 80 \times 10^3 \times 17.66 \times 10^6}{6000^2} = 387.33kN$

From Rankine Formulae $P_R = \frac{f_c A}{1+a\left(\frac{l_e}{r_{min}}\right)^2} = \frac{567 \times 8167.41}{1+\frac{1}{1600}\left(\frac{6000}{46.5}\right)^2} = 406.01kN$