GRAPH THEORY AND COMBINATORICS

Sub code:10CS42 **IA Marks:25**
Hours/Week:04 **Exam hours:03**
Total hours:52 **Exam marks:100**

UNIT-1

Introduction to Graph Theory: Definition and Examples Subgraphs Complements, and Graph Isomorphism Vertex Degree, Euler Trails and Circuits.

UNIT-2

Introduction the Graph Theory Contd.: Planner Graphs, Hamiloton Paths and Cycles, Graph Colouring and Chromatic Polynomials.

UNIT-3

Trees : definition, properties and examples, rooted trees, trees and sorting, weighted trees and prefix codes

UNIT-4

Optimization and Matching**:** Dijkstra's Shortest Path Algorithm, Minimal Spanning Trees – The algorithms of Kruskal and Prim, Transport Networks – Max-flow, Min-cut Theorem, Matching Theory

UNIT-5

Fundamental Principles of Counting: The Rules of Sum and Product, Permutations, Combinations – The Binomial Theorem, Combinations with Repetition, The Catalon Numbers

UNIT-6

The Principle of Inclusion and Exclusion: The Principle of Inclusion and Exclusion, Generalizations of the Principle, Derangements – Nothing is in its Right Place, Rook Polynomials

UNIT-7

Generating Functions: Introductory Examples, Definition and Examples – Calculational Techniques, Partitions of Integers, The Exponential Generating Function, The Summation Operator

UNIT-8

Recurrence Relations: First Order Linear Recurrence Relation, The Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients, The Non-homogeneous Recurrence Relation, The Method of Generating Functions

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UNIT 1
INTRODUCTION

This topic is about a branch of discrete mathematics called graph theory. Discrete mathematics – the study of discrete structure (usually finite collections) and their properties include combinatorics (the study of combination and enumeration of objects) algorithms for computing properties of collections of objects, and graph theory (the study of objects and their relations).

Many problem in discrete mathematics can be stated and solved using graph theory therefore graph theory is considered by many to be one of the most important and vibrant fields within discrete mathematics.

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DISCOVERY

It is no coincidence that graph theory has been independently discovered many times, since it may quite properly be regarded as an area of applied mathematics .Indeed the earliest recorded mention of the subject occurs in the works of Euler, and although the original problem he was considering might be regarded as a some what frivolous puzzle, it did arise from the physical world.

Kirchhoff's investigations of electric network led to his development of the basic concepts and theorems concerning trees in graphs. While Cayley considered trees arising from the enumeration of organic chemical isomer's. Another puzzle approach to graphs was proposed by Hamilton. After this, the celebrated four color conjecture came into prominence and has been notorious ever since. In the present century, there have already been a great many rediscoveries of graph theory which we can only mention most briefly in this chronological account.

WHY STUDY GRAPH?

The best way to illustrate the utility of graphs is via a "cook's tour" of several simple problem that can be stated and solved via graph theory. Graph theory has many practical applications in various disciplines including, to name a few, biology, computer science, economics, engineering, informatics, linguistics, mathematics, medicine, and social science, (As will become evident after reading this chapter) graphs are excellent modeling tools, we now look at several classic problems.

We begin with the bridges of Konigsberg. This problem has a historical significance, as it was the first problem to be stated and then solved using what is now known as graph theory. Leonard euler fathered graph theory in 1973 when his general solution to such problems was published euler not only solved this particular problem but more importantly introduced the terminology for graph theory.

1. THE KONIGSBERG BRIDGE PROBLEM

Euler (1707-- 1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem of his day called the Konigsberg bridge problem. The city of

Konigsberg was located on the Pregel river in Prussia, the city occupied two island plus areas on both banks. These region were linked by seven bridges as shown in fig(1.1).

The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point one can easily try to solve this problem empirically but all attempts must be unsuccessful, for the tremendous contribution of Euler in this case was negative.

In proving that the problem is unsolvable, Euler replaced each land area by a point and each bridge by a line joining the corresponding points these by producing a “graph” this graph is shown in fig(1.2) where the points are labeled to correspond to the four land areas of fig(1.1) showing that the problem is unsolvable is equivalent to showing that the graph of fig(1.2) cannot be traversed in a certain way.

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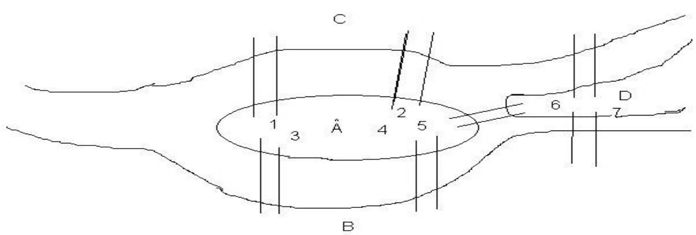


Figure1.1: A park in Konigsberg 1736

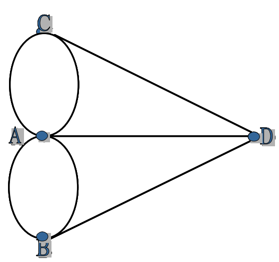


Figure1.2: The Graph of the Konigsberg bridge problem

Rather than treating this specific situation, Euler generalized the problem and developed a criterion for a given graph to be so traversable; namely that it is connected and every point is incident with an even number of lines. While the graph in fig(1.2) is connected, not every point incident with an even number of lines.

2. **ELECTRIC NETWORKS**

Kirchhoffs developed the theory of trees in 1847 in order to solve the system of simultaneous linear equations linear equations which gives the current in each branch and around each circuit of an electric network..

Although a physicist he thought like a mathematician when he abstracted an electric network with its resistances, condensers, inductances, etc, and replaced it by its corresponding combinatorial structure consisting only of points and lines without any indication of the type of electrical element represented by individual lines. Thus, in effect, Kirchhoff replaced each electrical network by its underlying graph and showed that it is not necessary to consider every cycle in the graph of an electric network separating in order to solve the system of equation.

Instead, he pointed out by a simple but powerful construction, which has since became std procedure, that the independent cycles of a graph determined by any of its "spanning trees" will suffice. A contrived electrical network N, its underlying graph G, and a spanning tree T are shown in fig(1.3)

N:

G:

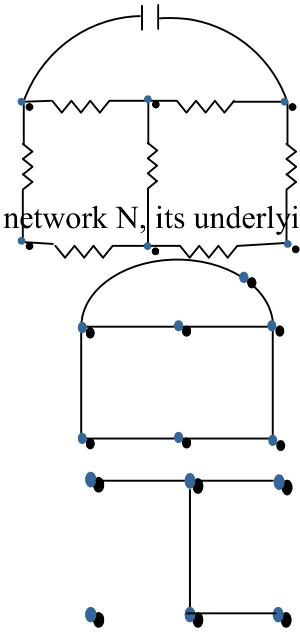
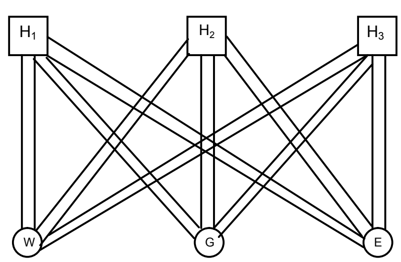


Fig (1.3)- A network N, its underlying graph G, and a spanning tree T

3. UTILITIES PROBLEM

These are three houses fig(1.4) H_1, H_2, and H_3, each to be connected to each of the three utilities water(w), gas(G), and electricity(E)- by means of conduits, is it possible to make such connection without any crossovers of the conduits?

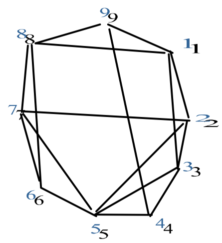


Fig(1.4)- three – utilities problem

Fig(1.4) shows how this problem can be represented by a graph – the conduits are shown as edges while the houses and utility supply centers are vertices

4. SEATING PROBLEM

Nine members of a new club meet each day for lunch at a round table they decide to sit such that every members has different neighbors at each lunch



Fig(1.5) – Arrangements at a dinner table

How many days can this arrangement lost?

This situation can be represented by a graph with nine vertices such that each vertex represent a member, and an edge joining two vertices represents the relationship of sitting next to each other. Fig(1.5) shows two possible seating arrangement – these are 1 2 3 4 5 6 7 8 9 1 (solid lines), and 1 3 5 2 7 4 9 6 8 1 (dashed lines) it can be shown by graph – theoretic considerations that there are only two more arrangement possible. They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is (n-1)/2, if n is odd. (n-2)/2, if n is even

WHAT IS A GRAPH?

A linear graph (or simply a graph) G = (V,E) consists of a set of objects V = {v_1, v_2,…..} called vertices, and another set E = {e_1, e_2,…..} whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i , v_j) of vertices. The vertices v_i , v_j associated with edge e_k are called the end vertices of e_k . The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices

The object shown in fig (a)

The Object Shown in Fig.(a)

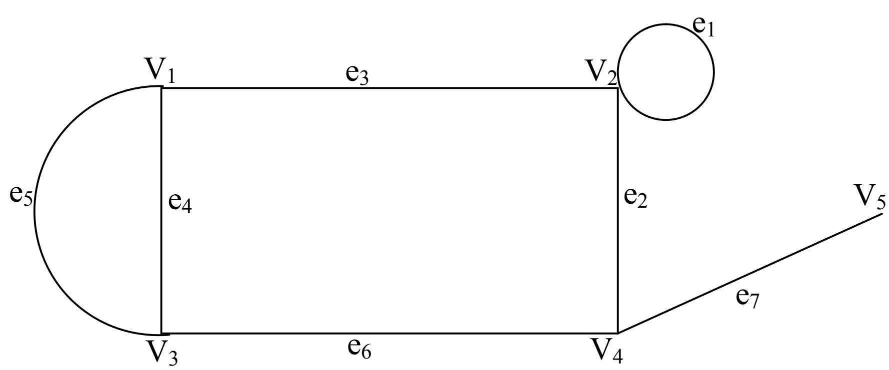
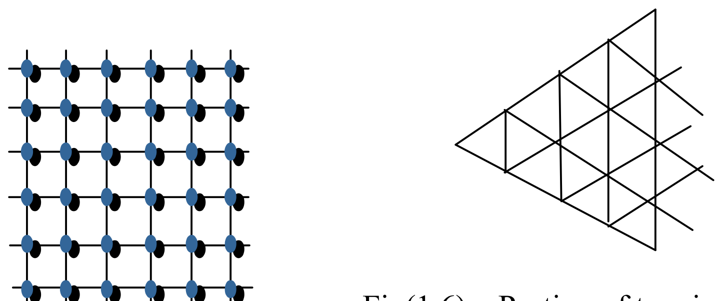


Fig (a) – Graph with five vertices and seven edges

Observe that this definition permits an edge to be associated with a vertex pair (v_i , v_j) such an edge having the same vertex as both its end vertices is called a self-loop. Edge e_1 in fig (a) is a self-loop. Also note that the definition allows more one edge associated with a given pair of vertices, for example, edges e_4 and e_5 in fig (a), such edges are referred to as 'parallel edges'. A graph that has neither self-loops nor **<u>parallel edges</u>** is called a '**<u>simple graph</u>**'.

FINITE AND INFINITE GRAPHS

Although in our definition of a graph neither the vertex set V nor the edge set E need be finite, in most of the theory and almost all application these sets are finite. A graph with a finite number of vertices as well as a finite number of edge is called a '**finite graph**': otherwise it is an **infinite graph**. The graphs in fig (a), (1.2), are all examples of finite graphs. Portions of two infinite graphs are shown below



Fig(1.6) – Portion of two infinite graphs

INCIDENCE AND DEGREE

When a vertex v_i is an end vertex of same edge e_j , v_i and e_j are said to be incident with (on or to) each other. In fig (a), for examples, edges e_2, e_6 and e_7 are incident with vertex v_4. Two nonparallel edges are said to be adjacent if there are incident on a common vertex. For example, e_2 and e_7 in fig (a) are adjacent. Similarly, two vertices are said to be adjacent if they are the end vertices of the same edge in fig (a), v_4 and v_5 are adjacent, but v_1 and v_4 are not.

The number of edges incident on a vertex v_i , with self-loops counted twice, is called the degree, d (v_i), of vertex v_i , in fig (a) for example $d(v_1) = d(v_2) = d(v_3) = 3$, $d(v_2) = 4$ and $d(v_5) = 1$. The degree of a vertex is same times also referred to as its valency.

Let us now considered a graph G with e edges and n vertices $v_1, v_2 ,\ldots.v_n$ since each edge contributes two degrees

The sum of the degrees of all vertices in G is twice the number of edges in G that is

$$\sum_{i=1}^{n} d(v_i) = 2e----(1.1)$$

Taking fig (a) as an example, once more $d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) = 3 + 4 + 3 + 3 + 1 = 14$ = twice the number of edges.

From equation (1.1) we shall derive the following interesting result.

THEOREM 1.1

"The number of vertices of odd degree in a graph is always even".

Proof : If we consider the vertices with odd and even degree separately, the quantity in the left side of equation (1.1) can be expressed as the sum of two sum, each taken over vertices of even and odd degree respectively, as follows.

$$\sum_{i=1}^{n} d(v_i) = \sum_{even} d(vj) + \sum_{odd} d(v_k) \;---(1.2)$$

Since the left hand side in equation (1.2) is even, and the first expression on the right hand side is even (being a sum of even numbers), the second expression must also be even

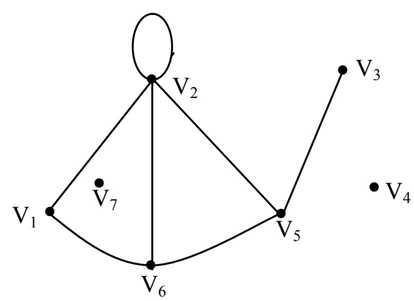
$$\sum_{odd} d(v_k) = an - even - number ----(1.3)$$

Because in equation (1.3) each $d(v_k)$ is odd, the total number of terms in the sum must be even to make the sum an even number. Hence the theorem.

A graph in which all vertices are of equal degree is called a **'regular graph'** (or simply a regular).

DEFINITION:

ISOLATED VERTEX , PENDANT VERTEX AND NULL GRAPH



Fig(1.7) – Graph containing isolated vertices, series edges, and a pendent vertex.

A vertex having no incident edge is called an '**isolated vertex**'. In other words, isolated vertices are vertices with zero degree. Vertices v_4 and v_7 in fig(1.7), for example, are isolated vertices a vertex of degree one is called a pendent vertex or an end vertex v_3 in fig(1.7) is a pendent vertex. Two adjacent edges are said to be in series if their common vertex is of degree two in fig(1.7), the two edges incident on v_1 are in series.

In the definition of a graph G = (V,E), it is possible for the edge set E to be empty. Such a graph , without any edges is called a '**null graph'**. In other words, every vertex in a null graph is an isolated vertex. A null graph of six vertices is shown in fig (1.8). Although the edge set E may empty the vertex set V must not be empty; otherwise there is no graph. In other words, by definition, a graph must have atleast one vertex

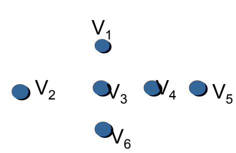


Fig 1.8: Null graph of Six Vertices

A BRIEF HISTORY OF GRAPH THEORY

As mentioned before, graph theory was born in 1736 with Euler's paper in which he solved Konigsberg bridge problem. For the next 100 years nothing more was done in the field.

In 1847,G.R.Kirchhoff (1824-1887) developed the theory of trees for their applications in Electrical network. Ten years later, A. Cayley (1821-1895) discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons C_nH_{2n+2}.

About the time of Kirchhoff and Cayley,two other milestones in graph theory were laid. One was the four-color conjecture, which states that four colors are sufficient for coloring any atlas(a map on a plane)such that the countries with common boundaries have different colors.

It is believed that A.F. Mobius (1790-1868) first presented four-color problem in one of his lectures in 1840.

About 10 years later A.De Morgan(1806-1871) discussed this problem with his fellow mathematicians in London.De Morgan's letter is the first authenticated reference to the four-color problem.The problem became well known after Cayley published it in 1879 in the first volume of the **Proceedings of the Royal Geographic Society .**To this day ,the four-color conjecture is by far the most famous unsolved problem in Graph theory. It has stimulated an enormous of research in the field.

The other milestone is due to Sir W.R. Hamilton (1805-1865). In the year 1859,he invented a puzzle and sold it for 25 guineas to a game manufacturer in Dublin. The puzzle consisted of a wooden ,regular Dodecahedron (A polyhedron with 12 faces and 20 corners, each face being a regular pentagon and three edges meeting at each corner). The corners were marked with the names of 20 important cities; London, Newyork, Delhi, Paris and so on .The object in the puzzle was to find a route along the edges of the Dodecahedron, passing through each of the 20 cities exactly once.

Although the solution of this specific problem is easy to obtain, to date no one has found a necessary and sufficient condition for the existence of such a route (called Hamiltonian circuit) in an arbitrary graph.

This fertile period was followed by half a century of relative inactivity. Then a resurgence of interest in graphs started during the 1920's.One of the pioneers in this period was D. Konig. He organized the work of other mathematicians and his own and wrote the first book on the subject which was published in 1936.

The past 30 years has been a period of intense activity in graph theory both pure and applied. A great deal of research has been done and is being done in this area. Thousands of papers have been published and more than hundred of books written during the past decade. **Among the current leaders in the field are Claude Berg, Oystein Ore, Paul Erdos, William Tutte and Frank Harary.**

DIRECTED GRAPHS AND GRAPHS:

DIRECTED GRAPHS :

Look at the diagram shown below. This diagram consists of four vertices A,B,C,D and three edges AB,CD,CA with directions attached to them .The directions being indicated by arrows.

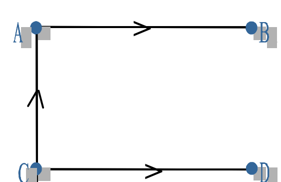


Fig.

Because of attaching directions to the edges, the edge AB has to be interpreted as an edge from the vertex A to the vertex B and it cannot be written as BA. Similarly the edge CD is from C to D and cannot be written as DC and the edge CA is from C to A and cannot be written as AC .Thus here the edges AB, CD, CA are directed edges.

The directed edge AB is determined by the vertices A and B in that order and may therefore be represented by the ordered pair (A,B). similarly, the directed edge CD and CA may be represented by the ordered pair(C,D) and (C,A) respectively. Thus the diagram in fig(1.1) consists of a nonempty set of vertices, namely {A,B,C,D} and a set of directed edges represented by ordered pairs {(A,B),(C,D),(C,A) }.Such a diagram is called a diagram of a directed graph.

DEFINITION OF A DIRECTED GRAPH :

A directed graph (or digraph) is a pair (V,E), where V is a non empty set and E is a set of ordered pairs of elements taken from the set V.

For a directed graph (V, E), the elements of V are called **Vertices (**points or nodes) and the elements of E are called **"Directed Edges".** The set V is called the **vertex set** and the set E is called the **directed** edge set

The directed graph (V,E) is also denoted
by D=(V,E) or D =D(V,E).
The geometrical figure that depicts a directed graph for which the vertex set is
V={A,B,C,D} and the edge set is
E={AB,CD,CA}={(A,B),(C,D),(C,A)}

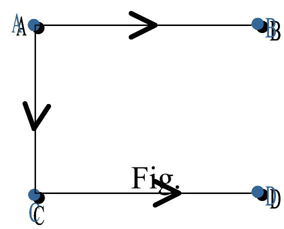


Fig.

Fig(1.2) depicts the directed graph for which the
vertex set is V={A,B,C,D} and the edge set is
E={AB,CD,AC}={(A,B),(C,D),(A,C)}.

It has to be mentioned that in a diagram of a directed graph the directed edges need not be straight line segments, they can be curve lines (arcs)Also.

For example, a directed edge AB of a directed graph can be represented by an arbitrary arc drawn from the vertex A to the vertex B as shown in fig(1.3).

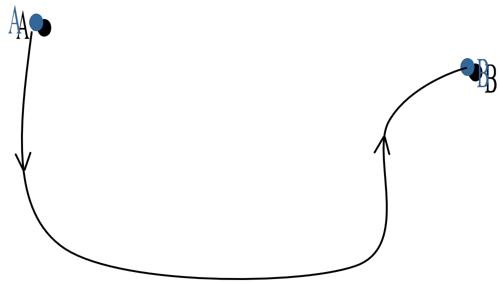
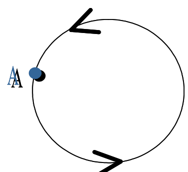


Fig.

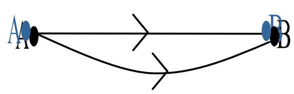
In fig (1.1) every directed edge of a digraph (directed graph) is determined by two vertices of the diagraph- a vertex from which it begins and a vertex at which it ends. Thus ,if AB is a directed edge of a digraph D. Then it is understood that this directed edge begins at the vertex A of D and terminates at the vertex B of D. Here we say that A is the **initial vertex** and B is the **terminal vertex** of AB.

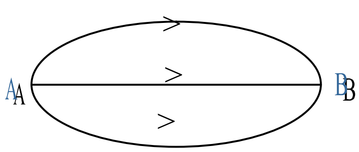
It should be mentioned that for a directed edge (in a digraph) the initial vertex and the terminal vertex need not be different. A directed edge beginning and ending at the same vertex A is denoted by AA or (A,A) and is called **directed loop.** The directed edge shown in Fig.(1.4) is a directed loop which begins and ends at the vertex A.



A digraph can have more than one directed edge having the same initial vertex and the same terminal vertex. Two directed edges having the same initial vertex and the same terminal vertex are called parallel directed edges.

Two parallel directed edges are shown in fig(1.5)(a).





Two or more directed edges having the same initial vertex and the same terminal vertex are called **"multiple directed edges".** Three multiple edges are shown in fig(1.5)(b).

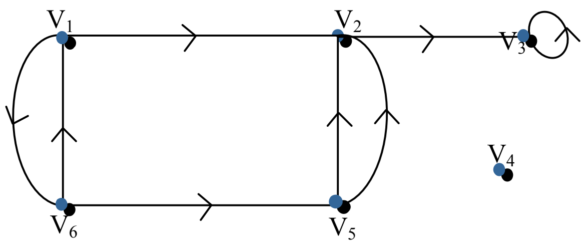
IN- DEGREE AND OUT –DEGREE

If V is the vertex of a digraph D, the number of edges for which V is the initial vertex is called the **outgoing degree** or the **out degree** of V and the number of edges for which V is the terminal vertex is called the **incoming degree** or the **in degree** of V. The out degree of V is denoted by d^+ (v) or o d (v) and the in degree of V is denoted by d^- (v) or i d (v).

It follows that

i. $d^+(v) = 0$, if V is a sink
ii. $d^-(v) = 0$, if V is a source
iii. $d^+(v) = d^-(v) = 0$, if V is an isolated vertex.

For the digraph shown in fig(1.6) the out degrees and the in degrees of the vertices are as given below



$d^+(v_1) = 2$	$d^-(v_1) = 1$
$d^+(v_2) = 1$	$d^-(v_2) = 3$
$d^+(v_3) = 1$	$d^-(v_3) = 2$
$d^+(v_4) = 0$	$d^-(v_4) = 0$
$d^+(v_5) = 2$	$d^-(v_5) = 1$
$d^+(v_6) = 2$	$d^-(v_6) = 1$

We note that ,in the above digraph, there is a directed loop at the vertex v_3 and this loop contributes a count 1 to each of $d^+(v_3)$ and $d^-(v_3)$.

We further observe that the above digraph has 6 vertices and 8 edges and the sums of the out-degrees and in-degrees of its vertices are

$$\sum_{i=1}^{6} d^+(v_j) = 8, \ \sum_{i=1}^{6} d^-(v_i) = 8$$

Example 1: Find the in- degrees and the out-degrees of the vertices of the digraph shown in fig (1.8)

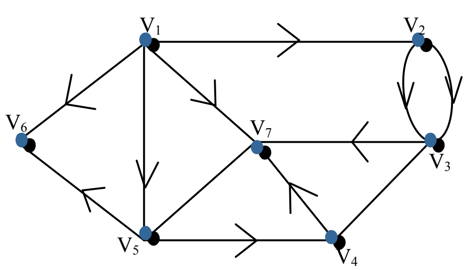


Fig.

SOLUTION:

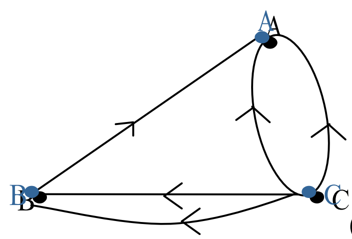
The given digraph has 7 vertices and 12 directed edges. The out-degree of a vertex is got by counting the number of edges that go out of the vertex and the in-degree of a vertex is got by counting the number of edges that end at the vertex. Thus we obtain the following data

Vertex	V_1	V_2	V_3	V_4	V_5	V_6	V_7
Out-degree	4	2	2	1	3	0	0
In-degree	0	1	2	2	1	2	4

This table gives the out-degrees and in-degrees of all the vertices. We note that v_1 is a source and v_6 and v_7 are sinks.

We also check that sum of out-degrees = sum of in – degrees = 12 = No of edges.

Example 2: Write down the vertex set and the directed edge set of each of the following digraphs.



(i)

Fig.

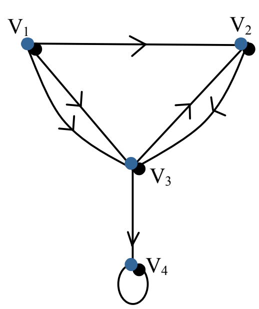


Fig. (ii)

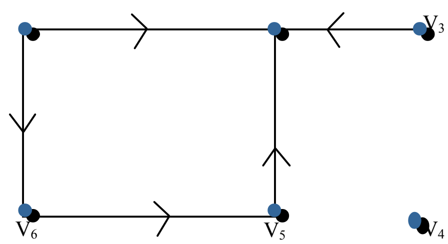
Solution of graph (i) & (ii):

i) This is a digraph whose vertex set is
$V=\{A,B,C\}$ and the directed edge set
$E=\{(B,A),(C,A),(C,A),(C,B),(C,B)\}$.

ii) This is a digraph whose vertex set is
$V=\{V_1,V_2,V_3, V_4\}$ and the directed edge set
$E=\{(V_1,V_2),(V_1,V_3),(V_1,V_3),$
$(V_2,V_3),(V_3,V_2)),(V_3,V_4) ,(V_4,V_4)\}$.

Example 3: For the digraph shown in fig, determine the out-degrees and in-degrees of all the vertices

V_1 V_2



Solution: $d^-(V_1) = 0$, $d^-(v_2) = 3$, $d^-(v_3) = 0$, $d^-(v_4) = 0$, $d^-(v_5) = 1$, $d^-(v_6) = 1$

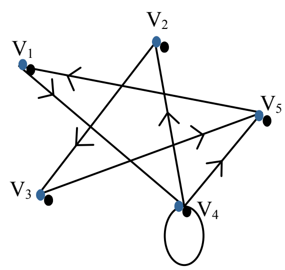
$d^+(v_1) = 2$, $d^+(v_2) = 0$, $d^+(v_3) = 1$, $d^+(v_4) = 0$, $d^+(v_5) = 1$, $d^+(v_6) = 1$

Example 4: Let D be the digraph whose vertex set

$V = \{V_1, V_2, V_3, V_4, V_5\}$ and the directed edge set is

$E = \{(V_1,V_4), (V_2,V_3), (V_3,V_5), (V_4,V_2), (V_4,V_4), (V_4,V_5), (V_5,V_1)\}$.

Write down a diagram of D and indicate the out-degrees and in-degrees of all the vertices



vertices	V_1	V_2	V_3	V_4	V_5
D^+	1	1	1	3	1
d^-	1	1	1	2	2

DEFINITION :

SIMPLE GRAPH :

A graph which does not contain loops and multiple edges is called simple graph.

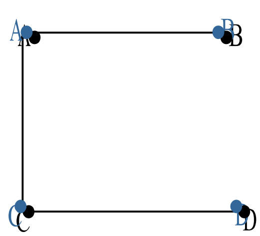


Fig. Simple Graph

LOOP FREE GRAPH.

A graph which does not contain loop is called loop free graph.

MULTIGRAPH

A graph which contain multiple edges but no loops is called <u>multigraph</u>.

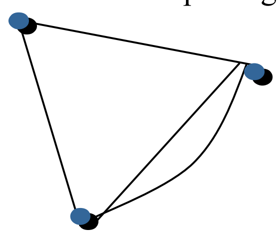
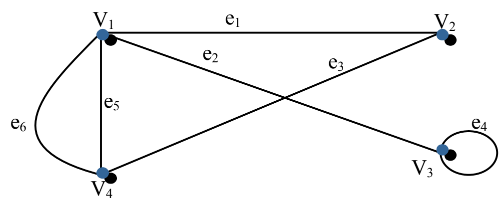


Fig. Multigraph

GENERAL GRAPH

A graph which contains multiple edges or loops (or both) is called <u>general graph.</u>



<u>COMPLETE GRAPH :</u>

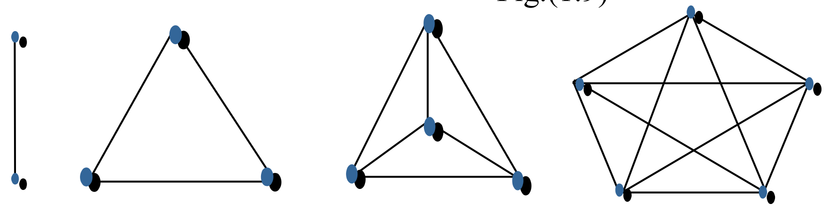
A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a complete graph (or a full graph).

In other words a complete graph is a simple graph in which every pair of distinct vertices are adjacent.

A complete graph with $n \geq 2$ vertices is denoted by K_n .

A complete graph with 2,3,4,5 vertices are shown in fig (1.9)(a) to (1.9)(d) respectively. Of these complete graphs ,the complete graph with 5 vertices namely K_5(shown in fig.1.9 (d),is of great importance. This graph is called the Kuratowski's first graph

Fig.(1.9)



(a) : K_2 (b) : K_3 (c) : K_4 (d) : K_5

BIPARTITE GRAPH

Suppose a simple graph G is such that its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 and V_2 which are such that each edge in G joins a vertex inV_1 and a vertex inV_2.Then G is called a **bipartite graph**. If E is the edge set of this graph, the graph is denoted by G = (V_1, V_2: E), or G = G(V_1, V_2: E). The sets $V_1$and V_2 are called **bipartites** (or partitions) of the vertex set V.

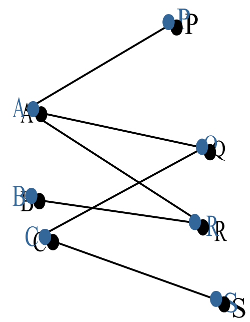


Fig. (1.10)

For example, consider the graph G in fig(1.10) for which the vertex set is
V={A,B,C,P,Q,R,S} and the edge set is
E= {AP,AQ,AR,BR,CQ,CS}. Note that the set V is the union of two of its subsets V_1={ A,B,C} and V_2={P,Q,R,S} which are such that

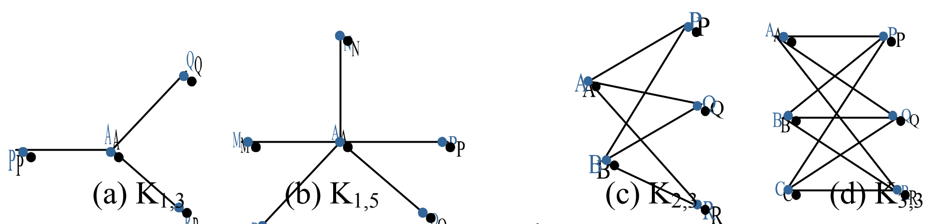
i) V_1 and V_2 are disjoint.
ii) Every edge in G joins a vertex in V_1 and a vertex in V_2.
iii) G contains no edge that joins two vertices both of which are in V_1 or V_2. This graph is a bipartite graph with V_1={ A,B,C} and V_2={P,Q,R,S} as bipartites.

COMPLETE BIPARTITE GRAPH

A bipartite graph G= {V_1, V_2 ; E} is called a complete bipartite graph, if there is an edge between every vertex in V_1 and every vertex in V_2 .

The bipartite graph shown in fig (1.10) is not a complete bipartite graph. Observe for example that the graph does not contain an edge joining A and S.

A complete bipartite graph G={ V_1, V_2 ; E} in which the bipartites $V_1$and V_2 contain r and s vertices respectively, with $r \le s$ is denoted by $K_{r,s}$ .In this graph each of r vertices in V= is joined to each of s vertices in V_2 .Thus $K_{r,s}$ has r+ s vertices and rs edges. That is $K_{r,s}$ is of order r+s and size rs. It is therefore a (r+ s,rs) graph



(a) $K_{1,3}$ (b) $K_{1,5}$ (c) $K_{2,3}$ (d) $K_{3,3}$

Fig. 1.11

Fig 1.11 (a) to (d) depict some bipartite graphs. Observe that in fig 1.11(a),the bipartites are V_1={ A } and V_2={P,Q,R}; the vertex A is joined to each of the vertices P,Q,R by an edge. In fig 1.11(b) ,the bipartites are V_1={A} and

V_2={M,N,P,Q,R}; the vertex A is joined to each of the vertices M,N,P,Q,R by an edge. In fig 1.11(c) ,the bipartites are V_1={ A,B } and V_2={ P,Q,R}; each of the vertices A and B is joined to each of the vertices P,Q,R by an edge. In fig 1.11(d),the bipartites are V_1={ A,B,C } and V_2={P,Q,R}; each of the vertices A,B,C is joined to each of the vertices P,Q,R. Of these complete bipartite graph the graph $K_{3,3}$ shown in fig 1.11(d),is of great importance. This is known as **Kuratowski's second graph.**

Example 1. Draw a diagram of the graph G = (V,E) in each of the following cases.

a) V= { A,B,C,D} ,E={AB,AC,AD,CD}

b) V={V_1,V_2,V_3, V_4 ,V_5 },

E={V_1V_2 ,V_1V_3,V_2V_3,V_4V_5}.

c) V= {P,Q,R,S,T} ,E={PS,QR,QS}

d) V={ V_1,V_2,V_3, V_4 ,V_5,V_6},

E={V_1V_4,V_1V_6,V_4V_6,V_3V_2,V_3V_5,V_2V_5}

Solution : The required diagram are shown below

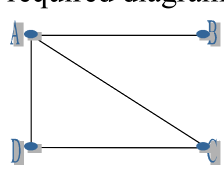


Fig: (a)

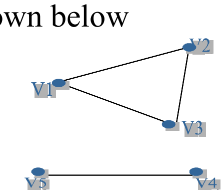


Fig: (b)

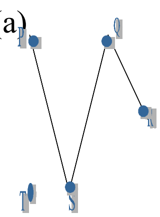


Fig: (c)

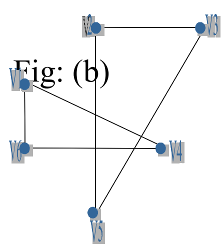
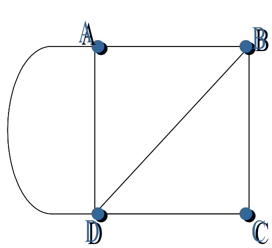
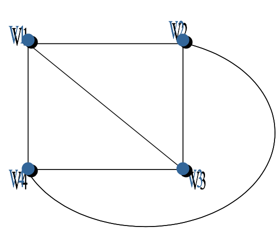


Fig: (d)

Example 2: Which of the following is a complete graph?



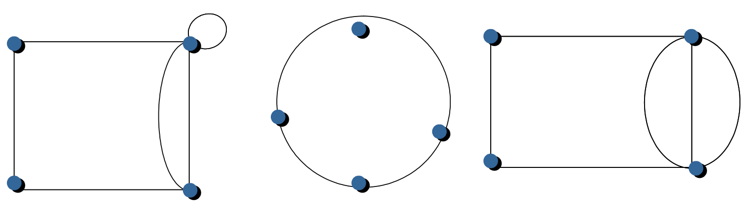
(a)



(b)

Solution: The first of the graph is not complete. It is not simple on the one hand and there is no edge between A and C on the other hand. The second of the graphs is complete. It is a simple graph and there is an edge between every pair of vertices.

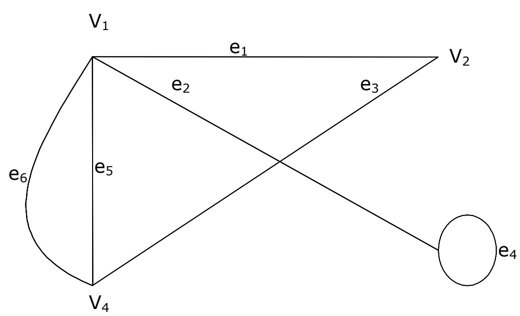
Example 3: Which of the following graphs is a simple graph? a multigraph ? a general graph ?



(i) (ii) (iii)

Solution: (i) General Graph,
(ii) Simple Graph,
(iii) Multigraph

Example 4: Identify the adjacent vertices and adjacent edges in the graph shown in Figure.



Solution :
Adjacent Vertices : V_1 & V_2, V_1 & V_3, V_1 & V_4, V_2 & V_4.
Adjacent edges : e_1 & e_2, e_1 & e_3, e_1 & e_5, e_1 & e_6, e_2 & e_4, e_2 & e_5, e_2 & e_6, e_3 & e_5, e_3 and e_6.

***VERTEX DEGREE AND HANDSHAKING PROPERTY* :**

Let G = (V,E) be a graph and V be a vertex of G. Then the number of edges of G that are incident on V (that is, the number of edges that join V to other vertices of G) with the loops counted twice is called the ***degree*** of the vertex V and is denoted by deg(v) or d(V).

The degree of the vertices of a graph arranged in non-decreasing order is called the ***degree sequence*** of the graph. Also, the minimum of the degree of a graph is called the ***degree of the graph***

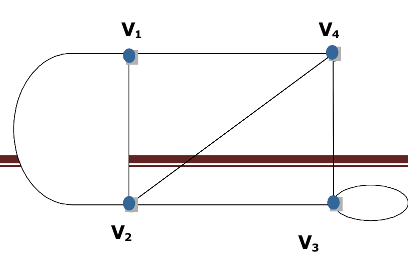
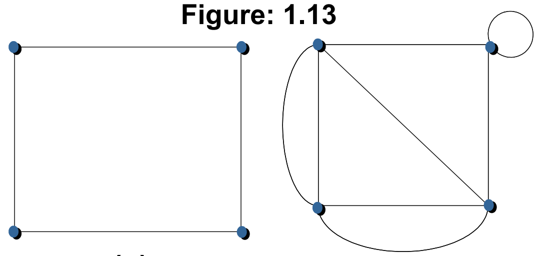


Figure (1.12)

For example, the degrees of vertices of the graph shown in fig are as given below
$d(V_1) = 3$, $d(V_2) = 4$, $d(V_3) = 4$, $d(V_4) = 3$
Therefore,the degree sequence of the graph is 3,3,4,4 and the degree of the graph is 3.
Regular Graph* :** A graph in which all the vertices are of the same degree K is called a regular graph of degree K, or a K- regular graph.In particular, a 3-regular graph is called a ***cubic graph.
The graph shown in figures 1.13 (a) and (b) are 2- regular and 4 - regular graph respectively.

Figure: 1.13



(a) **(b)**

The graph shown in fig1.13 (c) is a 3-regular graph (cubic graph). This particular cubic graph, which contains 10 vertices and 15 edges, is called the ***Peterson Graph***.

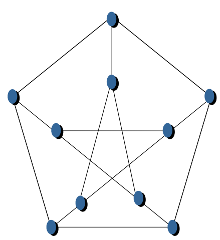


Figure (c)

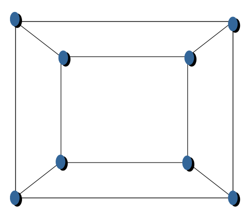


Figure (d)

The graph shown in fig (d) is a cubic graph with $8 = 2^3$ vertices. This particular graph is called the three dimentional hyper cube and is denoted by Q_3.

Handshaking property :
Let us refer back to degree of the graph shown in fig 1.14. we have, in this graph,

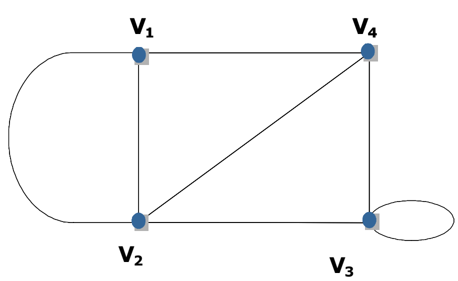


fig (1.14)

$d(V_1) = 3$, $d(V_2) = 4$, $d(V_3) = 4$, $d(V_4) = 3$

Also,the graph has 7 edges, we observe that deg (V_1) + deg (V_2) + deg (V_3) + deg (V_4) = 14 = 2 x 7

Property: The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the number of edges in the graph.

In an alternative form, this property reads as follows:

For a graph G = (V,E) $\sum_{v \in V} \deg(v) = 2|E|$

This property is obvious from the fact that while counting the degree of vertices, each edge is counted twice (once at each end).

The aforesaid property is popularly called the **'*handshaking property*'**

Because, it essentially states that if several people shake hands, then the total number of hands shaken must be even, because just two hands are involved in each hand shake.

Theorem : In every graph the number of vertices of odd degrees is even

Proof : Consider a graph with n vertices. Suppose K of these vertices are of odd degree so that the remaining n-k vertices are of even degree. Denote the vertices with odd degree by $V_1, V_2, V_3, \ldots\ldots, V_k$ and the vertices with even degree by $v_{k+1}, v_{k+2}, \ldots.., v_n$ then the sum of the degrees of vertices is

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) ----(1)$$

In view of the hand shaking property, the sum on the left hand side of the above expression is equal to twice the number of edges in the graph. As such, this sum is even. Further, the second sum in the right hand side is the sum of the degrees of the vertices with even degrees. As such this sum is also even. Therefore, the first sum in the right hand side must be even; that is,

$\deg(V_1) + \deg(V_2) + \text{--} + \deg(V_k) = \text{Even}$—(ii)

But, each of $\deg(V_1)$, $\deg(V_2)$,…….,deg (V_k) is odd. Therefore, the number of terms in the left hand side of (ii) must be even; that is, K is even

Example : For the graph shown in fig 1.15 indicating the degree of each vertex and verify the handshaking property

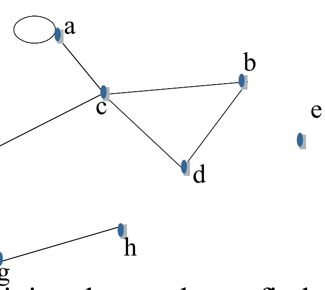


Fig (1.15)

Solution : By examining the graph, we find that the degrees of its vertices are as given below:

deg (a) = 3, deg (b) = 2, deg (c) = 4, deg (d) = 2, deg(e) = 0, deg (f) = 2, deg (g) = 2, deg (h) = 1.

We note that e is an isolated vertex and h is a pendant vertex.

Further, we observe that the sum of the degrees of vertices is equal to 16. Also, the graph has 8 edges. Thus, the sum of the degrees of vertices is equal to twice the number of edges.

This verifies the handshaking property for the given graph.

Example : For a graph with n-vertices and m edges, if δ is the minimum and Δ is the maximum of the degrees of vertices, show that $\delta \leq \frac{2m}{n} \leq \Delta$

Solution : Let $d_1, d_2, \ldots, d_n$, be the degrees of the vertices. Then, by handshaking property, we have

$d_1 + d_2 + d_3 + \text{-----} + d_n = 2m$ ---------(i)

Since $\delta = \min(d_1, d_2, \ldots, d_n)$, we have $d_1 \geq \delta$,

$d_2 \geq \delta$,……., $d_n \geq \delta$.

Adding these n inequalities, we get

$d_1 + d_2 + \text{----} + d_n \geq n\,\delta$ ------(ii)

Similarly, since $\Delta = \max(d_1, d_2, \ldots, d_n)$, we get

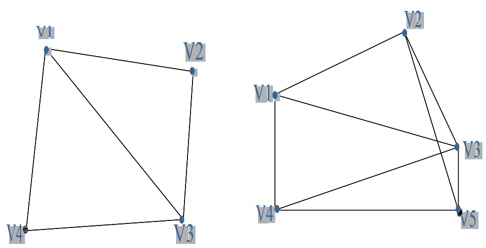
$d_1 + d_2 + \text{----} + d_n \leq n\Delta$ ----(iii)

From (i), (ii) and (iii), we get $2m \geq n\,\delta$ and $2m \leq n\Delta$, so that $n\,\delta \leq 2m \leq n\Delta$,

or

$$\delta \leq \frac{2m}{n} \leq \Delta$$

SUBGRAPHS



(a) : G_1 (b) : G

Fig. (1.16)

Given two graphs G and G_1, we say that G_1 is a ***subgraph*** of G if the following conditions hold:

(1). All the vertices and all the edges of G_1 are in G.

(2). Each edges of G_1 has the same end vertices in G as in G_1.

Essentially, a subgraph is a graph which is a part of another graph. Any graph isomorphic to a subgraph of a graph G is also referred to as a subgraph of G.

Consider the two graphs G_1 and G shown in figures 1.16(a) and 1.16(b) respectively, we observe that all vertices and all edges of the graph G_1 are in the graphs G and that every edge in G_1 has same end vertices in G as in G_1.Therefore G_1 is a subgraph of G .In the diagram of G ,the part G_1 is shown in thick lines.

The following observation can be made immediately.

i) Every graph is a sub-graph of itself.

ii) Every simple graph of n vertices is a subgraph of the complete graph K_n.

iii) If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G,then G_1 is a subgraph of a graph G.

iv) A single vertex in a graph G is a subgraph of a graph G.

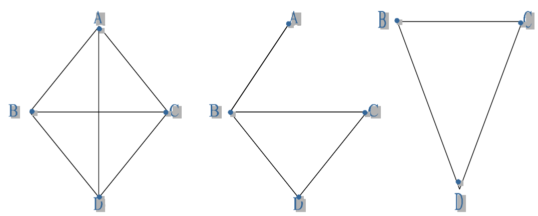
v) A single edge in a graph G together with its end vertices,is a subgraph of G

SPANNING SUBGRAPH :

Given a graph G=(V, E), if there is a subgraph $G_1=(V_1,E_1)$ of G such that $V_1=V$ then G_1 is called a spanning subgraph of G.

In other words , a subgraph G_1 of a graph G is a spanning subgraph of G whenever the vertex set of G_1 contains all vertices of G. Thus a graph and all its spanning subgraphs have the same vertex set. Obviously every graph is its own spanning subgraph.

Figure (1.17)



(a) (b) (c)

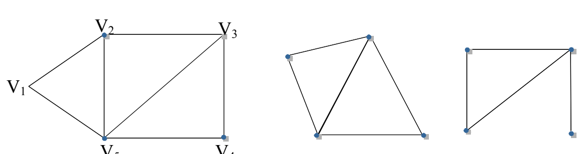
For example, for the graph shown in fig1.17(a), the graph shown in fig 1.17(b) is a spanning subgraph where as the graph shown in fig1.17(c) is a subgraph but not a spanning subgraph

INDUCED SUBGRAPH

Given a graph G=(V,E), suppose there is a subgraph $G_1=(V_1,E_1)$ of G such that every edge {A,B} of G, where AB ϵ V_1 is an edge of G_1 also .then G_1 is called an induced subgraph of G (induced by V_1) and is denoted by $< V_1 >$.

It follows that a subgraph $G_1=(V_1,E_1)$ of a graph G=(V,E) is not an induced subgraph of G, if for some A,B ϵ V_1,there is an edge{A,B} which is in G but not in G_1.

For example, for the graph shown in the figure 1.18 (a), the graph shown in the figure 1.18 (b), is an induced subgraph, induced by the set of vertices $V_1= \{v_1,v_2,v_3,v_5\}$ where as the graph shown in the figure 1.18 (c) is not an induced subgraph



(a) (b) (c)

Figure 1.18 (a, b & c)

EDGE-DISJOINT AND VERTEX-DISJOINT SUBGRAPHS

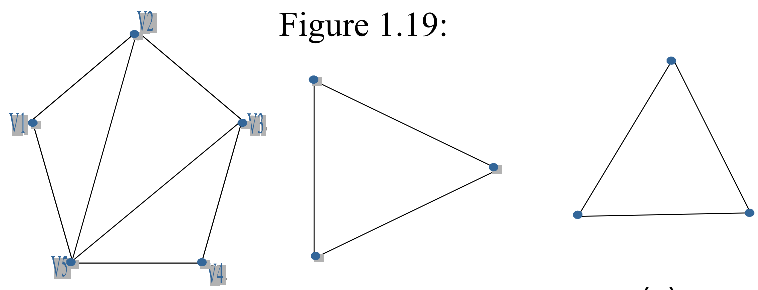
Let G be a graph and G_1 and G_2 be two subgraphs of G. then

G_1 and G_2 are said to be edge disjoint if they do not have any common edge.

G_1 and G_2 are said to be vertex disjoint if they do not have any common edge and any common vertex.

It is to be noted that edge disjoint subgraphs may have common vertices. Subgraphs that have no vertices in common cannot possibly have edges in common.
For example ,for the graph shown in the figure 1.19 (a), the graph shown in the figure1.19 (b) and 1.19 (c) are edge disjoint but not vertex disjoint subgraphs.

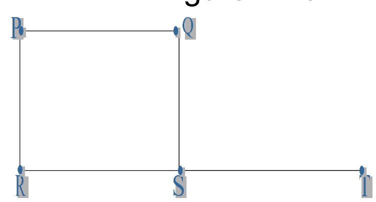
Figure 1.19:



(a) (b) (c)

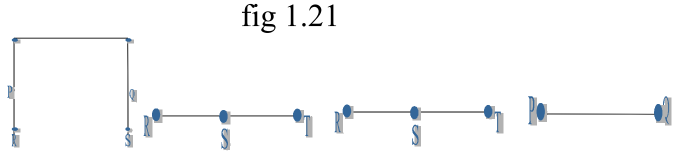
Example : For the graph shown in fig 1.20 ,find two edge-disjoint subgraphs and two vertex-disjoint subgraphs .

Figure 1.20



Solution:for the given graph, two edge-disjoint subgraphs are shown in fig 1.21(a) and two vertex-disjoint subgraphs are shown in fig 1.21(b).

fig 1.21



(a) (b)

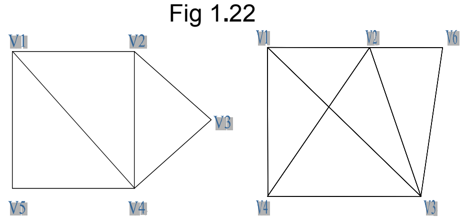
OPERATIONS ON GRAPHS

Consider two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ then the graph whose vertex set is $V_1 U V_2$ and edge set is $E_1 U E_2$ is called the union of G_1 and G_2 and is denoted by $G_1 U G_2$.
Thus $G_1 U G_2=(V_1 U V_2, E_1 U E_2)$.
Similarly, if $V_1 \cap V_2 \neq \phi$,the graph whose vertex set is $V_1 \cap V_2$ and the edge set $E_1 \cap E_2$ is called intersection of G_1 and G_2.It is denoted by $G_1 \cap G_2$.Thus $G_1 \cap G_2=(V_1 \cap V_2, E_1 \cap E_2)$, if $V_1 \cap V_2 \neq \phi$.

Next suppose we consider the graph whose vertex set is V_1UV_2 and edge set is $E_1\Delta E_2$ where $E_1\Delta E_2$ is the symmetric difference of E_1 and E_2.This graph is called the ring sum of G_1 and G_2.It is denoted by $G_1\Delta G_2$. Thus $G_1\Delta G_2 = (V_1UV_2, E_1\Delta E_2)$.

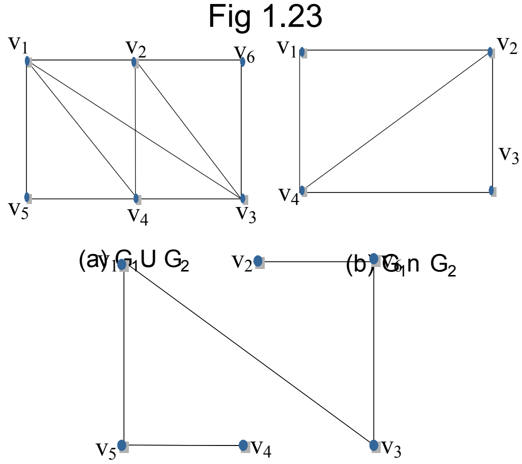
For the two graphs G_1 and G_2 shown in figures 1.22 (a) and (b), their union ,intersection and ring sum are shown in figures 1.23 (a), (b) and (c) respectively.

Fig 1.22



(a) : G_1 (b) : G_2

Fig 1.23



(a) $G_1 U G_2$ (b) $G_1 n G_2$

Fig 1.23: (c) $G_1\Delta G_2$

DECOMPOSITION

We say that a graph G is decomposed (or partitioned) in to two subgraphs G_1 & G_2 if G_1 Ù G_2 = G & $G_1 \cap G_2$ = null graph

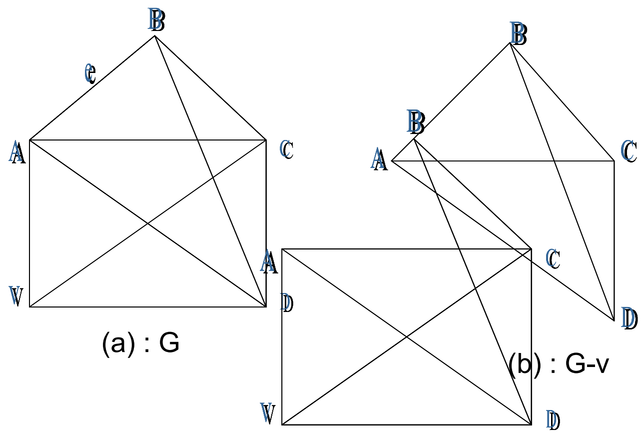
DELETION:

If V is a vertex in a graph G, then G – V denotes the subgraph of G obtained by deleting V and all edges incident in V, from G this subgraph G-u, is refered to as **vertex deleted subgraph** of G.

It should be noted that, the deletion of a vertex always results in the deletion of all edges incident on that vertex.

If e is an edge in a graph G, then G-e denotes the subgraph of G obtained by deleting e (but not its end vertices) from G. This subgraph, G-e, is referred to as **edge – deleted subgraph** of G. For the graph G shown in figure 1.24 (a), the subgraphs G-V and G-e are shown in figure 1.24 (b) and 1.24 (c) respectively.

Figure 1.24 (a, b, c)



(c) : G-e

COMPLEMENT OF A SUBGRAPH

Given a graph G and a subgraph G_1 of G ,the subgraph of G obtained by deleting from all the edges that belongs to G_1 is called the complement of G_1 in G;it is denoted by $G-G_1$ or $\overline{G_1}$

In other words ,if E_1 is the set of all edges of G_1 then the complement of G_1 in G is given by $\overline{G_1} = G-E_1$.We can check that $\overline{G_1}=G\Delta G_1$.

For example :

Consider the graph G shown in fig 1.25(a) .Let G_1 be the subgraph of G shown by thick lines in this figure. The complement of G_1 in G, namely $\overline{G_1}$, is as shown in fig 1.25(b)

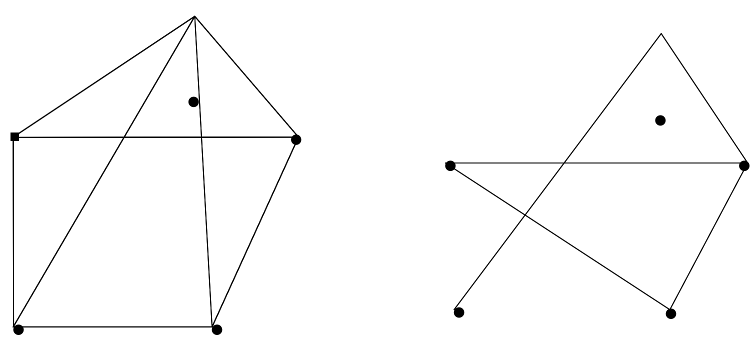


Fig. 1.25(a) Fig.1.25 (b)

COMPLEMENT OF A SIMPLE GRAPH

Earlier we have noted that every simple graph of order n is a subgraph of the complete graph Kn .If G is a simple graph of order n ,then the complement of G in K_n is called the **complement of G**, it is denoted by $\overline{G}$.

Thus, the complement $\overline{G}$ of a simple graph G with n vertices is that graph which is obtained by deleting those edges of Kn which belongs to G. Thus $\overline{G} = K_n - G = K_n \Delta G$.

Evidently K_n, G and $\overline{G}$ have the same vertex set and two vertices are adjacent in G if and only if they are not adjacent in G. Obviously, $\overline{G}$ is also a simple graph and the complement of $\overline{G}$ is G that is $\overline{\overline{G}}$ = **G**

In fig 1.26(a), the complete graph K_4 is shown. A simple graph G of order 4 is shown in fig 1.26(b). The complement $\overline{G}$, of G is shown in fig 1.26(c).

Observe that G, $\overline{G}$ & K_4 have the same vertices and that the edges of $\overline{G}$ are got by deleting those edges from K_4 which belong to G.

Figure 1.26 a, b & c

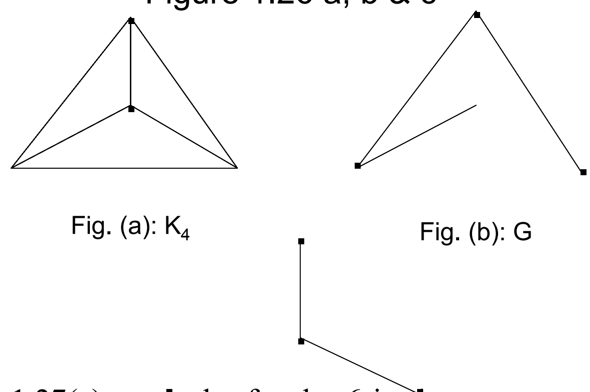
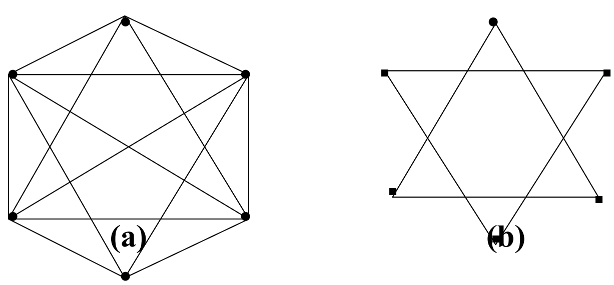


Fig. (a): K_4 Fig. (b): G Fig. (c): $\overline{G}$

In fig1.27(a) ,a graph of order 6 is shown as a subgraph of K_6 ,the edges of G being shown in thick lines .Its complement $\overline{G}$, is shown in fig1.27(b).The graph shown in fig1.27(b) is known as **<u>David Graph</u>**.

Fig. 1.27



(a) **(b)**

Example 1.Show that the complement of a bipartite graph need not be a bipartite graph.

Solution: Fig 1.28(a) shows a bipartite graph which is of order 5.The complement of this graph is shown in fig1.28(b),this is not a bipartite graph.

Fig. 1.28(a)

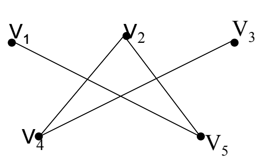
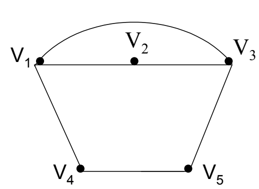


Fig. 1.28(b)



Example 2.Let G be a simple graph of order n.If the size of graph G is 56 and size of is G 80.What is n?

Solution: We know that $\overline{G} = K_n - G$ therefore

Size of $\overline{G}$ = (Size of K_n)- (Size of G)

Since size of K_n (ie the number of edges in K_n) is ½(n)(n-1),this yields

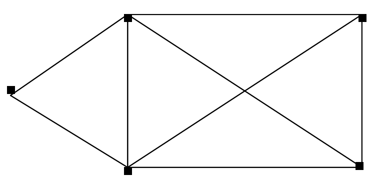
$80 = \frac{1}{2} n(n-1) - 56$

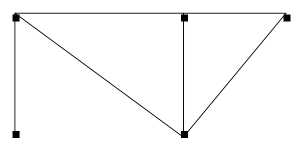
or $n(n-1) = 160 + 112 = 272 = 17 \times 16$

thus, n = 17, (that is, G is of order 17)

Example 3: Find the union, intersection and the ring sum of the graph G1 and G2 shown below.

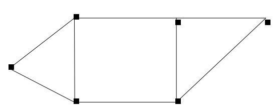
Fig. 1.29(G) Fig. 1.29($\overline{G}$)



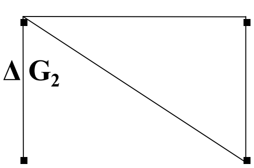


Solution :

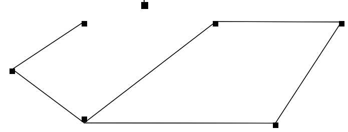
Union :- **G_1 U G_2**



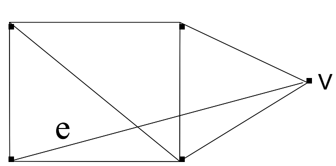
Intersection : - **G_1 n G_2**



Ring Sum :- **$G_1 \Delta G_2$**



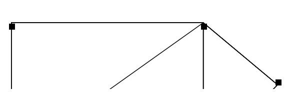
Example 4: For the graph G shown below, find G-v and G-e .



Solution :

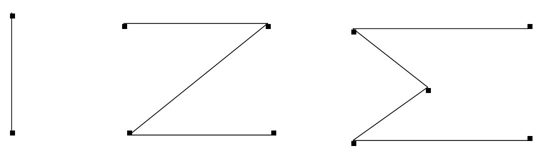
Fig 1.30

Fig. 1.31



Example 5: Find the complement of each of the following simple graphs

Fig. 1.32



(a) (b) (c)

Example 6: Find the complement of the complete bipartite graph $K_{3,3}$

Solution :

Fig. 1.34

WALKS AND THEIR CLASSIFICATION

WALK:

Let G be a graph having atleast one edge. In G, consider a finite, alternating sequence of vertices and edges of the form v_i e_j v_{i+1} e_{j+1} v_{i+2},…..,e_k v_m which begin and ends with vertices and which is such that each edge in the sequence is incident on the vertices preceding and following it in the sequence. Such a sequence is called a **walk** in G. In a walk, a vertex or an edge (or both) can appear more than once.

The number of edges present in a walk is called its '**length**'.

For example : Consider the graph shown below;

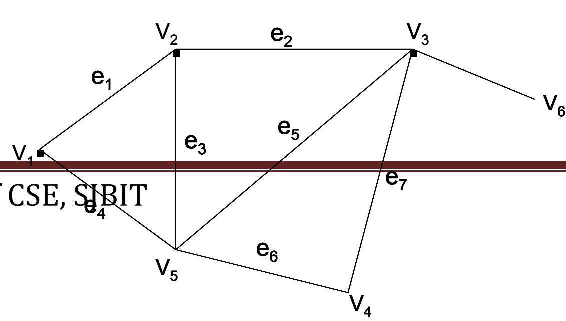


Fig.1.35

In this graph,

i) The sequence v_1e_1 v_2c_2 v_3,e_8v_6 is a walk of length 3 (because this walk contains 3 edges; e_1,e_2,e_8). In this walk, no vertex and no edge is repeated.

ii) The sequence v_1,e_4 v_5e_3 $v_2c_2v_3e_5$ $v_5e_6v_4$ is a walk of length 5. In this walk, the vertex v5 is repeated; but no edge is repeated.

iii) The sequence $v_1e_1v_2e_3v_5e_3v_2e_2v_3$ is a walk of length 4. In this walk, the edge e_3 is repeated and the vertex v_2 is repeated

A walk that begins and ends at the same vertex is called a **closed walk**. In other words, a closed walk is a walk in which the terminal vertices are coincident.

A walk which is not closed is called an **open walk**. In other words, an open walk is a walk that begins and ends at two different vertices.

For Example, in the graph shown in figure (1.35) $v_1e_1v_2c_3v_5e_4v_1$ is a **closed walk** and $v_1e_1v_2e_2v_3e_5v_5$ is our **open walk**.

TRAIL AND CIRCUIT:

In a walk, vertices and /or edges may appear more than once, if in an open walk no edge appears more than once, than the walk is called **a trail**. A closed walk in which no edge appears more than once is called a **circuit**.

For example: In fig (1.35), the open walk $v_1e_1v_2e_3v_5e_3v_2e_2v_3$ (shown separately in figure 1.36(a) is not a trail (because, in this walk, the edge e_3 is repeated) where as

Fig. 1.36 (a) :Not a trail

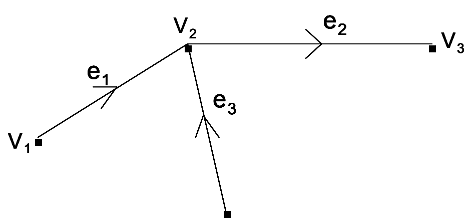
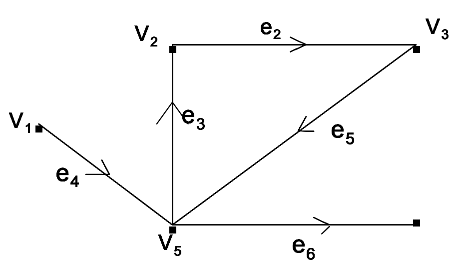


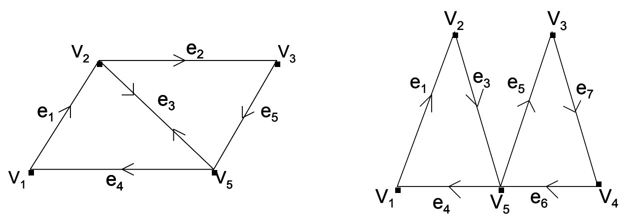
Fig. 1.36 (b): trail



The open walk $v_1e_4v_5e_3v_2v_2v_3e_5v_5e_6v_4$ (shown separately in fig 1.36(b) is trail.

Also, in the same fig (ie., in fig1.35), the closed walk v_1 e_1v_2 e_3 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_4 v_1 (shown separately in fig 1.37(a) is not a circuit (because e_3 is repeated) where as the closed walk $v_1e_1v_2e_3v_5e_5v_3e_7v_4e_6v_5e_4v_1$ (shown separately in fig1.37(b)) is a circuit.

Fig. 1.37(a) Fig. 1.37(b)



PATH AND CYCLE:

(a): Not a circuit (b): Circuit

A trail in which no vertex appears more than once is called a **path**.

A Circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a '**cycle**'.

Fig. 1.38

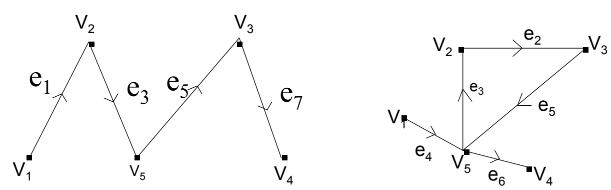
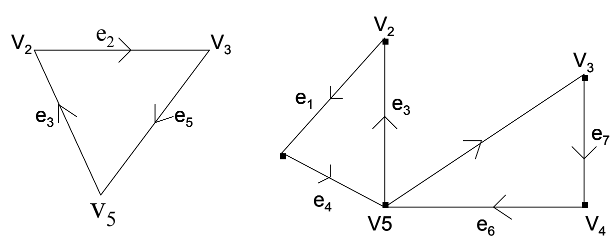


Fig. 1.39

(a): Path (a): Not a path



(a): Cycle **(b): Not a Cycle**

For example, in figure (1.35), the trail $v_1e_1e_3v_5e_5v_3e_7v_4$ (shown separately in fig 1.38(a)) is a path whole as the trail $v_1e_4v_5e_3v_2e_2e_5v_5e_6v_4$ (shown separately in fig 1.38(b) is not a path (because in this trail, v5 appears twice).

Also, in the same fig, the circuit $v_2e_2v_3e_5v_5e_3v_2$ (shown separately in fig 1.39(a)) is a cycle where as the circuit $v_2e_1v_1e_4v_5e_5v_3e_7v_4e_6v_5e_3v_2$ (shown separately in fig 1.39(b) is not a cycle (because, in this circuit, v5 appears twice)

The following facts are to be emphasized.

1. A walk can be open or closed. In a walk (closed or open), a vertex and / or an edge can appear more than once.
2. A trail is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.
3. A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.
4. A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail; but a trail need not be a path.
5. A cycle is a closed walk in which neither a vertex nor an edge can appear ore than once.

Every cycle is a circuit; **but, a circuit need not be a cycle.**

Example:

For the graph shown in figure1.40 indicate the nature of the following walks.

$v_1e_1v_2e_2v_3e_2v_2$

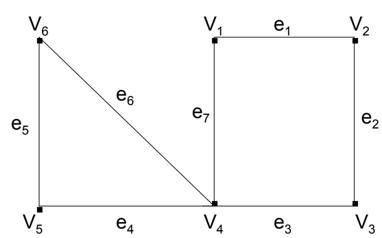
$v_4e_7v_1e_1v_2e_2v_3e_3v_4e_4v_5$

$v_1e_1v_2e_2v_3e_3v_4e_4v_5$

$v_1e_1v_2e_2v_3e_3v_4e_7v_1$

$v_6e_5v_5e_4v_4e_3v_3e_2v_2e_1v_1e_7v_4e_6v_6$

Fig. 1.40



Solution:

1. Open walk which is not a trail the edge e_2 is repeated.
2. Trail which is not a path (the vertex v_4 is repeated)
3. Trail which is a path
4. Closed walk which is a cycle.
5. Closed walk which is a circuit but not a cycle (the vertex v_4 is repeated)

EULER CIRCUITS AND EULER TRAILS.

Consider a connected graph G. If there is a circuit in G that contains all the edges of G. Than that circuit is called an **Euler circuit** (or Eulerian line, or Euler tour) in G. If there is a trail in G that contains all the edges of G, than that trail is called an **Euler trail**.

Recall that in a trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails and Euler Circuits also.

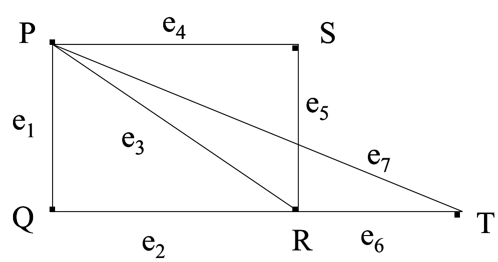
Since Euler circuits and Euler trails include all edge, then automatically should include all vertices as well.

A connected graph that contains an Euler circuit is called a **Semi Euler graph** (or a Semi Eulerian graph).

For Example, in the graph shown in figure 1.41 closed walk.

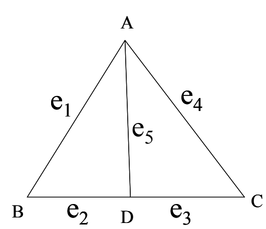
$Pe_1Qe_2Re_3Pe_4Se_5Re_6Te_7P$ is an Euler circuit. Therefore, this graph is a an Euler graph.

Fig 1.41



Consider the graph shown in fig.1.41. We observe that, in this graph, every sequence of edges which starts and ends with the same vertex and which includes all edges will contain at least one repeated edge. Thus, the graph has no Euler circuits. Hence this graph is not an Euler graph.

Fig. 1.42



It may be seen that the trail $Ae_1Be_2De_3Ce_4Ae_5D$ in the graph in fig 1.42 is an Euler trail. This graph therefore a Semi – Euler Graph.

Example 1: Show that the following graph contains an Euler Circuits

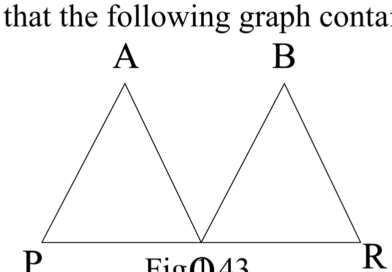
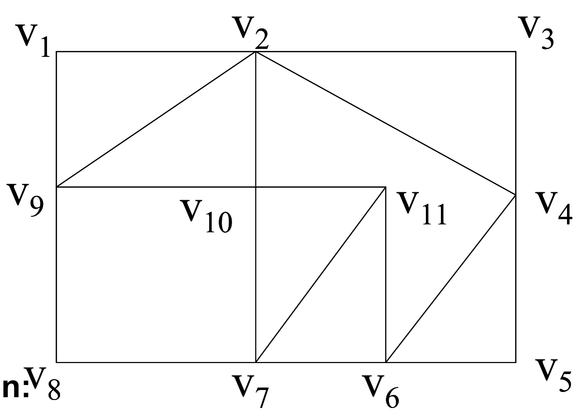


Fig.1.43

Solution: The graph containas an Euler Circuit PAQBRQP

Example 2: find an Euler circuit in the graph shown below.

Fig.1.44

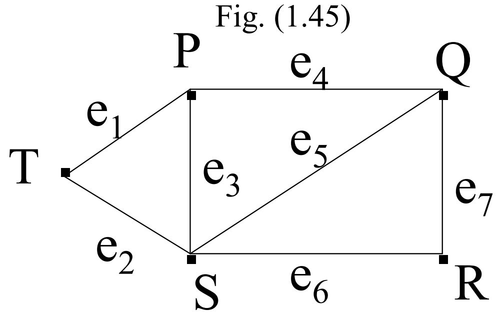


Solution:

$V_1V_2V_9V_{10}V_2V_{11}V_7V_{10}V_{11}V_6V_4V_2V_3V_4V_5V_6V_7V_8\,V_9V_1$

Example 3: show that the following graph contains an Euler trail.

Fig. (1.45)



Solution: the graph contains $Pe_1Te_2Se_3Pe_4Qe_5Se_6Re_7Q$as an **Euler trail**.

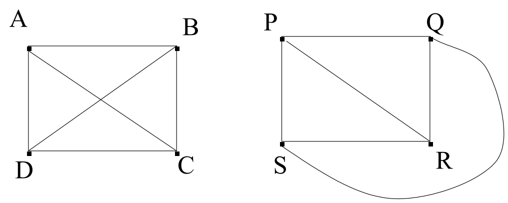
ISOMORPHISM :

Consider two graphs G = (V, E) and G' = (V', E') suppose their exists a function f : V → V' such that (i) f is a none to one correspondence and(ii) for all vertices A, B of G {A, B} is an edge of G if and only if { f(A), f(B)} is an edge of G', then f is called as **isomorphism** between G and G', and we say that G and G' are **isomorphic graphs**.

In other words, two graphs G and G' are said to be isomorphic (to each other) if there is a one to one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved such graphs will have the same structures, differing only in the way their vertices and edges

are labelled or only in the way they are represented geometrically for any purpose, we regard them as essentially the same graphs.
When G and G' are isomorphic we write $G \cong G'$
Where a vertex A of G corresponds to the vertex A' = f(A) of G' under a one to one correspondence f : G → G', we write A ←→ A' Similarly , we write {A, B} ←→ {A', B'} to mean that the edge AB of G and the edge A'B' of G' correspond to each other, under f.
For example, look at the graphs shown in fig1.46

Fig. 1.46



Consider the following one to one correspondence between between the vertices of these two graphs.
A←→ P, B ←→Q, C ←→ R, D ←→ S
Under this correspondence, the edges in two graphs correspond with each other as indicated below:
{A, B} ←→ {P, Q}, {A, C} ←→ {P, R}, {A, D} ←→ {P, S}
{B, C} ←→ {Q, R}, {B, D} ←→ {Q, S}, {C, D} ←→ {R, S},
We check that the above indicated one to cone correspondence between the
Vertices / edges of the two graphs. Preserves the adjacency of the vertices. The existence of this correspondence proves that the two graphs are isomorphic (note that both the graphs represent the complete graph K_4).
Next, consider the graphs shown in figures 1.47 (a) and 1.47(b)

Fig. 1.47

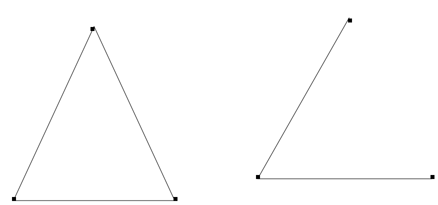


Fig. (a) Fig. (b)

We observe that the two graphs have the same mumber of vertices but different number of edges. Therefore, although there can exist one-to-one correspondence between the vertices, there cannot be a one-to-one correspondence between the edges. The two graphs are therefore not isomorphic.

From the definition of isomorphism of graphs, it follows that if two graphs are isomorphic, then they must have

1. The same number of vertices.
2. The same number of edges.
3. An equal number of vertices with a given degree.

These conditions are necessary but not sufficient. This means that two graphs for which these conditions hold need not be isomorphic.

In particular, two graphs of the same order and the same size need not be isomorphic. To see this, consider the graphs shown in figures 1.48(a) and (b).

Fig.1.48(a)

Fig. (a) Fig. (b)

We note that both graphs are of order 4 and size 3. But the two graphs are not isomorphic. Observe that there are two pendant vertices in the first graph where as there are three pendant vertices in the second graph. As such, under any one-to-one correspondence between the vertices and the edges of the two graphs, the adjacency of vertices is not preserved

Example 1:

Prove that the two graphs shown below are isomorphic.

Fig.1.49

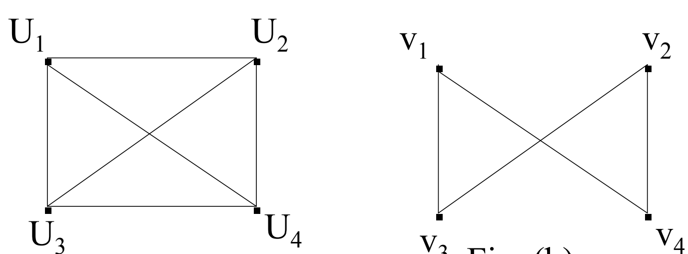


Fig. (a) Fig. (b)

Solution: We first observe that both graphs have four vertices and four edges. Consider the following one – to- one correspondence between the vertices of the graphs.

$u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_3$, $u_4 \leftrightarrow v_2$.

This correspondence give the following correspondence between the edges.

$\{u_1,u_2\} \leftrightarrow \{v_1,v_4\}$, $\{u_1,u_3\} \leftrightarrow \{v_1,v_3\}$

$\{u_2,u_4\} \leftrightarrow \{v_4,v_2\}$, $\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$.

These represent one-to-one correspondence between the edges of the two graphs under which the adjacent vartices in the first graph correspond to adjacent vertices in the second graph and vice-versa.

Example 2: Show that the following graphs are not isomorphic.

Fig. 1.50

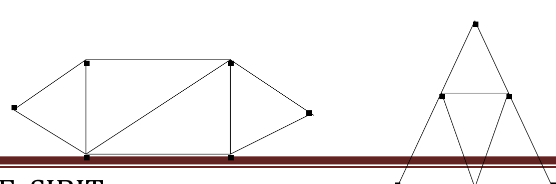


Fig. (a) Fig. (b)

Solution: We note that each of the two graphs has 6 vertices and nine edges. But, the first graph has 2 vertices of degree 4 where as the second graph has 3 vertices of degree 4. Therefore, there cannot be anyone-to-one correspondence between the vertices and between the edges of the two graphs which preserves the adjacency of vertices. As such, the two graphs are not isomorphic.

UNIT 2

PLANAR GRAPHS:

It has been indicated that a graph can be represented by more than one geometrical drawing. In some drawing representing graphs the edges intersect (cross over) at points which are not vertices of the graph and in some others the edges meet only at the vertices. A graph which can be represented by at least one plane drawing in which the edges meet only at vertices is called a '**planar graph**'

On the other hand, a graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a **non planar graph.**

In other words, a non planar graph is a graph whose every possible plane drawing contains at least two edges which intersect each other at points other than vertices.

Example 1

Show that (i) a graph of order 5 and size 8, and (ii) a graph of order 6 and size 12, are planar graphs.

Solution: A graph of order 5 and size 8 can be represented by a plane drawing

Fig. 2.1

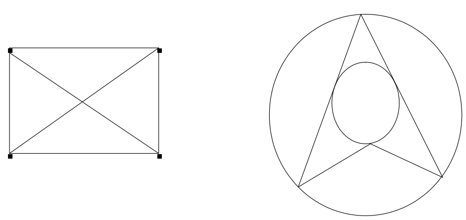


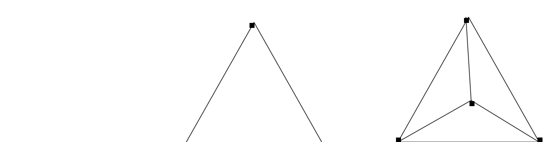
Fig. (a) Fig. (b)

In which the edges of the graph meet only at the vertices, as shown in fig. 2.1 (a) therefore, this graph is a planar graph. Similarly, fig. 2.1(b) shows that a graph of order 6 and size 12 is a planar graph.

Example 2:

Show that the complete graphs K_2,K_3 and K_4 are planar graphs.

Fig. 2.2



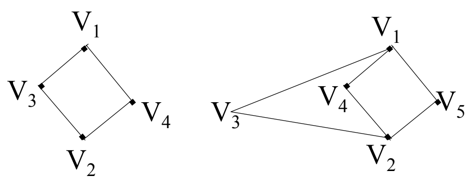
K_2 K_3 K_4

Solution: the diagrams in fig 2.2 represent the graphs K_2,K_3,K_4. In none of these diagrams, the edge meet at points other than the vertices. Therefore K_2, K_3, K_4 are all planar graphs.

Example 3:

Show that the bipartite graphs $K_{2.2}$ and $K_{2,3}$ are planar graphs.

Fig. 2.3



(a): $K_{2.2}$ (a): $K_{2.3}$

Solution: In $K_{2,2}$, the vertex set is made up of two bipartites V_1,V_2, with V_1 containing two vertices say V_1,V_2 and V_2 containing two vertices, say V_3,V_4, and there is an edge joining every vertex in V_1 with every vertex in V_2 and vice-versa. Fig 2.3(a) represents this graph. In this fig. the edges meet only at the vertices therefore, $K_{2,2}$ is a planar graph.

In $K_{2,3}$ the vertex set is made up of two bipartites V_1 and V_2, with V_1 containing two vertices, say V_1,V_2, and V_2 containing three vertices, say V_3,V_4,V_5 and there is an edge joining every vertex in V_1 with every vertex in $V_2$and Vice Versa. Fig. 2.3(b) represents this graph. In this figure the edges meet only at the vertices, therefore $K_{2,3}$ is a planar graph.

Example4:

Show that the complete graph K_5 (viz., the Kuratowskis first graph) is a non planar graph.

Solution:

We first recall that in the complete graph K_5 there are 5 vertices and there is an edge between every pair of vertices, totaling to 10 edges. (see fig. Ref. complete graph). This fig is repeated below with the vertices named as V_1,V_2,V_3,V_4,V_5 and the edges named e_1,e_2,e_3,……e_{10}

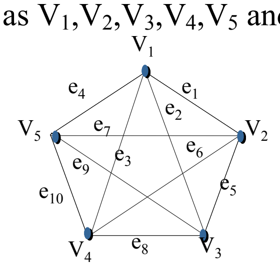


Fig. 2.4

In the above drawing of K_5, the five edges e_1,e_5,e_8 e_{10},e_4 form a pentagonal cycle and the remaining five edges e_2,e_3,e_6,e_7,e_9 are all

Inside this cycle and intersect at points other than the vertices.

Let us try to draw a diagram of K_5 in which the edges meet ony at the vertices. In the pentagonal cycle present in fig (2.4) the edges meet only at the vertices. Let us start our new drawing of K_5 with this cycle: the cycle is shown in fig. 2.5 (a)

Fig. 2.5

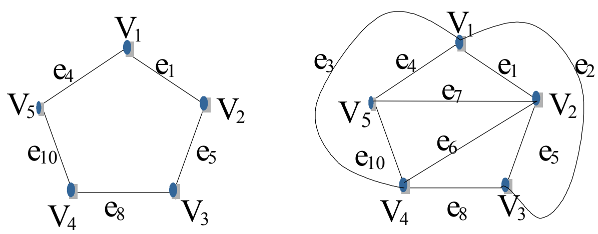


Fig. (a) Fig. (b)

Consider the edge $e_7 = \{V_2V_5\}$. This edge can be drawn either inside or outside the pentagonal cycle. Suppose we draw it inside, as shown in fig. 2.5 (b) the other case is similar now, consider the edges $e_2 = \{V_1V_3\}$ & $e_3 = \{V_1V_4\}$. If we draw these edges also inside the pentagon, they will intersect e_7, that is, they cross e_7 at points, which are not vertices, therefore, let us draw of them outside: see fig. 2.5 (b).

Next consider the edge $e_6 = \{V_2,V_4\}$ if we draw this edge outside the pentagon intersects the edge e_2; see fig 2.5(b) therefore let us draw e6 inside the pentagon.

Lastly, consider the edge $e_9 = \{V_3,V_5\}$If we draw this edge outside the pentagon, it intersects the edge e_3, and if we draw it inside, it intersects the edge e_6.

This demonstrates that in every possible plane drawing of K_5 at least two edges of K_5 intersect at a point which is not a vertex of K_5.

This proves that K_5 is a non planar graph.

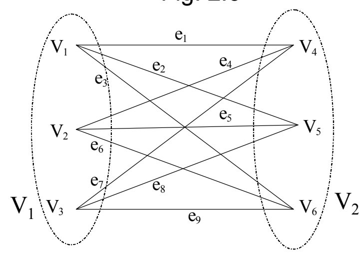
Example 5:

Show that the complete bipartite graph $K_{3,3}$ (namely the Kuratowski's second graph) is a non-planar graph.

Solution:

by definition, $K_{3,3}$ is a graph with 6 vertices and 9 edges, in which the vertex set is made up of two bipartites V_1 and V_2 each containing three vertices such that every vertex in V_1 is joined to every vertex in V_2 by an edge and vice-versa.

Fig. 2.6



Let us name the vertices in V_1 as v_1,v_2,v_3 and the vertices inV_2 as v_4,v_5,v_6. Also let the edges be named as $e_1,e_2,e_3,\ldots\ldots.e_9$.

A diagram of the graph is shown in fig (2.6). In this diagram of $K_{3,3}$. the six edges $e_1 = \{v_1,v_4\}$, $e_4=\{v_4v_2\}$, $e_5 = \{v_2v_5\}$, $e_8=\{v_5,v_3\}$ $e_9=\{v_3,v_6\}$ and $e_3=\{v_6,v_1\}$ form a hexagonal cycle and the remaining three edges e_2,e_6,e_7 either intersect these edges or intersect among themselves at points other than the vertices.

Let us try to draw a diagram of $K_{3,3}$ in which no two of its edges intersect. The hexagonal cycle present in fig.2.6 does not contain any mutually intersecting edges. Let us start our new drawing of $K_{3,3}$ with this cycle. This cycle is exhibited separately in fig. 2.7 (a)

Fig. 2.7

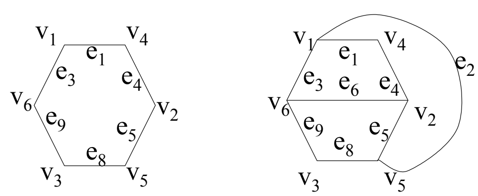


Fig. (a) Fig. (b)

Consider three edge $e_6=\{v_2,v_6\}$ this edge can be drawn either inside the hexagonal cycle or outside it. Let us draw it inside (as shown in fig.2.7 (b) the other case is similar. Now consider the edge $e_2 = \{v_1,v_5\}$. If we draw this edge the hexagon, it intersects the edges e_6. Therefore, let us draw it outside the hexagon see fig. 2.7 (b)

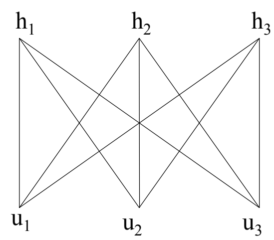
Next consider the edge $e_7 = \{v_3,v_4\}$. If this edge is drawn inside the hexagon, it intersects the edgte e_6, and if it is drawn outside the hexagon, it intersects the edge e_2

This demonstrates that in every possible plane drawing of $K_{3,3}$, at least two edges of $K_{3,3}$ intersect at a point which is not a vertex of $K_{3,3}$. this proves that $K_{3,3}$ is a non planar graph.

Example 6

Suppose there are three houses and three utility points (electricity, water sewerage, say) which are such that each utility point is joined to each house. Can the lines of joining be such that no two lines cross each other ?

Fig. 2.8



Solution:

Consider the graph in which the vertices are the three houses (h_1,h_2,h_3) and the three utility points (u_1,u_2,u_3). Since each house is joined to each utility point. The graph has to be $K_{3,3}$ (see fig. 2.8). This graph is non-planar and therefore, in its plane drawing, at lest two of its edges cross each other. As such, it is not possible to have the lines joining the houses and the utility points such that no two lines cross each other.

HAMILTON CYCLES AND HAMILTON PATHS

Let G be a connected graph. If there is a cycle in G that contains all the vertices of G, then that cycle is called a '**Hamilton Cycle**' in G.

A Hamilton cycle in a graph of n vertices consists of exactly n edges, because, a cycle with n vertices has n edges.

By definition, a Hamilton cycle in Graph G must include all vertices in G, This does not mean that it should include all edges of G.

A graph that contains a Hamilton cycle is called a **Hamilton graph** (or Hamiltonian graph).

For example, in the graph shown in fig. (2.7), the cycle shown in thick lines is a Hamilton cycle. (observe that this cycle does not include the edge BD). the graph is therefore a Hamilton graph.

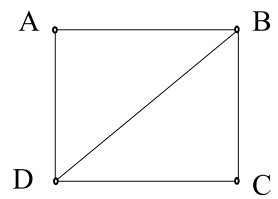


Fig.2.7

A path (if any) in a connected graph which includes every vertex (but not necessarily every edge) of the graph is called a Hamilton / Hamiltonian path in the graph.

For example: In the graph shown in fig (2.8), The path shown in thick lines is a Hamilton path.

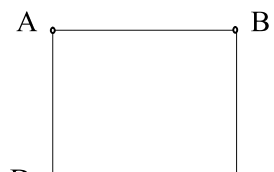


Fig. 2.8

In the graph shown in fig. (2.9), the path ABCFEDGHI is a Hamilton path. We check that this graph does not contain a Hamilton cycle.

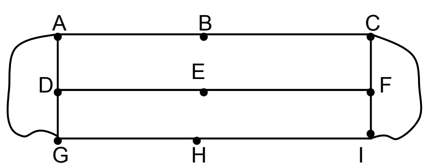


fig. (2.9)

Since a Hamilton path in a graph G meets every vertex of G, the length of a Hamilton path (if any) in a connected graph of n vertices is n-1 (a path with n vertices has n-1 edges)

Theorem 1:

If in a simple connected graph with n vertices (where $n \geq 3$) The sum of the degrees of every pair of non-adjacent vertices is greater than or equal to n, than the graph is Hamiltonian.

Theorem 2:

If in a simple connected graph with n vertices (where $n \geq 3$) the degree of every vertex is greater than or equal to n/2. then the graph is Hamiltonian .

Proof:If in a simple connected graph with n vertices, the degree of each vertex is greater than or equal to n/2. then the sum of the degrees of every pair of adjacent or non-adjacent vertices is greater than or equal to n, therefore, the graph is Hamiltonian (by Them 1).

Example 1:

Prove that the complete graph K_n where $n \geq 3$, is a hamilton graph.
Solution: In K_n, the degree of every vertex is n-1, if $n \geq 3$, we have $n-2 > 0$, or $2n-2 > n$, or $(n-1) > n/2$. Thus, in K_n, where $n \geq 3$, the degree of every vertex is greater than n/2. Hence K_n is Hamiltonian by Them. 2.

Example 2:

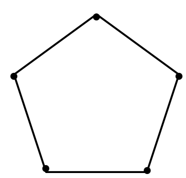
Show that every simple K - Regular graph with 2K-1 vertices is Hamiltonian.
Solution: In a K - Regular graph , the degree of every vertex is K, and $K > K - 1/2 = 1/2\ (2K - 1) = 1/2$ n. Where n = 2K-1 is the number of vertices, therefore, by Them. 2, the graph considered is Hamiltonian if it is simple.

Example 3:

Disprove the converses of theorems 1 and 2.

Solution: Consider a 2 – Regular graph with n=5, vertices, shown in fig. (2.10)

Fig. 2.10



Evidently, this graph is Hamiltonian. But the degree of every vertex is 2 which is less than n/2 and the sum of the degrees of every pair of vertices is 4 which is less than n. Thus, the converses of theorems 1 & 2 are not necessarily true.

Example 4:

Let G be a simple graph with n vertices and m edges where m is at least 3. if $m \geq 1/2\ (n-1)(n-2)+2$. Prove that G is Hamiltonian. Is the converse true?

Solution :

Let u & v be any two non-adjacent vertices in G. Let x & y be their respective degrees. If we delete u,v from G, we get a subgraph with n-2 vertices. If this subgraph has q edges, then $q \leq 1/2\ (n-2)(n-3)$. [in a simple graph of order n, the number of edges is $\leq 1/2n(n-1)$] since u and v are non adjacent.

$m = q + x + y$, Thus

$x + y = m - q$

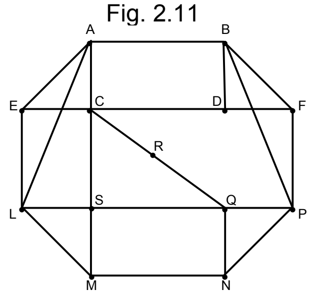
$\geq \{1/2\,(n-1)(n-2)+2\} - \{1/2(n-2)(n-3)\}$

$= n$

Therefore, by Theorem 1, the graph is Hamiltonian. The converse of the result just proved is not always true. Because, a 2- Regular graph with five vertices shown in fig (2.10) is Hamiltonian but the inequality does not hold.

Example 5: Show that the graph shown in fig (2.11) is a Hamilton graph.

Fig. 2.11



Solution:

By examining the given graph, we notice that in the graph there is a cycle AELSMNPQRCDFBA which contains all the vertices of the graph. this cycle is a hamiltonian cycle. since the graph has Hamiltonian cycle in it. The graph is a Hamiltonian graph.

Example 6:

Exhibit the following.

(a): A graph which has both an Euler Circuit and a Hamilton cycle.

Solution:

The graph shown is the required graph.

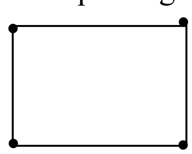


Fig. (a)

(b) : A graph which has an Euler circuit but no Hamilton cycle.

Solution: The graph shown is the required graph.

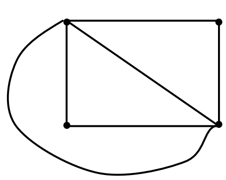


Fig. (b)

(C) A graph which has a Hamilton cycle but no Euler Circuit.

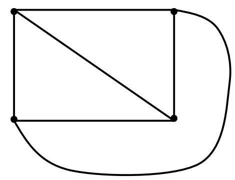


Fig. (c)

(d): A graph which has neither a Hamilton cycle nor an Euler circuit.

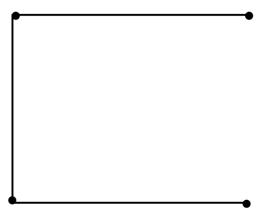


Fig. (d)

The following theorem contains useful information on the existence of Hamilton cycle in the complete graph K_n.

Theorem 3: In the complete graph with n vertices, where n is an odd number ≥ 3, there are (n-1) / 2 edge - disjoint Hamiltonian cycles.

Proof:

Let G be a complete graph with n vertices, where n is odd and ≥ 3. Denote the vertices of G by 1,2,3…..n and Represent them as points as shown in fig. (2.12)

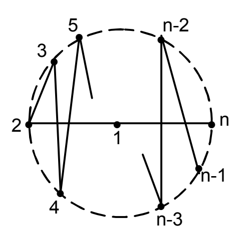


Fig. 2.12

We note that the polygonal pattern of edges from vertex 1 to vertex n as depicted in the fig is a cycle that includes all the vertices of G. This cycle is therefore a Hamilton cycle. This representation demonstrates that G has at least one Hamilton cycle. (In the fig (2.12)), the vertex 1 is at the centre of a circle and the other vertices are on its circumference. The circle is dotted.

Now, rotate the polygonal pattern clockwise by $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha k$ degrees where $\alpha_1 = 360^0/n-1$, $\alpha_2 = 2.\ 360^0/n-1$, $\alpha_3 = 3.360^0/n-1, \ldots \alpha_k=(n-3)/2.\quad 360^0\ /n-1$

Each of these K = (n-3)/2 rotations gives a Hamilton cycle that has no edge in common with any of the preceding ones. Thus, there exists k = (n-3)/2, new Hamilton cycles, all edge - disjoint from the one shown in fig (2.12) and also edge - disjoint among themselves thus, in G, there are exactly.

1+K = 1 + (n-3)/2 = 1/2 (n-1)

Mutually edge –disjoint Hamilton cycle.

This completes the proof of the theorem.

Example 7:

How many edge - disjoint Hamilton cycles exist in the complete graph with seven vertices? Also, draw the graph to show these Hamilton cycles.

Solution:

According to theorem 3, the complete graph K_n has (n-1)/2 edge - disjoint Hamilton cycles when $n \geq 3$ and n is add. When n = 7, their number is (7-1)/2 = 3. As indicated in the proof of Theorem 3 . One of these Hamilton cycles appears as shown in fig (2.13)

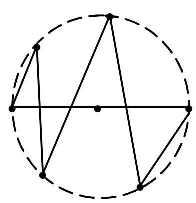


Fig. 2.13

The other two cycles are got by rotating the above shown cycle clock wise through angles.

$\alpha_1 = 360^0/7-1, = 60$, and $\alpha_2 = 2(360^0)/7-1, = 120^0$

TRAVELING –SALESMAN PROBLEM :

A problem closely related to the question of Hamiltonian circuits is the traveling sales man problem, stated as follows: A sales man is required to visit a number of cities during a trip, given the distances between the cities, in what order should be travel so as to visit every city precisely once and return home, with the minimum mileage traveled ?

Representing the cities by vertices and the roads between them by edges, we get a graph. In this graph, with every edge e_i there is associated a real number (the distance in miles, say), $w(e_i)$ such a graph is called a weighted graph; $w(e_i)$ being the weight of edge e_i.

In our problem, if each of the cities has a road to every other city, we have a 'complete weighted graph'. This graph has numerous Hamiltonian circuits, and we are to pick the one that has the smallest sum of distances (or weights)

The total number of different (not edge - disjoint, of course) Hamiltonian circuits in a complete graph of n vertices can be shown to be (n-1)!/2.

This follows from the fact that starting from any vertex we have n-1 edges to choose from the first vertex, n-2 from the second, n-3 from the third, and so on. These being independent choices.

We get (n-1)! possible number of choices. This number is, however, divided by 2, because each Hamiltonian circuit has been counted twice.

Theoretically, the problem of the traveling salesman can always be solved by enumerating all (n-1)! /2 Hamiltonian circuits, calculating the distance traveled in each, and then picking the shortest one. However for a large value of n, the labor involved is too great even for a digital computer (try solving it for the 50 state capitals in the united states: n = 50).

The problem is to prescribe a manageable algorithm for finding the shortest route. No efficient algorithm for problems of arbitrary size has yet been found, although many attempts have been made. Since this problem has applications in operations research, some specific large - scale examples have been worked out. There are also available several heuristic methods of solution that give a route very close to the shortest one.

Various types of walks

Discussed in this chapter are summarized in fig (2.14). The arrows point in the direction of increasing restriction.

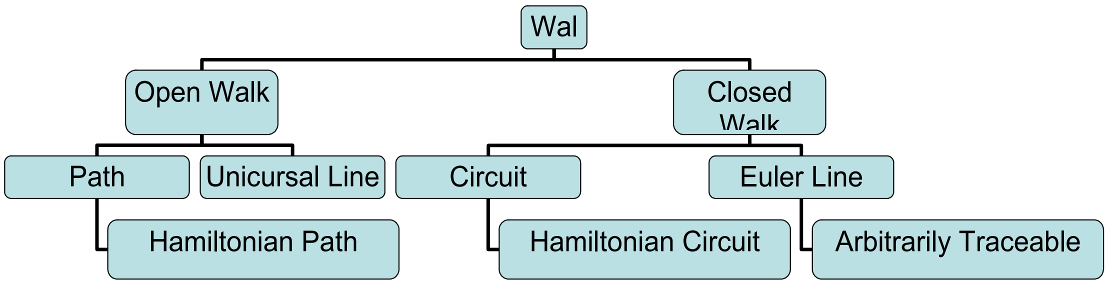


Fig. 2.14 Different Types of Walks

GRAPH COLORING:

Given a planar or non-planar graph G, if we assign colors (colours) to its vertices in such a way that no two adjacent vertices have (receive) the some color , then we say that the graph G is Properly colored.

In otherwords, proper coloring of a graph means assigning colors to its vertices such that adjacent vertices have different colors.

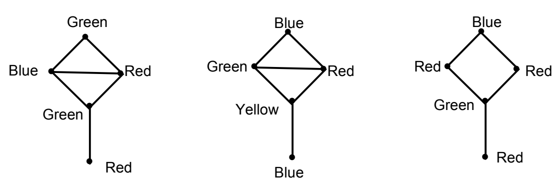


Fig. 2.15

In fig. (2.15), the first two graphs are properly colored where as the third graph is not properly colored.

By Examining the first two graphs in fig (2.15) which are properly colored , we note the following

i) A graph can have more than one proper coloring.

ii) Two non–adjacent vertices in a properly colored graph can have the same color.

CHROMATIC NUMBER:

A graph G is said to be K–colorable if we can properly color it with K (number of) colors. A graph G which is K–colorable but not (K-1) – colorable is called a

'K – Chromatic graph'.

In otherwords, a K–Chromatic graph is a graph that can be properly colored with K colors but not with less than K colors.

If a graph G is K–Chromatic, then K is called the chromatic number of G. Thus, the chromatic number of a graph is the minimum number of colors with which the graph can be properly colored. The chromatic number of a graph G is usually denoted by χ (G).

SOME RESULTS:

i) A graph consisting of only isolated vertices (ie., Null graph) is 1–Chromatic (Because no two vertices of such a graph are adjacent and therefore we can assign the same color to all vertices).

ii) A graph with one or more edges is at least 2 -chromatic (Because such a graph has at least one pair of adjacent vertices which should have different colors).

iii) If a graph G contains a graph G_1 as a subgraph, then

$\chi(G) \geq \chi(G_1)$.

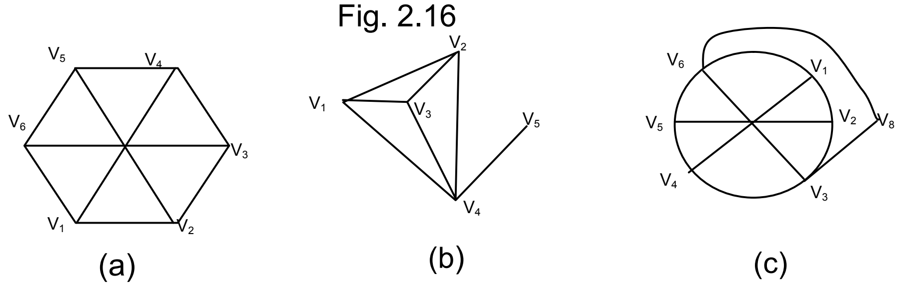
iv. If G is a graph of n vertices, then $\chi(G) \leq n$.

v. $\chi(K_n) = n$, for all $n \geq 1$. (Because, in K_n, every two vertices are adjacent and as such all the n vertices should have different colors)

vi. If a graph G contains K_n as a subgraph, then $\chi(G) \geq n$.

Example 1: Find the chromatic number of each of the following graphs.

Fig. 2.16



Solution :

i) For the graph (a), let us assign a color α to the vertex V_1, then for a proper coloring, we have to assign a different color to its neighbors V_2,V_4,V_6, since V_2, V_4, V_6 are mutually non-adjacent vertices, they can have the same color as V_1, namely α.

Thus, the graph can be properly colored with at lest two colors, with the vertices V_1,V_3,V_5 having one color α and V_2,V_4,V_6 having a different color β . Hence, the chromatic number of the graph is 2.

ii) For the graph (b) , let us assign the color α to the vertex V_1. Then for a proper coloring its neighbours V_2,V_3 & V_4 cannot have the color α.

Further more, V_2, V_3,V_4 must have different colors, say β, γ, δ .Thus, at least four colors are required for a proper coloring of the graph.

Hence the chromatic number of the graph is 4.

iii) For the graph (c) , we can assign the same color, say α, to the non-adjacent vertices V_1, V_3, V_5.

Then the vertices V_2,V_4,V_6 consequently V_7 and V_8 can be assigned the same color which is different from both α and β . Thus, a minimum of three colors are needed for a proper coloring of the graph. Hence its chromatic number is 3.

Example 2: Find the chromatic numbers of the following graphs.

Fig. 2.17

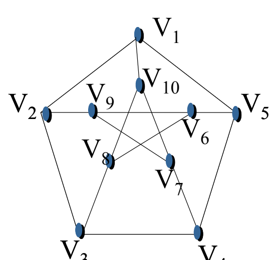


Fig. 2.17 (a)

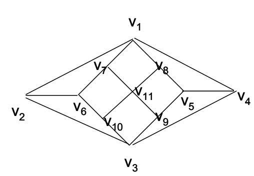


Fig. 2.17 (b)

Solution (i):

We note that the graph (a) is the Peterson graph. By observing the graph, we note that the vertices V_1,V_3,V_6 and V_7 can be assigned the same color, say α . Then the vertices V_2,V_4, V_8 and V_{10} can be assigned the same color, β (other than α) . Now, the vertices V_5 and V_9 have to be assigned colors other than α and β ; they can have the same color γ. Thus, a minimum of three colors are required for a proper coloring of this graph. Hence, the chromatic number of this graph is 3.

Solution (ii) :

By observing the graph (b),(this graph is called the **Herscher graph**),we note that the vertices V_1, V_3, V_5, V_6 and V_{11} can be assigned the same color α and all the remaining vertices: V_2,V_4, V_7,V_8, V_9 and V_{10} can be assigned the same color β (other than α). Thus two colors are sufficient
(one color is not sufficient) for proper coloring of the graph. Hence its chromatic number is 2.

Example (3):

Prove that a graph of order n ($\geq$ 2) consisting of a single cycle is 2–chromatic if n is even and 3 – chromatic if n is odd.

Solution:

The graph being considered is shown as below.

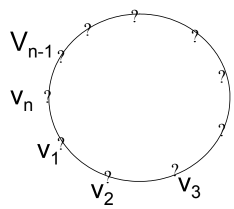


Fig. 2.18

Obviously, the graph cannot be properly colored with a single color. Assign two colors alternatively to the vertices, starting with V_1. Then, the odd vertices, V_1, V_3, V_5 etc., will have a color α and the even vertices V_2, V_4, V_6 will have a different color β. Suppose n is even, then the vertex V_n is an even vertex and therefore will have the color β, and the graph gets properly colored therefore, the graph is 2–chromatic

Suppose n is odd, then the vertex V_n is an odd vertex and therefore will have the color α and the graph is not properly colored (because, then the adjacent vertices V_n and V_1 will have the same color α). To make it properly colored, it is enough if V_n is a assigned a third color, γ. Thus, in this case, the graph is 3-chromatic.

Example 4:

Prove that a graph G is 2–chromatic if and only if it is non – null bipartite graph.

Solution:

Suppose a graph G is 2 - chromatic. Then it is non-null and some vertices of G have one color, say α and the rest of the vertices have another color, say β .Let V_1 be the set of vertices having color α and V_2 be the set of vertices having color β .Then $V_1 \cup V_2 = V$. The vertex set of G , and $V_1 \cap V_2 = \Phi$. Also, no two vertices of V_1 can be adjacent and no two vertices of V_2 can be adjacent. As such, every edge in G has one end in V_1 and the other end in V_2. Hence G is bipartite graph.

Conversely, suppose G is a non- null bipartite graph. Then the vertex set of G has two bipartites V_1 and V_2 such that every edge in G has one end in V_1 and another end in V_2. Consequently, G cannot be properly colored with one color ; because then vertices in V_1 and V_2 will have the same color and every edge has both of its ends of the same color. Suppose we assign a color α to all vertices in V_1 and a different color β to all vertices in V_2. This will make a proper coloring of V. Hence G is 2- Chromatic.

Example 5 :

If Δ (G) is the maximum of the degrees of the vertices of a graph G, then prove that $\chi(G) \leq 1+\Delta(G)$. ………….. (i)

Solution:

Suppose G contains n = 2 vertices, then the degrees of both the vertices is 1, so that $\Delta(G) = 1$,also $\chi(G) = 2$ .Hence $\chi(G) = 1+\Delta(G)$.

Thus, the required inequality (i) is verified for n=2.

Assume that the inequality is true for all graphs with K- vertices. Consider a graph G′ with K + 1 vertices. If we remove any vertex v from G′ then the resulting graph H will have K vertices and $\Delta(H) \leq \Delta(G')$. since H has K vertices, the inequality (i) holds for H (by the assumption made). Therefore, $\chi(H) \leq 1+\Delta(H)$. since $\Delta(H) \leq \Delta(G')$, this yields $\chi(H) \leq 1+\Delta(G')$

Now, a proper coloring of G′ can be achieved by retaining the colors assigned to the vertices in H and by assigning a color to V that is different from the colors assigned to the vertices adjacent to it.

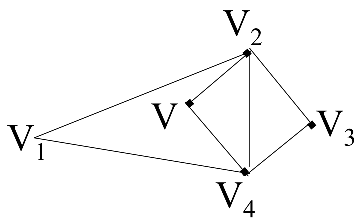


Fig. (a): G'

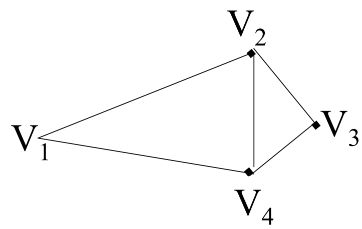


Fig. (b): H

The color to be assigned to V can be one of the colors already assigned to a vertex in H that is not adjacent to V. Thus, a proper
Coloring of G´can be done without the use of a new color.

Hence χ (G') = χ (H) $\leq$ 1+ Δ(G').

Thus, if the inequality (1) holds for all graphs with K vertices, it holds for a graph with K + 1 vertices.Hence, by induction, it follows that the inequality (1) holds for all graphs .

EULER'S FORMULA

If G is a planar graph, then G can be represented by a diagram in a plane. In which the edges meet only at the vertices. Such a diagram divides the plane in to a number of parts called **regions (or faces)**, of which exactly one part is unbounded. The number of edges that form the boundary of a region is called the **degree** of that region.

For example, in the diagram of a planar graph shown in fig. (2.20) the diagram divides the plane into 6 regions R_1,R_2,R_3,R_4,R_5,R_6. We observe that each of the regions R_1 to R_5 is bounded and the region R6 is unbounded. That is, R_1 to R_5 are in the interior of the graph while R_6 is in the **Exterior**.

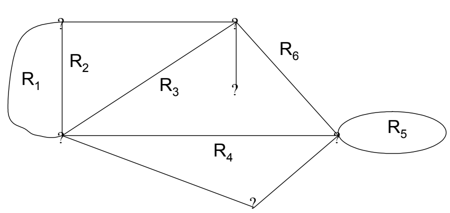


Fig. 2.20

We further observe that, the fig (2.20) the boundary of the region R_1 is made up of two edges. Therefore, the degree of R_1 is 2. We write this as $d(R_1) = 2$. The boundary of each of the regions R_2 and

R_4 is made up of 3 edges; therefore, $d(R_2) = d(R_4) = 3$. The boundary of the region R_3 consists of 4 edges of which one is pendant edge.

Therefore, $d(R_3) = 5$. The region R_5 is bounded by a single edge (loop) therefore, $d(R_5)=1$.The boundary of the exterior region R_6 consists of six edges;therefore,$d(R_6) = 6$.

We note that

$d(R_1) + d(R_2) + d(R_3) + d(R_4) + d(R_5) + d(R_6) = 20$.

Which is twice the number of edges in the graph. This property is analogous to the handshaking property and is true for all planar graphs.

It should be pointed out that the regions are determined by a diagram of a planar graph and not by the graph itself. This means that if we change the diagram of the graph, the regions determined by the new diagram will be generally different from those determined by the old one in the sense that the unbounded region in the old diagram need not be unbounded in the new diagram. However, the interesting fact is **that the total number of regions in the two diagrams remains the same**.

The proof of this fact is contained in the following **Euler's fundamental theorem** on planar graphs.

Theorem:

A connected planar graph G with n vertices and m edges has exactly $m - n + 2$ regions in all of its diagrams.

Proof:

Let r denote the number of regions in a diagram of G. The theorem states that,

$r = m - n + 2$, or $n - m + r = 2$ …………(1)

We give the proof by induction on m.

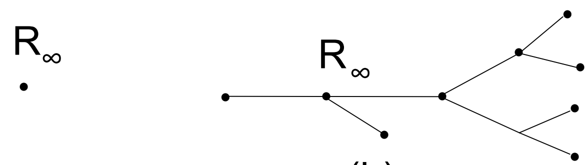
If $m = 0$, then n , must be equal to 1. Because , if $n > 1$, then G will have

at least two vertices and there must be an edge connecting them (because G is connected), so that $m \neq 0$, which is a contradiction.

If $n = 1$, a diagram of G determines only one region – the entire plane region (as shown in fig 2.21 (a)).

Thus , if $m = 0$, then $n = 1$ and $r = 1$, so that $n - m + r = 2$. This verifies the theorem for $m = 0$.

Fig. 2.21



(a) (b)

Now, assume that the theorem holds for all graphs with m=k number of edges, where k is a non-negative integer.

Consider a graph G_{k+1} with k +1 edges and n vertices. First, suppose that G_{k+1} has no cycles in it. Then a diagram of G_{k+1} will be of the form shown in fig. 2.21 (b) in which the number of vertices will be exactly one more than the number of edges, and the diagram will determine only one region–the entire plane region (as in fig. 2.21 (b)). Thus for G_{k+1}, we have, in this case, n = (k+1)+1 and r = 1, so that

n – (k+1)+r = 2.

This means that the result (i) is true when m=k+ 1 as well, if G k+1 contains no cycles in it.

Next, suppose G_{K+1} contains at least one cycle. Let r be the number of regions which a diagram of G_{k+1} determine. Consider an edge 'e ' in a cycle and remove it from G_{k+1}. The resulting graph, G_{k+1} – e, will have n vertices and (k+1)–1=k edges, and its diagram will determine r-1 regions. Since G_{k+1} – e has k edges, the theorem holds for this graph (by the induction assumption made).

That is we have

r - 1 = k – n +2, or n – (k + 1)+r = 2

This means that in this case also the result (1) is true when m = k + 1 as well.

Hence, by induction, it follows that the result (1) is true for all **non– negative** integers m. This completes the proof of the theorem.

Corollary I :

If G is connected simple planar graph with n(≥3) vertices, m (>2) edges and r regions, then (i) m ≥ **(3/2)r** and (ii) m ≤ 3n-6.

Proof:

Since the graph G is simple, it has no multiple edges and no loops. As such, every region must be bounded by three or more edges. Therefore, the total number of edges that bound all the regions is greater then or equal to 3r.On the other hand, an edge is in the boundary of at most two regions.

Therefore, the total number of edges that bound all regions is less than or equal to 2m.Thus,$3r \le 2m$. or $m \ge$ **(3/2)r**

This is required result (i) .

Now, substituting for r from Euler's formula in the result just proved, we get $m \ge 3/2\ (m-n+2)$

Which simplifies to $m \le 3n-6$. This is required result (ii)

Corollary 2:

Kuratowski's first graph, K5, is non-planar.

Proof:

The graph K_5 is simple, connected and has n = 5 vertices and m = 10 edges; refer to figure Kuratowski's first graph. If this graph is planar, then by result (ii) of Corollary 1, we should have $m \le 3n - 6$; that is $10 \le 15 - 6$, which is not true. Therefore, K5 is non – planar

Corollary 3:

Kuratowski's second graph, $K_{3,3}$, is non-planar.

Proof: We first note that $K_{3,3}$ is simple, connected and has n = 6 vertices and m = 9 edges; see fig Kuratowski's second graph.

Suppose $K_{3,3}$ is planar. By examining the figure Kuratowski's graph, we note that $K_{3,3}$ has no cycles of length 3. Therefore by result (iii) of Corollary 1, we should have $m \le 2n - 4$; that is, $9 \le 12 - 4$, which is not true. Hence, $K_{3,3}$ is non – planar.

Corollary 4:

Every connected simple planar graph G contains a vertex of degree less than 6.

Proof:

Suppose every vertex of G is of degree greater than or eual to 6.Then,if $d_1,d_2,\ldots\ldots dn$ are the degrees of the n vertices of G,we have $d_1 \ge 6$, $d_2 \ge 6,\ldots\ldots d_n \ge 6$.

Adding these, we get

$d_1+d_2+\ldots\ldots+d_n \ge 6$ n.

By handshaking property, the left hand side of this inequality is equal to 2m,where m is the number of edges inG,thus,$2m \ge 6n$, or $3n \le$ **m.**

On the other hand ,by the result(ii)of corollary 1,(Result (ii) ie $m \le 3n-6$**).**

We should have $m \le 3n-6$.Thus, $3n \le m \le 3n-6$.This cannot be true.

Therefore, G must have a vertex of degree less than 6.

Example 1:

Verify Euler's formula for the planar graph shown in figure 2.20.

Solution:

The given graph has n=6 vertices, m=10 edges and r=6 regions. Thus,

$n - m + r = 6 - 10 + 6 = 2$.

The Euler's formula is thus verified for the given graph.

Example 2:

Verify Euler's formula for the planar graphs shown below:

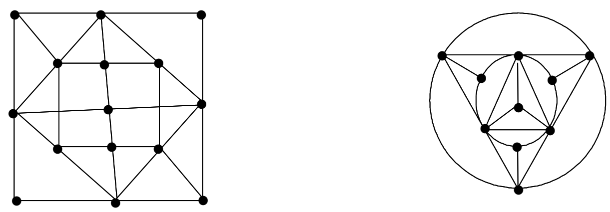


Fig. 2.22

Fig. (a) Fig. (b)

Solution:

We observe that the first of the given graphs has n = 17 vertices, m = 34 edges and r = 19 regions. Thus, $n - m + r = 17 - 34 + 19 = 2$.

In the second of the given graphs, there are n = 10 vertices, m = 24 edges and r = 16 regions, so that $n - m + r = 10 - 24 + 16 = 12$.

Thus, for both of the given graphs, Euler's formula is verified.

Example 3:

For the diagram of a planar graph shown below, find the degrees of regions and verify that the sum of these degrees is equal to twice the number of edges

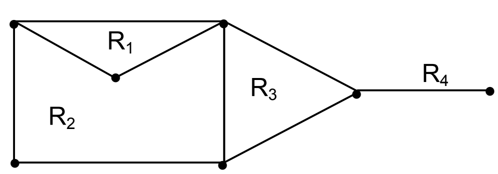


Fig. 2.23

Solution:

The diagram has 9 edges and 4 regions. The region R_1 is bound by three edges. Therefore, $d(R_1)=3$. Similarly, $d(R_2)=5$, $d(R_3)=3$.

The infinite region R_4 is bound by 5 edges plus a pendant edge. Therefore,$d(R_4)=7$. (Recall that while determining the degree of a region, a pendant edge is counted twice).

Accordingly,

$$d(R_1) + d(R_2) + d(R_3) + d(R_4) = 18$$

$$= \text{twice the no. of edges.}$$

Example 4:

A connected planar graph has 9 vertices with degrees 2,2,3,3,3,4,5,6,6.Find the number of regions of G.

Solution:

The given graph has $n = 9$ vertices. Let m be the number of edges and r be the number of regions.

Therefore by the Handshaking property, we have

$2m$ = sum of degrees of vertices

$= 2+2+3+3+3+4+5+6+6$

$= 34$.

Therefore, $m = 17$.

By using Euler's formula, we find that

$r = m - n + 2$.

$= 17-9+2 = 10$

Thus, the given graph has 10 regions.

Example 5:

Show that every connected simple planar graph G with less than 12 vertices must have a vertex of degree ≤ 4.

Solution:

Suppose every vertex of G has degree greater than 4. Then, if $d_1, d_2, d_3\ d_4, \ldots\ldots\ldots\ldots d_n$ are the degrees of n vertices of G, we have

$d_1 \geq 5, d_2 \geq 5, \ldots\ldots\ d_n \geq 5$ so that,

$d_1+ d_2+ d_3 +d_4 \ldots\ldots\ldots\ldots\ldots d_n \geq 5n$, or $2m \geq 5n$,by hand shaking property,

or $5n / 2 \leq m$…………(i)

On the other hand, Corollary 1 requires $m \leq 3n-6$. Thus, we should have, in view of (i), $5n/2 \leq 3n–6$ or $n \geq 12$……..(ii)

Thus, if every vertex of G has degree greater than 4, then G must have at least 12 vertices. Hence, if G has less than 12 vertices, it must have a vertex of degree $\leq$ 4.

Example 6:

Show that if a planar graph G of order n and size m has r regions and k components, then $n - m + r = k + 1$.

Solution:

let $H_1, H_2, \ldots\ldots H_k$ be the k components of G. Let the number of vertices, the number of edges and the number of non – exterior regions in H_i be n_i, m_i, r_i respectively, $i = 1, 2\ldots\ldots\ldots, k$. the exterior region is the same for all components. Therefore. $\sum n_i = n$, $\sum m_i = m$, $\sum r_i = r - 1$.

If the exterior region is not considered, then the Euler's formula applied to Hi yields

$n_i - m_i + r_i = 1$.

On summation (from i = 1 to i = k), this yields

$n - m + (r - 1) = k$, or $n - m + r = k + 1$.

2.5.1 Chromatic Polynominals:

Given a connected graph G & λ number of different colors, let us take up the problem of finding the number of different ways of properly coloring G with these λ colors.

First, consider the null graph N_n with n vertices. In this graph, no two vertices are adjacent. Therefore, a proper coloring of this graph can be done by assigning a single color to all the vertices. Thus, if there are λ number of colors, each vertex of the graph has λ possible choices of colors assigned to it, and as such the graph can be properly colored in λ^n different ways

Next consider the complete graph K_n. In this graph, every two vertices are adjacent, and as such there must be at least n colors for a proper coloring of the graph. If the number of different colors available is λ, then the number of ways of properly coloring K_n is

(i) Zero if $\lambda < n$,

(ii) One if $\lambda = n$,

(iii) Greater than 1 if $\lambda > n$.

Let $v_1, v_2, v_3 \ldots vn$ be the vertices of K_n and suppose $\lambda > n$.

For a proper coloring of K_n, the vertex v_1 can be assigned any of the λ colors, the vertex v_2 can be assigned any of the remaining $\lambda - 1$ colors, the vertex v_3 can be assigned any of the remaining $\lambda - 2$

colors and finally the vertex vn can be assigned any of the λ - n+1 colors. Thus, K_n can be profperly colored in λ (λ -1)(λ -2) ... (λ –n+1) different ways if λ >n.

Lastly, consider the graph L_n which is a path consisting of n vertices v_1, v_2, v_3….v_n shown below:

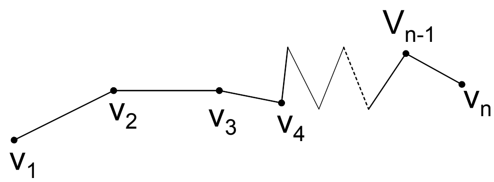


Figure 2.44

This graph cannot be properly colored with one color, but can be properly colored with 2 colors – by assigning one color to v_1, v_3, v_5…….. and another color to v_2, v_4, v_6…. Suppose there are $\lambda \geq 2$ number of colors available. Then, for a proper coloring of the graph, the vertex v_1 can be assigned any one of the λ colors and each of the remaining vertices can be assigned any one of λ-1 colors.

(Bear in mind that alternative vertices can have the same color). Thus, the graph L_n can be properly colored in $\lambda(\lambda-1)^{n-1}$ different ways.

The number of different ways of properly coloring a graph G with λ number of colors is denoted by P(G, λ). Thus, from what is seen in the above three illustrate examples, we note that

(i) $P(N_n, \lambda) = \lambda^n$,

(ii) $P(K_n, \lambda) = 0$ if $\lambda < n$,

$P(K_n, n) = 1$ if $\lambda = n$, and

$P(K_n, \lambda) = \lambda (\lambda -1)(\lambda -2) ... (\lambda -n+1)$ if $\lambda > n$,

(iii) $P(L_n, \lambda) = \lambda (\lambda -1)^{n-1}$ if $\lambda \geq 2$,

We observe that in each of the above cases, P(G, λ) is a polynominal. Motivated by these cases, we take that P(G, λ) is polynomial for all connected graph G. This polynomial is called the **Chromatic Polynomial.**

It follows that if a graph G is made up of n parts, G_1,G_2….G_n, then P(G, λ) is given by the following

PRODUCT RULE:

$P (G, \lambda) = P (G_1, \lambda). P (G_2, \lambda)……… P (G_n, \lambda)$

In particular, If G is made up of two parts G_1 and G_2, then we have $P(G, \lambda) =P (G_1, \lambda)$.

P (G2, λ) so that

$P(G_2, \lambda) = P(G, \lambda) / P(G_1, \lambda)$

DECOMPOSITION THEOREM:

Let G be a graph and e = {a,b} be an edge of G. Let $G_e = G - e$ be that subgraph of G which is obtained by deleting e from G without deleting vertices a and b*. Suppose we construct a new graph G_e' by coalescing (identifying / merging) the vertices a and b in G_e. Then G_e' is subgraph of G_e as well as G.

The process of obtaining G_e and G_e' from G is illustrated in Figure 2.45.

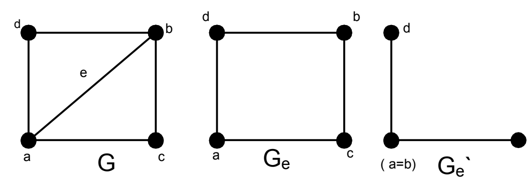


Figure 2.45

The following theorem called the Decomposition theorem for chromatic polynomials given an expression for $P(G, \lambda)$ in terms of $P(G_e, \lambda)$ and $P(G_e', \lambda)$ for a connected graph G.

Theorem 1:

If G is a connected graph and e = {a,b} is an edge of G, then

$$P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$$

Proof: In a proper coloring of G_e, the vertices a and b can have the same color or different colors. In every proper coloring of G, the vertices a and b have different colors and in every proper coloring of G_e' these vertices have the same color. Therefore, the number of proper colorings of G_e is the sum of the number of proper colorings of G and the number of proper colorings of G_e'. That is, $P(G_e, \lambda) = P(G, \lambda) + P(G_e', \lambda)$

This completes the proof of the theorm.

MULTIPLICATION THEOREM

The following theorem gives an expression for $P(G, \lambda)$ for a special class of graphs.

Theorem 2: If a graph G has sub graphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_n$ for some posistive intger n, then

$P(G, \lambda) = P(G_1, \lambda) . P(G_2, \lambda) / \lambda^{(n)}$

Where $\lambda^{(n)} = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$

Given $\lambda > n$ number of different colors, there are $\lambda(n) = \lambda (\lambda -1) (\lambda -1) (\lambda -2) \ldots\ldots (\lambda - n+1)$ number of proper colorings of K_n. For each of these $\lambda(n)$ proper colorings of K_n, the product rule yields $P(G_1,\lambda)/\lambda^{(n)}$ ways of properly coloring the remaining vertices of G_1. Similarly, there are $P(G_2, \lambda)/\lambda^{(n)}$ ways of properly coloring the remaining vertices of G. As such

$$P(G, \lambda) = P(K_n, \lambda) . P(G_1, \lambda) / \lambda^{(n)} . P(G_2, \lambda) / \lambda^{(n)}$$

$$= \lambda^{(n)} . P (G_1, \lambda) / \lambda^{(n)} . P (G_2, \lambda) / \lambda^{(n)}$$

$$= P (G_1, \lambda) . P (G_2, \lambda) / \lambda^{(n)}$$

This completes the proof of the theorem.

Example 1: Find the chromatic polynomial for the graph shown in Figure 2.46. What is its chromatic number ?

We observe that the given graph G is a path of length $n = 5$, namely L_5. Therefore, its chromatic polynomial is

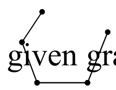


Figure 2.46

$$P (G, \lambda) = \lambda(\lambda-1)^{n-1} = \lambda(\lambda-1)^4$$

Next, we note that the chromatic number of the graph is $\chi(G)=2$. (Because, thse graph cannot be properly colored with one color but can be properly colored with 2 colors by assigning two colors to the alternative vertices).

Example 2:

Find the chromatic number and the chromatic polynomial for the graph $K_{1,n}$.

We note that $K_{1,n}$ is the complete bipartite graph wherein one bipartite of the vertex set has only one vertex, say v, and the other bipartite has n vertices, say $v_1,v_2,\ldots\ldots v_n$. A proper coloring of this graph cannot be done with just one color and but can be done with two colors – by assigning one color to v and another color to all of $v_1,v_2,\ldots\ldots v_n$. Thus, the chromatic number of this graph is 2.

If λ colors are available, then the vertex v can be colored in λ ways and each of the vertices $v_1,v_2\ldots\ldots v_n$ can be colored in $\lambda-1$ ways. Therefore, the number of ways of properly coloring the graph is $\lambda(\lambda-1)^n$. This is the chromatic polynomial for the graph.

Example 3:

(a) consider the graph $K_{2,3}$ shown in Figure 2.47. Let λ denote the number of colors available to properly color the vertices of this graph. Find:

(i) how many proper colorings of the graph have vertices a, b colored the same.

(ii) how many proper colorings of the graph have vertices a,b colored with different colors.

(iii) The chromatic polynomial of the graph.

(b) For the graph $K_{2,n}$ what is the chromatic polynomial?

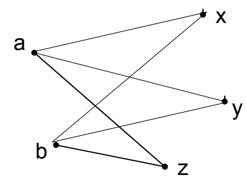


Figure 2.47

(a): (i) If the vertices a and b are to have the same color, then there are λ choices for coloring the vertex a and only one choice for the vertex b (or vice versa). Consequently, there are $\lambda-1$ choices for each of the vertices x,y,z. Hence, the number of proper colorings (in this case) is $\lambda (\lambda-1)^3$

(ii) If the vertices a and b are to have different colors, then there are λ choices for coloring the vertex a and λ-1 choices for the vertex b (or vice versa). Consequently, there are $\lambda-2$ choices for each of the vertices x,y,z. Hence the number of proper colorings (in this case) is $\lambda (\lambda-1) (\lambda-2)^3$.

(iii) Since the two cases of the vertices a and b have the same color or different colors are exhaustive and mutually exclusive, the chromatic polynomial of the graph is

$P(K_{2,3}, \lambda) = \lambda (\lambda-1)^3 + \lambda(\lambda-1) (\lambda-2)^3$.

(b): Let $V_1 = \{a,b\}$ and $V_2 = \{x_1,x_2,x_3,\ldots\ldots x_n\}$ be the two bipartites of $K_{2,n}$. Then, if a and b are to have the same color, the number of proper colorings of $K_{2,n}$ is $\lambda (\lambda-1)^n$ as in case (i) above, If a and b are to have different colors, the number of proper colorings is $\lambda(\lambda-1)(\lambda-2)^n$, as in case (ii) above. Consequently, the chromatic polynomial for $K_{2,n}$ is

$$P(K_{2,n}, \lambda) = \lambda (\lambda-1)^n + \lambda(\lambda-1) (\lambda-2)^n .$$

Example 4: Find the chromatic polynomial for the cycle C_4 of length 4.

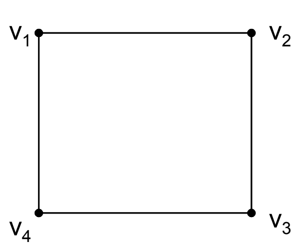


Figure 2.48

A cycle of length 4, namely C_4, is shown in Figure 2.48. Let us redesignate it as G and denote the edge $\{v_2,v_3\}$ as e. Then the graph G_e and G_e' would be as shown below..

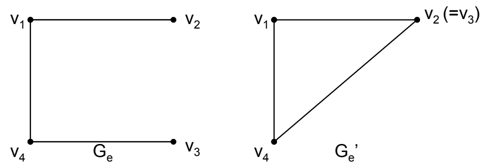


Fig. 2.49

We note that the graph G_e is a path with 4 vertices. Therefore, $P(G_e, \lambda) = \lambda\,(\lambda-1)^3$ Also, the graph G_e' is the graph K_3. Therefore $P(G_e', \lambda) = \lambda(\lambda-1)(\lambda-2)$Accordingly, using the decomposition theorem, we find that

$$P(C_4, \lambda) = P(G,\lambda) = P(G_e, \lambda) - P(G_e', \lambda)$$

$$= \lambda\,(\lambda-1)^3 - \lambda\,(\lambda-1)\,(\lambda-2)$$

$$= \lambda^4 - 4\,\lambda^3 + 6\,\lambda^2 - 3\,\lambda\,.$$

This is the chromatic polynomial for the given cycle.

Example 5: Find the chromatic polynomial for the graph shown below. If 5 colors are available, in how many ways can the vertices of this graph be properly colored?.

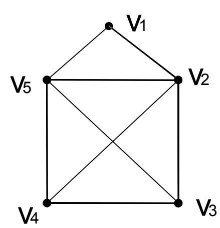


Figure 2.50

Let us denote the given graph by G and the edge $\{v_1,v_2\}$ by e. Then the graph G_e and G_e' would be as shown in Figure 2.51.

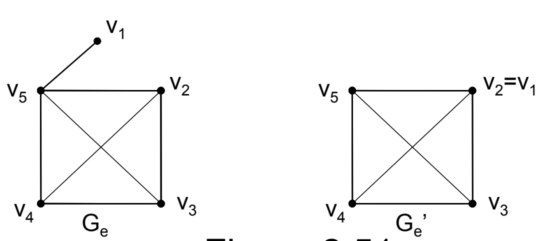


Figure 2.51

Let us redesignate the graph G_e as H and denote the edge $\{v_1,v_5\}$ as f. Then the graph H_f and H_f' would appear as shown below:

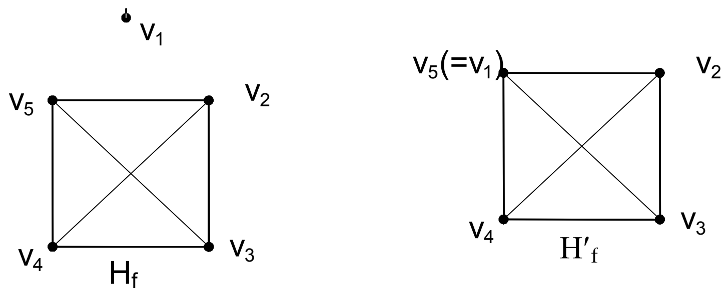


Figure 2.52

Applying the decomposition theorem to the graphs G and H we note that

$P(G, \lambda) = P(G_e,\lambda) - P(G_e', \lambda)$

$= P(H,\lambda) - P(G_e', \lambda)$

$= \{ P(H_f,\lambda) - P(H_f', \lambda)\} - P(G_e', \lambda)$ ------------- (1)

We observe that both of the graphs Ge' and H_f' are the graph K_4 and the graph H_f is a deconnected graph having N_1) - null graph of order 1 consisting of the single vertex v_1) and K_4 as components. Accordingly,

$P(G_e', \lambda) = P(H_f' \lambda) = P(K_4, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$

And $P(H_f, \lambda) = P(N_1, \lambda) . P(K_4, \lambda)$

$= \lambda. \lambda (\lambda-1)(\lambda-2)(\lambda-3).$

Consequently, expression (i) gives

$P(G, \lambda) = \lambda. \lambda (\lambda-1)(\lambda-2)(\lambda-3) - 2 \lambda(\lambda-1)(\lambda-2)(\lambda-3)$

$= \lambda (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-2)$

$= \lambda (\lambda-1)(\lambda-2)^2 (\lambda-3).$

This is the chromatic polynomial for the given graph.

For $\lambda = 5$, this polynomial gives

$P(G, \lambda) = 5 \times 4 \times 3^2 \times 2=360.$

This means that if 5 colors are available, the vertices of the graph can be properly colored in 360 different ways.

Example 6: Use the multiplication theorem to find $P(G, \lambda)$ for the graph shown in Figure (2.50).

The graph G in figure 2.50 can be regarded as the union of the graphs G_1 and G_2 shown in figures 2.53 (a) and 2.53(b) .

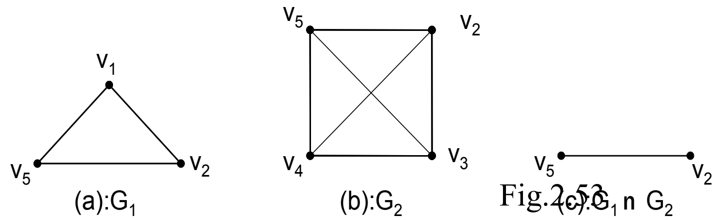


Fig.2.53

Then $G_1 \cap G_2 = \{v_5, v_2\}$ Shown in Figure 2.53 (c).

WE note that G_1 is the same as K_3, G_2 is the same as K_4 and $G_1 \cap G_2$ is the same as K_2. Hence, using the multiplication theorem (Theorem 2), we get

$P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda) / \lambda^{(2)}$.

$= P(K_3, \lambda) \cdot P(K_4, \lambda) / \lambda^{(2)}$

$= \lambda(\lambda-1)(\lambda-2) \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3) / \lambda(\lambda-1)$

$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)$

As the chromatic polynomial for the give G. (This result agrees with the result proved in example 5)

Example 7: Find the chromatic polynomial for the graph shown below:

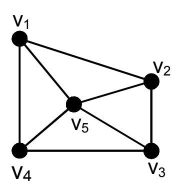


Figure 2.54

Let us denote the given graph by G and the edge $\{v_1, v_5\}$ as e. Then the graph G_e and G_e' would be as shown below.

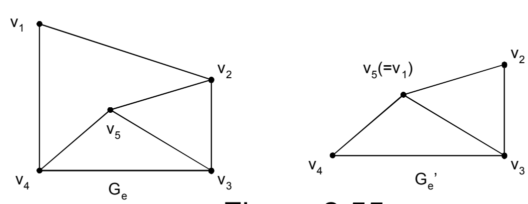


Figure 2.55

Let us redesignate G_e as H and denote the edge $\{v_5, v_2\}$ by f. Then the graphs H_f and H_f' are as shown below.

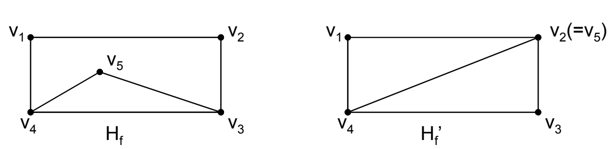


Figure 2.56

Now, we note that H_f' is the union of the cycles $v_1v_4v_2v_1$ and $v_2v_3v_4v_2$ each of which is the same as K_3, and that the intersection of these cycles is the edge $\{v_4,v_2\}$ which is the same as K_2. Therefore, by the multiplication theorem, we have

$P(H_f',\lambda)=P(K_3, \lambda).P(K_3,\lambda)/\lambda^{(2)}$ __________ (i)

Similarly,

$P(G_e',\lambda)=P(K_3,\lambda).P(K_3,\lambda)/\lambda^{(2)}$ __________ (ii)

Next, we note that Hf is the union of the cycles $v_1v_2v_3v_4v_1$ and $v_5v_3v_4v_5$ and that the intersection of these cycles is the edge $\{v_4,v_3\}$. The first of these cycles is C_4, the second cycle is K_3 and the edge $\{v_4,v_3\}$ is K_2. Therefore, by the multiplication theorem, we have

$P(H_f,\lambda)=P(C_4,\lambda).P(K_3,\lambda)/\lambda^{(2)}$ _________ (iii)

Now, by using the decomposition theorem and the fact that $H \equiv G_e$, we get.

$P (G, \lambda) = P (G_e, \lambda) - P(G_e', \lambda)$

$= P (H, \lambda) - P(Ge', \lambda)$

$= P (H_f, \lambda) - P(H_f', \lambda) - P(G_e', \lambda)$

$=1/\lambda^{(2)}\{P(C_4,\lambda).P(K_3,\lambda)-2P(K_3,\lambda)P(K_3,\lambda)\}$,

using (i) – (iii)

$= P(K_3,\lambda)/ \lambda^{(2)} \{ P(C_4,\lambda) - 2P(K_3,\lambda)\}$

Using the result of Example 4 and the expressions for $P(K_3,\lambda)$ & $\lambda^{(2)}$ this becomes

$P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)/\lambda(\lambda-1)\{\lambda\{\lambda(\lambda-1)^3-(\lambda-1)(\lambda-2)\}-2\lambda(\lambda-1)(\lambda-2)\}$

$= \lambda (\lambda-1) (\lambda-2) \{ (\lambda-1)^2 - 3(\lambda-2)\}$

$= \lambda (\lambda-1) (\lambda-2) (\lambda^2 - 5\lambda+7)$.

Example 8: Let G = G(V,E) be a graph with a,b € V but {a,b} = e ∉ E. Let Ge^+ denote the graph obtained by including e into G and Ge^{++} denote the graph obtained by coalescing (merging) the vertices a and b. Prove that

$P(G, \lambda)= P (Ge^+, \lambda) + P (Ge^{++}, \lambda)$

Hence find the chromatic polynomial for the graph shown in figure 2.57.

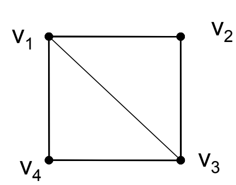


Figure 2.57

Let us redesignate Ge^+ as H. Then, from the definitions of Ge^+ and $Ge^{++,}$ we find that H_e = G and H_e' = $Ge^{++.}$ Now, applying the decomposition theorem to H, we get

$P(H_e, \lambda)= P(H, \lambda) + P(H_e', \lambda)$

This is the same as

$P(G, \lambda)= P(G_e^+, \lambda) + P(G_e^{++}, \lambda)$

Which is the required result.

For the graph G shown in figure 2.57 , if e = {V_2 V_4}, the graphs Ge^+ and Ge^{++} are as shown below:

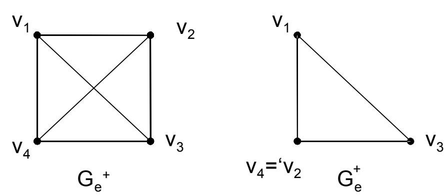


Figure 2.58

We note that Ge^+ is K_4 and Ge^{++} is K_3, Therefore,

$P(Ge^{+,} \lambda)= P(K_4, \lambda) = \lambda (\lambda\text{-}1) (\lambda\text{-}2) (\lambda\text{-}3)$

and $P(Ge^{++}, \lambda)= P(K_3, \lambda) = \lambda (\lambda\text{-}1) (\lambda\text{-}2)$

Accordingly, the chromatic polynomial for the given graph is

$P(G, \lambda)= P(Ge^+, \lambda) + P(Ge^{++,} \lambda)$

$= \lambda (\lambda\text{-}1) (\lambda\text{-}2) (\lambda\text{-}3) + \lambda (\lambda\text{-}1) (\lambda\text{-}2)$

$= \lambda (\lambda\text{-}1) (\lambda\text{-}2)^2$.

Example 9: Prove the following:

(a) for any graph G, the constant term in P(G, λ) is zero.

(b) For any graph G = G(V,E) with \E\ ≥ 1, the sum of the coefficients in P(G, λ) is zero.

Solution:

Let $P(G, \lambda)= a_0+a_1 \lambda+ a_2 \lambda_2+\ldots+ a_r \lambda_{r.}$ Then

$P(G,0)= a_0$ & $P(G,1)= a_0+a_1+ a_2+\ldots\ldots+ a_r$.

(a)For any graph G, P(G,0) represents the number of ways of properly coloring G with zero number of colors. Since a graph cannot be colored with no color on hand, it follows that P(G,0) = 0: that is a0 = 0.

(b) For any graph G, P(G,1) represents the number of ways of properly coloring G with 1 color. If G has at least one edge, G cannot be properly colored with 1 color. This means that, for G = G((V,E) with \E\ ≥ 1, we have

P(G,1)= 0, that is, $a_0+a_1+ a_2+\ldots\ldots\ldots+ a_r= 0$.

Exercises

01. Determine the chromatic polynomials for the graphs shown below:.

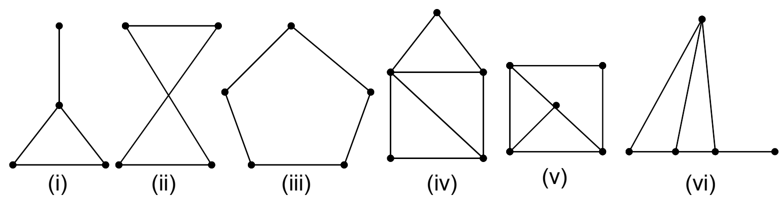


Figure 2.59

Ans 1. $\lambda (\lambda-1)^2(\lambda-2)$.

Ans 2. $\lambda (\lambda-1)^2 (\lambda-2)^2$.

Ans 3. $\lambda (\lambda-1) (\lambda-2) (\lambda^2 - 2 \lambda+2)$.

Ans 4. $\lambda (\lambda-1) (\lambda-2)^3$

Ans 5. $\lambda (\lambda-1) (\lambda-2) (2 \lambda-5)$.

Ans 6. $\lambda (\lambda-1)^2 (\lambda-2)^2$.

02. If 4 colors are available, in how many different ways can the vertices of each graph in Figure 2.59 be properly colored?

Ans: (i) 72 (ii) 144 (iii) 240 (iv) 96 (v) 72 (vi) 144

03. For n≥ 3,Let G_n be the graph obtained by deleting one edge from K_n. Determine $P(G_n, \lambda)$ and $\chi(G_n)$.

04. If C_n denotes a cycle of length n≥ 3, prove that $P(C_n, \lambda) = (\lambda-1)_n+(-1)n(\lambda-1)$

05. If C_n denotes a cycle of length n ≥ 4 , prove that $P(Cn, \lambda) + P(Cn-1, \lambda) = \lambda(\lambda-1)^{n-1}$

UNIT 3

TREES

Graphs

- Graph consists of two sets: set V of vertices and set E of edges.
- Terminology: endpoints of the edge, loop edges, parallel edges, adjacent vertices, isolated vertex, subgraph, bridge edge
- Directed graph (digraph) has each edge as an ordered pair of vertices

Special Graphs

- Simple graph is a graph without loop or parallel edges. A complete graph of n vertices K_n is a simple graph which has an edge between each pair of vertices. A complete bipartite graph of (n, m) vertices $K_{n,m}$ is a simple graph consisting of vertices, $v_1, v_2, \dots, v_m$ and $w_1, w_2, \dots, w_n$ with the following properties:
  - There is an edge from each vertex v_i to each vertex w_j
  - There is no edge from any vertex v_i to any vertex v_j
  - There is no edge from any vertex w_i to any vertex w_j

The Concept of Degree

- The degree of a vertex deg(v) is a number of edges that have vertex v as an endpoint. Loop edge gives vertex a degree of 2. In any graph the sum of degrees of all vertices equals twice the number of edges. The total degree of a graph is even. In any graph there are even number of vertices of odd degree

Paths and Circuits

- A walk in a graph is an alternating sequence of adjacent vertices and edges. A path is a walk that does not contain a repeated edge. Simple path is a path that does not contain a repeated vertex. A closed walk is a walk that starts and ends at the same vertex. A circuit is a closed walk that does not contain a repeated edge. A simple circuit is a circuit which does not have a repeated vertex except for the first and last

Connectedness

- Two vertices of a graph are connected when there is a walk between two of them. The graph is called connected when any pair of its vertices is connected. If graph is connected, then any two vertices can be connected by a simple path. If two vertices are part of a circuit and one edge is removed from the circuit then there still exists a path between these two vertices. Graph H is called a connected component of graph G when H is a subgraph of G, H is connected and H is not a subgraph of any bigger connected graph. Any graph is a union of connected components

Euler Circuit

- Euler circuit is a circuit that contains every vertex and every edge of a graph. Every edge is traversed exactly once. If a graph has Euler circuit then every vertex has even degree. If some vertex of a graph has odd degree then the graph does not have an Euler circuit. If every vertex of a graph has even degree and the graph is connected then the graph has an Euler circuit. A Euler path is a path between two vertices that contains all vertices and traverces all edge exactly ones. There is an Euler path between two vertices v and w iff vertices v and w have odd degrees and all other vertices have even degrees

Hamiltonian Circuit

Hamiltonian circuit is a simple circuit that contains all vertices of the graph (and each exactly once). Example: Traveling salesperson problem

Trees

- Connected graph without circuits is called a tree. Graph is called a forest when it does not have circuits. A vertex of degree 1 is called a terminal vertex or a leaf, the other vertices are called internal nodes. Examples: Decision tree, Syntactic derivation tree.
- Any tree with more than one vertex has at least one vertex of degree 1. Any tree with n vertices has n – 1 edges. If a connected graph with n vertices has n – 1 edges, then it is a tree

Rooted Trees

- Rooted tree is a tree in which one vertex is distinguished and called a root. Level of a vertex is the number of edges between the vertex and the root. The height of a rooted tree is the maximum level of any vertex. Children, siblings and parent vertices in a rooted tree. Ancestor, descendant relationship between vertices

Binary Trees

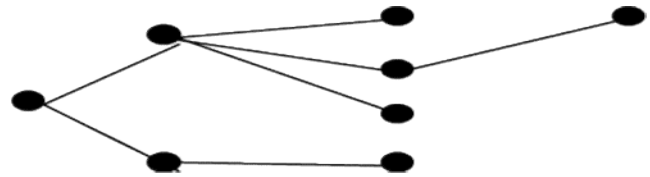
- Binary tree is a rooted tree where each internal vertex has at most two children: left and right. Left and right subtrees.
- Full binary tree: Representation of algebraic expressions
- If T is a full binary tree with k internal vertices then T has a total of 2k + 1 vertices and k + 1 of them are leaves. Any binary tree with t leaves and height h satisfies the following inequality: $t \leq 2^h$

Spanning Trees

- A subgraph T of a graph G is called a spanning tree when T is a tree and contains all vertices of G. Every connected graph has a spanning tree. Any two spanning trees have the same number of edges. A weighted graph is a graph in which each edge has an associated real number weight. A minimal spanning tree (MST) is a spanning tree with the least total weight of its edges.

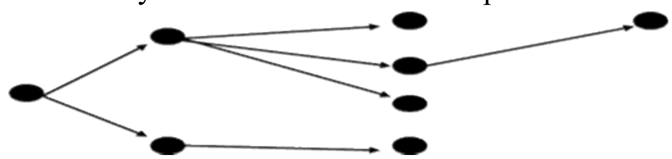
Trees: Definition & Applications

A tree is a connected graph with no cycles. A forest is a graph whose components are trees. An example appears below. Trees come up in many contexts: tournament brackets, family trees, organizational charts, and decision trees, being a few examples.



Directed Trees

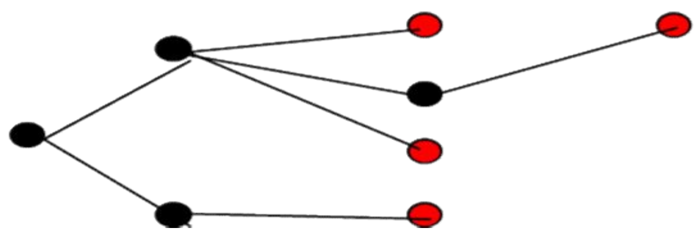
A directed tree is a digraph whose underlying graph is a tree and which has no loops and no pairs of vertices joined in both directions. These last two conditions mean that if we interpret a directed tree as a relation, it is irreflexive and asymmetric. Here is an example.



Theorem: A tree T(V,E) with finite vertex set and at least one edge has at least two leaves (a leaf is a vertex with degree one). Proof: Fix a vertex a that is the endpoint of some edge. Move from a to the adjacent vertex along the edge. If that vertex has no adjacent vertices then it has degree one, so stop. If not, move along another edge to another vertex. Continue building a path in this fashion until you reach a vertex with no adjacent vertices besides the one you just came from. This is sure to happen because V is finite and you never use the same vertex twice in the path (since T is a tree). This produces one leaf. Now return to a. If it is a leaf, then you are done. If not, move along a different edge than the one at the first step above. Continue extending the path in that direction until you reach a leaf (which is sure to happen by the argument above).

Trees: Leaves & Internal Vertices

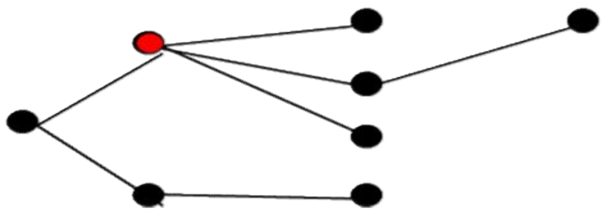
In the following tree the red vertices are leaves. We now know every finite tree with an edge has a least two leaves. The other vertices are internal vertices.

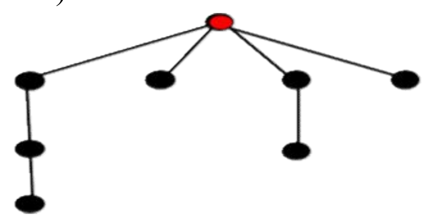


- Theorem: Given vertices a and b in a tree T(V,E), there is a unique simple path from a to b. Proof: Trees are connected, so there is a simple path from a to b. The book gives a nice example of using the contrapositive to prove the rest of the theorem.
- Theorem: Given a graph G(V,E) such that every pair of vertices is joined by a unique simple path, then G is a tree. This is the converse of Theorem 6.37. Proof: Since a simple path joins every pair of points, the graph is connected. Now suppose G has a cycle abc…a. Then ba and bc…a are distinct simple paths from b to a. This contradicts uniqueness of simple paths, so G cannot possess such a cycle. This makes G a tree.

Rooted Trees

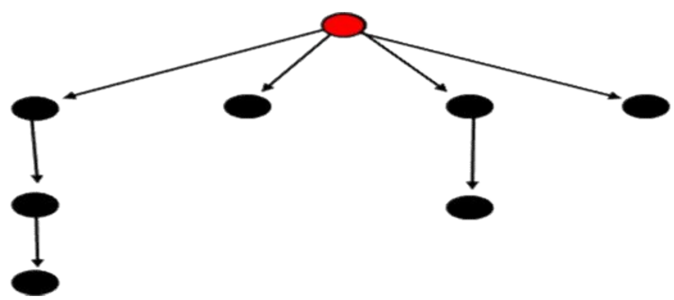
Sometimes it is useful to distinguish one vertex of a tree and call it the root of the tree. For instance we might, for whatever reasons, take the tree above and declare the red vertex to be its root. In that case we often redraw the tree to let it all “hang down” from the root (or invert this picture so that it all “grows up” from the root, which suits the metaphor better)





Rooted Directed Trees

It is sometimes useful to turn a rooted tree into a rooted directed tree T′ by directing every edge away from the root.



Rooted trees and their derived rooted directed trees have some useful terminology, much of which is suggested by family trees. The level of a vertex is the length of the path from it to the root. The height of the tree is the length of the longest path from a leaf to the root. If there is a directed edge in T′ from a to b, then a is the parent of b and b is a child of a. If there are directed edges in T′ from a to b and c, then b and c are siblings. If there is a directed path from a to b, then a is an ancestor of b and b is a descendant of a.

Binary & m-ary Trees

We describe a directed tree as binary if no vertex has outdegree over 2. It is more common to call a tree binary if no vertex has degree over 3. (In general a tree is m-ary if no vertex has degree over m+1. Our book calls a directed tree m-ary if no vertex has outdegree over m.) The directed rooted tree above is 4-ary (I think the word is quaternary) since it has a vertex with outdegree 4. In a rooted binary tree (hanging down or growing up) one can describe each child vertex as the left child or right child of its parent.

Trees: Edges in a Tree

Theorem: A tree on n vertices has n–1 edges. Proof: Let T be a tree with n vertices. Make it rooted. Then every edge establishes a parent-child relationship between two vertices. Every child has exactly one parent, and every vertex except the root is a child. Therefore there is exactly one edge for each vertex but one. This means there are n–1 edges.

Theorem: If G(V,E) is a connected graph with n vertices and n–1 edges is a tree.
Proof: Suppose G is as in the statement of the theorem, and suppose G has a cycle. Then we can remove an edge from the cycle without disconnecting G (see the next slide for why). If this makes G a tree, then stop. If not, there is still a cycle, so we can remove another edge without disconnecting G. Continue the process until the remaining graph is a tree. It still has n vertices, so it has n–1 edges by a prior theorem. This is a contradiction since G had n–1 vertices to start with. Therefore G has no cycle and is thus a tree.

(Why can we remove an edge from a cycle without disconnecting the graph? Let a and b be vertices. There is a simple path from a to b. If the path involves no edges in the cycle, then the path from a to be is unchanged. If it involves edges in the cycle, let x and y be the first and last vertices in the cycle that are part of the path from a to b. So there is a path from a to x and a path from y to b. Since x and y are part of a cycle, there are at least simple two paths from x to y. If we remove an edge from the cycle, at least one of the paths still remains. Thus there is still a simple path from a to b.)

Important Concepts, Formulas, and t heorems

1. Graph. A graph consists of a set of vertices and a set of edges with the property that each edge has two (not necessarily different) vertices associated with it and called its endpoints.

2. Edge; Adjacent. We say an edge in a graph joins its endpoints, and we say two endpoints are adjacent if they are joined by an edge.

3. Incident. When a vertex is an endpoint of an edge, we say the edge and the vertex are incident.

4. Drawing of a Graph. To draw a graph, we draw a point in the plane for each vertex, and then for each edge we draw a (possibly curved) line between the points that correspond to the endpoints of the edge. Lines that correspond to edges may only touch the vertices that are their endpoints.

5. Simple Graph. A simple graph is one that has at most one edge joining each pair of distinct vertices, and no edges joining a vertex to itself.

6. Length, Distance. The length of a path is the number of edges. The distance between two vertices in a graph is the length of a shortest path between them.

7. Loop; Multiple Edges. An edge that joins a vertex to itself is called a loop and we say we have multiple edges between vertices x and y if there is more than one edge joining x and y.

8. Notation for a Graph. We use the phrase "Let $G = (V, E)$" as a shorthand for "Let G stand for a graph with vertex set V and edge set E."

9. Notation for Edges. In a simple graph we use the notation $\{x, y\}$ for an edge from x to y. In any graph, when we want to use a letter to denote an edge we use a Greek letter like 6 so that we can save e to stand for the number of edges of the graph.

10. Complete Graph on fl vertices. A complete graph on fl vertices is a graph with fl vertices that has an edge between each two of the vertices. We use K_m to stand for a complete graph on fl vertices.

11. Path. We call an alternating sequence of vertices and edges in a graph a path if it starts and ends with a vertex, and each edge joins the vertex before it in the sequence to the vertex after it in the sequence.

12. Simple Path. A path is called a simple path if it has no repeated vertices or edges.

13. Degree of a Vertex. The degree of a vertex in a graph is the number of times it is incident with edges of the graph; that is, the degree of a vertex x is the number of edges from x to other vertices plus twice the number of loops at vertex x.

14. Sum of Degrees of Vertices. The sum of the degrees of the vertices in a graph with a finite number of edges is twice the number of edges.

15. Connected. A graph is called connected if there is a path between each two vertices of the graph. We say two vertices are connected if there is a path between them, so a graph is connected if each two of its vertices are connected. The relationship of being connected is an equivalence relation on the vertices of a graph.

16. Connected Component. If C is a subset of the vertex set of a graph, we use $E(C)$ to stand for the set of all edges both of whose endpoints are in C. The graph consisting of an equivalence class C of the connectivity relation together with the edges $E(C)$ is called a connected component of our original graph.

17. Closed Path. A path that starts and ends at the same vertex is called a closed path.

18. Cycle. A closed path with at least one edge is called a cycle if, except for the last vertex, all of its vertices are different.

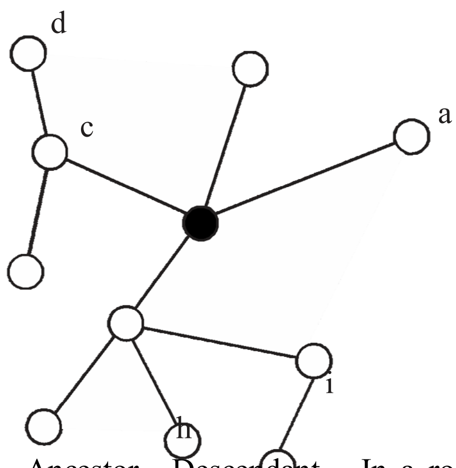
19. Tree. A connected graph with no cycles is called a tree.

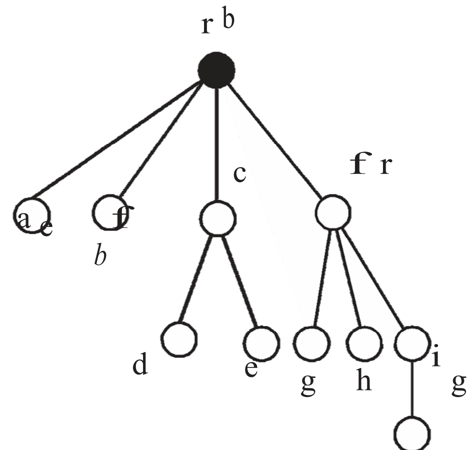
20. Important Properties of Trees.

    (a) There is a unique path between each two vertices in a tree. (b) A tree on V vertices has V — 1 edges.
    (c) Every finite tree with at least two vertices has a vertex of degree one.

Rooted trees

A rooted tree consists of a tree with a selected vertex, called a root, in the tree.





1. Ancestor, Descendant. In a rooted tree with root r, a vertex x is an ancestor of a vertex y, and vertex y is a descendant of vertex x if x and y are different and x is on the unique path from the root to y.

2. Parent, Child. In a rooted tree with root r, vertex x is a parent of vertex y and y is a child of vertex x in if x is the unique vertex adjacent to y on the unique path from r to y.

3. Leaf (External) Vertex. A vertex with no children in a rooted tree is called a leaf vertex or an external vertex.

4. Internal Vertex. A vertex of a rooted tree that is not a leaf vertex is called an internal vertex.

5. Binary Tree. We recursively describe a binary tree as

    • an empty tree (a tree with no vertices), or

    • a structure T consisting of a root vertex, a binary tree called the left subtree of the root and a binary tree called the right subtree of the root. If the left or right subtree is nonempty, its root node is joined by an edge to the root of T.

6. Full Binary Tree. A binary tree is a full binary tree if each vertex has either two nonempty children or two empty children.

7. Recursive Definition of a Rooted Tree. The recursive definition of a rooted tree states that it is either a single vertex, called a root, or a graph consisting of a vertex called a root and a set of disjoint rooted trees, each of which has its root attached by an edge to the original root.

Traversal Algorithms

A **traversal algorithm** is a procedure for systematically visiting every vertex of an ordered binary tree

• Tree traversals are defined recursively

• Three commonly used traversals are:

– **preorder**

– **inorder**

– **postorder**

PREORDER Traversal Algorithm

Let T be an ordered binary tree with root R

If T has only R then

R is the preorder traversal

Else

Let T1, T2 be the left and right subtrees at R

Visit R

Traverse T1 in preorder

Traverse T2 in preorder

INORDER Traversal Algorithm

Let T be an ordered binary tree with root R

If T has only R then

R is the inorder traversal

Else

Let T1, T2 be the left and right subtrees at R

Traverse T1 in inorder

Visit R

Traverse T2 in inorder

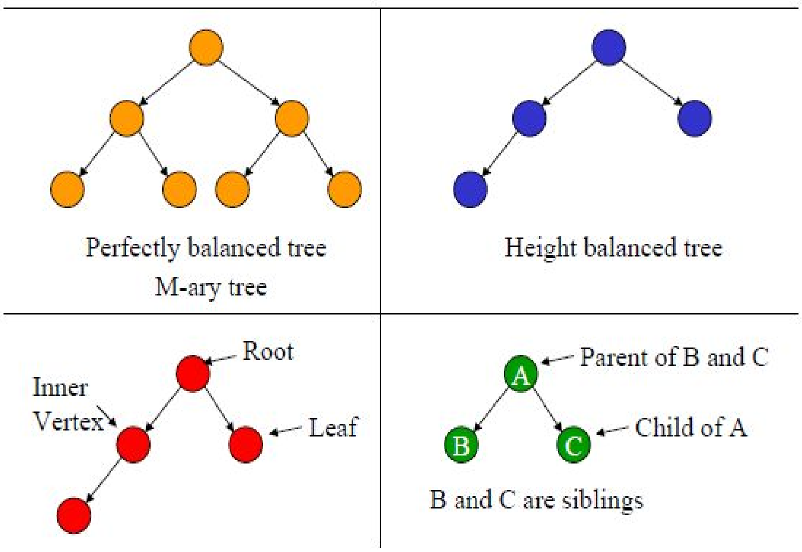
POSTORDER Traversal Algorithm

Let T be an ordered binary tree with root R

If T has only R then

R is the postorder traversal

Else
Let T1, T2 be the left and right subtrees at R
Traverse T1 in postorder
Traverse T2 in postorder
Visit R



Constructing an Optimal Huffman Code

An optimal Huffman code is a Huffman code in which the average length of the symbols is minimum. In general an optimal Huffman code can be made as follows. First we list the frequencies of all the codes and represent the symbols as vertices (which at the end will be leaves of a tree). Then we replace the two smallest frequencies f_1 and f_2 with their sum $f_1 + f_2$, and join the corresponding two symbols to a common vertex above them by two edges, one labeled 0 and the other one labeled 1. Than common vertex plays the role of a new symbol with a frequency equal to $f_1 + f_2$. Then we repeat the same operation with the resulting shorter list of frequencies until the list is reduced to one element and the graph obtained becomes a tree.

Spanning Trees of a Graph

If G(V,E) is a graph and T(V,F) is a subgraph of G and is a tree, then T is a spanning tree of G. That is, T is a tree that includes every vertex of G and has only edges to be found in G. Using the procedure in the previous paragraph (remove edges from cycles until only a tree remains), we can easily prove that every connected graph has a spanning tree.

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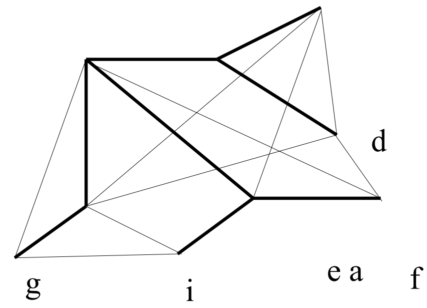


FIGURE : Spanning tree.

Every connected graph has a spanning tree which can be obtained by removing edges until the resulting graph becomes acyclic. In prac- tice, however, removing edges is not efficient because finding cycles is time consuming.

Next, we give two algorithms to find the spanning tree T of a loop- free connected undirected graph G — (V, E). We assume that the vertices of G are given in a certain order $v_1, v_2, \ldots, v_m$. The resulting spanning tree will be $T — (V^!, E^!)$.

Breadth-First Search Algorithm

The idea is to start with vertex v_1 as root, add the vertices that are adjacent to v_1, then the ones that are adjacent to the latter and have not been visited yet, and so on. This algorithm uses a queue (initially empty) to store vertices of the graph. In consists of the following:

1. Add v_1 to T, insert it in the queue and mark it as "visited".
2. If the queue is empty, then we are done. Otherwise let v be the vertex in the front of the queue.
3. For each vertex $v^!$ of G that has not been visited yet and is adjacent to v (there might be none) taken in order of increasing subscripts, add vertex $v^!$ and edge $(v, v^!)$ to T, insert $v^!$ in the queue and mark it as "visited".
4. Delete v from the queue. 5. Go to tep22

A pseudocode version of the algorithm is as follows:

```
procedure bfs(V,E)
    S := (v1)  {ordered list of vertices of a fix level}
    V' := {v1}  {v1 is the root of the spanning tree}
    E' := { }  {no edges in the spanning tree yet}
```

```
while true
  begin
    for each x in S, in order,
      for each y in V-V'
        if (x,y) is an edge then
          add edge (x,y) to E' and vertex y to V'
      if no edges were added then
        return T
    S := children of S
  end
end bfs
```

Figure below shows the spanning tree obtained using the breadth-first search algorithm on the graph with its vertices ordered lexicographi- cally: a, b, c, d, e, f, g, h, i.

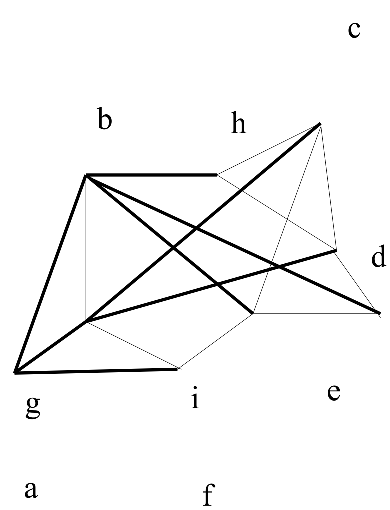


FIGURE Breadth-First Search.

Depth-First Search Algorithm

The idea of this algo- rithm is to make a path as long as possible, and then go back (back- track) to add branches also as long as possible.

This algorithm uses a stack (initially empty) to store vertices of the graph. In consists of the following:

1. Add v_1 to T, insert it in the stack and mark it as "visited".

2. If the stack is empty, then we are done. Otherwise let v be the vertex on the top of the stack.
3. If there is no vertex $v^{!}$ that is adjacent to v and has not been visited yet, then delete v and go to step 2 (backtrack). Oth- erwise, let $v^{!}$ be the first non-visited vertex that is adjacent to v.
4. Add vertex $v^{!}$ and edge $(v, v^{!})$ to T, insert $v^{!}$ in the stack and mark it as "visited".
5. Go to step 2.

An alternative recursive definition is as follows. We define recur- sively a process P applied to a given vertex v in the following way:

1. Add vertex v to T and mark it as "visited".
2. If there is no vertex $v^{!}$ that is adjacent to v and has not been visited yet, then return. Otherwise, let $v^{!}$ be the first non-visited vertex that is adjacent to v.
3. Add the edge $(v, v^{!})$ to T.
4. Apply P to $v^{!}$.
5. Go to step 2 (backtrack).

The Depth-First Search Algorithm consists of applying the process just defined to v_1.

A pseudocode version of the algorithm is as follows:

```
procedure dfs(V,E)
    V' := {v1}  {v1 is the root of the spanning tree}
    E' := { }  {no edges in the spanning tree yet}
    w := v1
    while true
      begin
        while there is an edge (w,v) that when added
                    to T does not create a cycle in T
          begin
            Choose first v such that (w,v)
            does not create a cycle in T
```

```
            add (w,v) to E'
            add v to V'
            w := v
          end
        if w=v1 then

          return T
        w := parent of w in T {backtrack}
      end
end
```

Figure shows the spanning tree obtained using the breadth-first search algorithm on the graph with its vertices ordered lexicographi- cally: a, b, c, d, e, f, g, h, i.

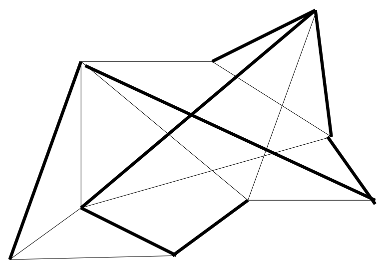


FIGURE Depth-First Search.

UNIT 4

Minimum spanning trees

Let G = (V,E) be a connected **graph** and let l : E −→ R be a function, called the length function. For any subset F of E, the length l(F) of F is, by definition:
$l(F) := \sum_{e \in F} l(e)$.

In this section we consider the problem of finding a spanning tree in G of minimum length. There is an easy algorithm for finding a minimum-length spanning tree, essentially due toBoruvka . There are a few variants. The first one we discuss is sometimes called the Dijkstra-Prim method.

Choose a vertex $v_1 \in V$ arbitrarily. Determine edges e_1, e_2 ... successively as follows. Let $U_1 := \{v_1\}$. Suppose that, for some $k \geq 0$, edges $e_1,\ldots,e_k$ have been chosen, spanning a tree on the set U_k. Choose an edge $e_{k+1} \in \delta(U_k)$ that has minimum length among all edges in $\delta(U_k)$.1 Let $U_{k+1} := U_k \cup e_{k+1}$.

By the connectedness of G we know that we can continue choosing such an edge until $U_k = V$. In that case the selected edges form a spanning tree T in G. This tree has minimum length, which can be seen as follows.

Call a forest F greedy if there exists a minimum-length spanning tree T of G that contains F.

Theorem: Let F be a greedy forest, let U be one of its components, and let $e \in \delta(U)$. If e has minimum length among all edges in $\delta(U)$, then $F \cup \{e\}$ is again a greedy forest.
Proof. Let T be a minimum-length spanning tree containing F. Let P be the unique path in T between the end vertices of e. Then P contains at least one edge f that belongs to $\delta(U)$. So $T := (T \setminus \{f\}) \cup \{e\}$ is a tree again. By assumption, $l(e) \leq l(f)$ and hence $l(T) \leq l(T)$. Therefore, T is a minimum-length spanning tree. As $F \cup \{e\} \subseteq T$, it follows that $F \cup \{e\}$ is greedy.

Corollary : The Dijkstra-Prim method yields a spanning tree of minimum length.
Proof. It follows inductively with Theorem above that at each stage of the algorithm we have a greedy forest. Hence the final tree is greedy — equivalently, it has minimum length.
The Dijkstra-Prim method is an example of a so-called greedy algorithm. We construct a spanning tree by throughout choosing an edge that seems the best at the moment. Finally we get a minimum-length spanning tree. Once an edge has been chosen, we never have to replace it by another edge (no 'back-tracking').There is a slightly different method of finding a minimum-length spanning tree, Kruskal'smethod . It is again a greedy algorithm, and again iteratively edges e1,e2,... are chosen, but by some different rule

Dijkstra's algorithm

Dijkstra's algorithm, conceived by Dutch computer scientist Edsger Dijkstra in 1956 and published in 1959, is a graph search algorithm that solves the single-source shortest path problem for a graph with nonnegative edge path costs, producing a shortest path tree. This algorithm is often used in routing. An equivalent algorithm was developed by Edward F. Moore in 1957.

For a given source vertex (node) in the graph, the algorithm finds the path with lowest cost (i.e. the shortest path) between that vertex and every other vertex. It can also be used for finding costs of shortest paths from a single vertex to a single destination vertex by stopping the algorithm once the shortest path to the destination vertex has been determined. For example, if the vertices of the graph represent cities and edge path costs represent driving distances between pairs of cities connected by a direct road, Dijkstra's algorithm can be used to find the shortest route between one city and all other cities. As a result, the shortest path first is widely used in network routing protocols, most notably IS-IS and OSPF (Open Shortest Path First).

Algorithm

Let the node at which we are starting be called the **initial node**. Let the **distance of node Y** be the distance from the **initial node** to Y. Dijkstra's algorithm will assign some initial distance values and will try to improve them step by step.

1. Assign to every node a distance value: set it to zero for our initial node and to infinity for all other nodes.
2. Mark all nodes as unvisited. Set initial node as current.
3. For current node, consider all its unvisited neighbors and calculate their tentative distance (from the initial node). For example, if current node (A) has distance of 6, and an edge connecting it with another node (B) is 2, the distance to B through A will be 6+2=8. If this distance is less than the previously recorded distance (infinity in the beginning, zero for the initial node), overwrite the distance.
4. When we are done considering all neighbors of the current node, mark it as visited. A visited node will not be checked ever again; its distance recorded now is final and minimal.
5. If all nodes have been visited, finish. Otherwise, set the unvisited node with the smallest distance (from the initial node, considering all nodes in graph) as the next "current node" and continue from step 3.

Description

Note: For ease of understanding, this discussion uses the terms **intersection**, **road** and **map** — however, formally these terms are **vertex**, **edge** and **graph**, respectively.

Suppose you want to find the shortest path between two intersections on a city map, a starting point and a destination. The order is conceptually simple: to start, mark the distance to every intersection on the map with infinity. This is done not to imply there is an infinite distance, but to note that that intersection has not yet been visited. (Some variants of this method simply leave the intersection unlabeled.) Now, at each iteration, select a current intersection. For the first iteration the current intersection will be the starting point and the distance to it (the intersection's label) will be zero. For subsequent iterations (after the first) the current intersection will be the closest unvisited intersection to the starting point—this will be easy to find.

From the current intersection, update the distance to every unvisited intersection that is directly connected to it. This is done by determining the sum of the distance between an unvisited intersection and the value of the current intersection, and relabeling the unvisited intersection with this value if it is less than its current value. In effect, the intersection is relabeled if the path to it through the current intersection is shorter than the previously known paths. To facilitate shortest path identification, in pencil, mark the road with an arrow pointing to the relabeled intersection if you label/relabel it, and erase all others pointing to it. After you have updated the distances to each neighboring intersection, mark the current intersection as visited and select the unvisited intersection with lowest distance (from the starting point) -- or lowest label—as the current intersection. Nodes marked as visited are labeled with the shortest path from the starting point to it and will not be revisited or returned to.

Continue this process of updating the neighboring intersections with the shortest distances, then marking the current intersection as visited and moving onto the closest unvisited intersection until you have marked the destination as visited. Once you have marked the destination as visited (as is the case with any visited intersection) you have determined the shortest path to it, from the starting point, and can trace your way back, following the arrows in reverse.

In the accompanying animated graphic, the starting and destination intersections are colored in light pink and blue and labelled a and b respectively. The visited intersections are colored in red, and the current intersection in a pale blue.

Of note is the fact that this algorithm makes no attempt to direct "exploration" towards the destination as one might expect. Rather, the sole consideration in determining the next "current" intersection is its distance from the starting point. In some sense, this algorithm "expands outward" from the starting point, iteratively considering every node that is closer in terms of shortest path distance until it reaches the destination. When understood in this way, it is clear how the algorithm necessarily finds the shortest path, however it may also reveal one of the algorithm's weaknesses: its relative slowness in some topologies.

Pseudocode

In the following algorithm, the code u := vertex in Q with smallest dist[], searches for the vertex *u* in the vertex set *Q* that has the least dist[*u*] value. That vertex is removed from the set *Q* and returned to the user. dist_between(*u*, *v*) calculates the length between the two neighbor-nodes *u* and *v*. The variable *alt* on line 15 is the length of the path from the root node to the neighbor node *v* if it were to go through *u*. If this path is shorter than the current shortest path recorded for *v*, that current path is replaced with this *alt* path. The previous array is populated with a pointer to the "next-hop" node on the source graph to get the shortest route to the source.

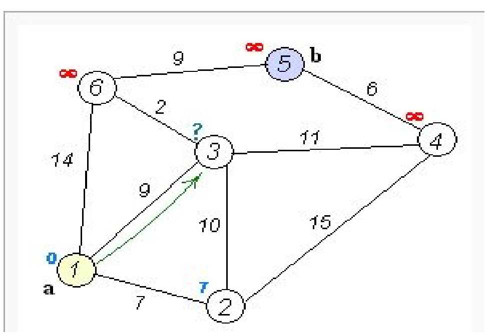
```
function Dijkstra(Graph, source):
    for each vertex v in Graph:          // Initializations
        dist[v] := infinity ;            // Unknown distance function from source to v
        previous[v] := undefined ;       // Previous node in optimal path from source
    end for ;
    dist[source] := 0 ;                  // Distance from source to source
    Q := the set of all nodes in Graph ;
    // All nodes in the graph are unoptimized - thus are in Q
    while Q is not empty:                // The main loop
        u := vertex in Q with smallest dist[] ;
        if dist[u] = infinity:
            break ;                      // all remaining vertices are inaccessible from source
        fi ;
        remove u from Q ;
        for each neighbor v of u:        // where v has not yet been removed from Q.
            alt := dist[u] + dist_between(u, v) ;
            if alt < dist[v]:            // Relax (u,v,a)
                dist[v] := alt ;
                previous[v] := u ;
            fi  ;
        end for ;
    end while ;
    return dist[] ;
end Dijkstra.
```

If we are only interested in a shortest path between vertices *source* and *target*, we can terminate the search at line 13 if *u* = *target*. Now we can read the shortest path from *source* to *target* by iteration:

```
S := empty sequence
u := target
while previous[u] is defined:
    insert u at the beginning of S
    u := previous[u]
```

Now sequence *S* is the list of vertices constituting one of the shortest paths from *target* to *source*, or the empty sequence if no path exists.

A more general problem would be to find all the shortest paths between *source* and *target* (there might be several different ones of the same length). Then instead of storing only a single node in each entry of previous[] we would store all nodes satisfying the relaxation condition. For example, if both *r* and *source* connect to *target* and both of them lie on different shortest paths through *target* (because the edge cost is the same in both cases), then we would add both *r* and *source* to previous[*target*]. When the algorithm completes, previous[] data structure will actually describe a graph that is a subset of the original graph with some edges removed. Its key property will be that if the algorithm was run with some starting node, then every path from that node to any other node in the new graph will be the shortest path between those nodes in the original graph, and all paths of that length from the original graph will be present in the new graph. Then to actually find all these short paths between two given nodes we would use a path finding algorithm on the new graph, such as depth-first search.



Kruskal's algorithm

Kruskal's algorithm is an algorithm in graph theory that finds a minimum spanning tree for a connected weighted graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a minimum spanning forest (a minimum spanning tree for each connected component). Kruskal's algorithm is an example of a greedy algorithm.

Description

- create a forest F (a set of trees), where each vertex in the graph is a separate tree
- create a set S containing all the edges in the graph
- while S is nonempty and F is not yet spanning
  - remove an edge with minimum weight from S
  - if that edge connects two different trees, then add it to the forest, combining two trees into a single tree
  - otherwise discard that edge.

At the termination of the algorithm, the forest has only one component and forms a minimum spanning tree of the graph.

Performance

Where E is the number of edges in the graph and V is the number of vertices, Kruskal's algorithm can be shown to run in O(E log E) time, or equivalently, O(E log V) time, all with simple data structures. These running times are equivalent because:

- E is at most V^2 and $\log V^2 = 2\log V$ is O(log V).
- If we ignore isolated vertices, which will each be their own component of the minimum spanning forest, $V \leq E+1$, so log V is O(log E).

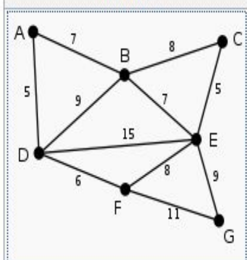
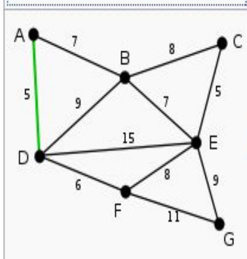
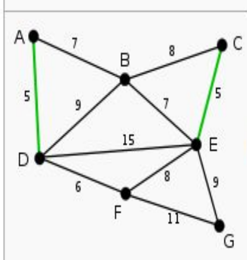
We can achieve this bound as follows: first sort the edges by weight using a comparison sort in O(E log E) time; this allows the step "remove an edge with minimum weight from S" to operate in constant time. Next, we use a disjoint-set data structure (Union&Find) to keep track of which vertices are in which components. We need to perform O(E) operations, two 'find' operations and possibly one union for each edge. Even a simple disjoint-set data structure such as disjoint-set forests with union by rank can perform O(E) operations in O(E log V) time. Thus the total time is O(E log E) = O(E log V).

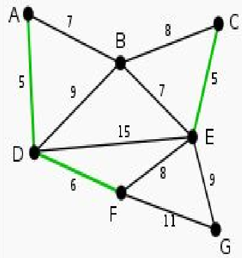
Provided that the edges are either already sorted or can be sorted in linear time (for example with counting sort or radix sort), the algorithm can use more sophisticated disjoint-set data structure to run in O(E α(V)) time, where α is the extremely slowly-growing inverse of the single-valued Ackermann function.

Pseudocode

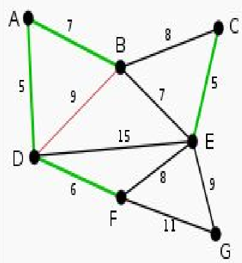
```
function Kruskal(G = <N, A>: graph; length: A → R+): set of edges
  Define an elementary cluster C(v) ← {v}.
  Initialize a priority queue Q to contain all edges in G, using the weights as keys.
  Define a forest T ← Ø       //T will ultimately contain the edges of the MST
   // n is total number of vertices
  while T has fewer than n-1 edges do
```

```
// edge u,v is the minimum weighted route from u to v
(u,v) ← Q.removeMin()
// prevent cycles in T. add u,v only if T does not already contain a path between u and v.
 // the vertices has been added to the tree.
 Let C(v) be the cluster containing v, and let C(u) be the cluster containing u.
 if C(v) ≠ C(u) then
   Add edge (v,u) to T.
   Merge C(v) and C(u) into one cluster, that is, union C(v) and C(u).
return tree T
```

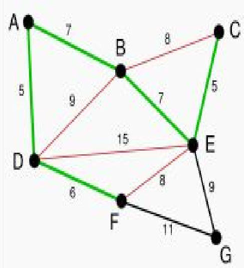
<table>
<tr><th>Image</th><th>Description</th></tr>
<tr><td>
</td><td>This is our original graph. The numbers near the arcs indicate their weight. None of the arcs are highlighted.</td></tr>
<tr><td>
</td><td>AD and CE are the shortest arcs, with length 5, and AD has been arbitrarily chosen, so it is highlighted.</td></tr>
<tr><td>
</td><td>CE is now the shortest arc that does not form a cycle, with length 5, so it is highlighted as the second arc.</td></tr>
</table>



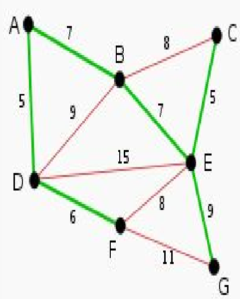
The next arc, **DF** with length 6, is highlighted using much the same method.



The next-shortest arcs are **AB** and **BE**, both with length 7. **AB** is chosen arbitrarily, and is highlighted. The arc **BD** has been highlighted in red, because there already exists a path (in green) between **B** and **D**, so it would form a cycle (**ABD**) if it were chosen.



The process continues to highlight the next-smallest arc, **BE** with length 7. Many more arcs are highlighted in red at this stage: **BC** because it would form the loop **BCE**, **DE** because it would form the loop **DEBA**, and **FE** because it would form **FEBAD**.



Finally, the process finishes with the arc **EG** of length 9, and the minimum spanning tree is found.

Proof of correctness

The proof consists of two parts. First, it is proved that the algorithm produces a spanning tree. Second, it is proved that the constructed spanning tree is of minimal weight.

Spanning tree

Let P be a connected, weighted graph and let Y be the subgraph of P produced by the algorithm. Y cannot have a cycle, since the last edge added to that cycle would have been within one subtree and not between two different trees. Y cannot be disconnected, since the first encountered edge that joins two components of Y would have been added by the algorithm. Thus, Y is a spanning tree of P.

Minimality

We show that the following proposition **P** is true by induction: If F is the set of edges chosen at any stage of the algorithm, there is some minimum spanning tree that contains F.

- Clearly **P** is true at the beginning, when F is empty: any minimum spanning tree will do.
- Now assume **P** is true for some non-final edge set F and let T be a minimum spanning tree that contains F. If the next chosen edge e is also in T, then **P** is true for F+e. Otherwise, T+e has a cycle C and there is another edge f that is in C but not F. Then T-f+e is a tree, and its weight is not more than the weight of T since otherwise the algorithm would choose f in preference to e. So T-f+e is a minimum spanning tree containing F+e and again **P** holds.
- Therefore, by the principle of induction, **P** holds when F has become a spanning tree, which is only possible if F is a minimum spanning tree itself.

Prim's algorithm

In computer science, Prim's algorithm is an algorithm that finds a minimum spanning tree for a connected weighted undirected graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. Prim's algorithm is an example of a greedy algorithm.

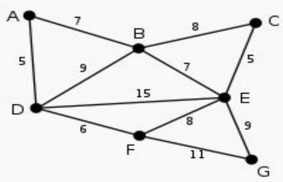
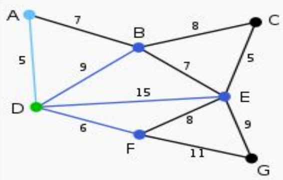
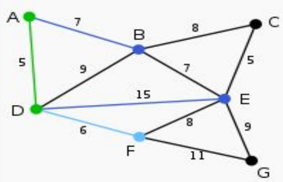
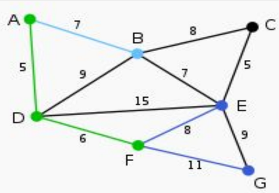
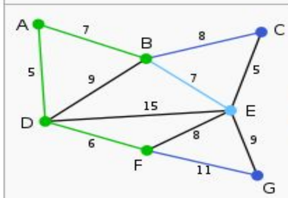
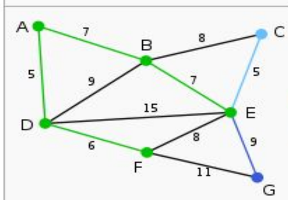
Description

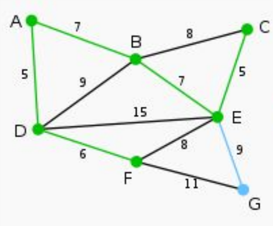
Prim's algorithm has many applications, such as in maze generation.

The only spanning tree of the empty graph (with an empty vertex set) is again the empty graph. The following description assumes that this special case is handled separately.

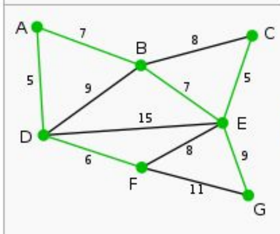
The algorithm continuously increases the size of a tree, one edge at a time, starting with a tree consisting of a single vertex, until it spans all vertices.

- Input: A non-empty connected weighted graph with vertices V and edges E (the weights can be negative).
- Initialize: $V_{new} = \{x\}$, where x is an arbitrary node (starting point) from V, $E_{new} = \{\}$
- Repeat until $V_{new} = V$:
  - Choose an edge (u, v) with minimal weight such that u is in V_{new} and v is not (if there are multiple edges with the same weight, any of them may be picked)
  - Add v to V_{new}, and (u, v) to E_{new}
- Output: V_{new} and E_{new} describe a minimal spanning tree

Image	Description
	This is our original weighted graph. The numbers near the edges indicate their weight.
	Vertex **D** has been arbitrarily chosen as a starting point. Vertices **A**, **B**, **E** and **F** are connected to **D** through a single edge. **A** is the vertex nearest to **D** and will be chosen as the second vertex along with the edge **AD**.
	The next vertex chosen is the vertex nearest to *either* **D** or **A**. **B** is 9 away from **D** and 7 away from **A**, **E** is 15, and **F** is 6. **F** is the smallest distance away, so we highlight the vertex **F** and the arc **DF**.
	The algorithm carries on as above. Vertex **B**, which is 7 away from **A**, is highlighted.
	In this case, we can choose between **C**, **E**, and **G**. **C** is 8 away from **B**, **E** is 7 away from **B**, and **G** is 11 away from **F**. **E** is nearest, so we highlight the vertex **E** and the arc **BE**.
	Here, the only vertices available are **C** and **G**. **C** is 5 away from **E**, and **G** is 9 away from **E**. **C** is chosen, so it is highlighted along with the arc **EC**.



Vertex **G** is the only remaining vertex. It is 11 away from **F**, and 9 away from **E**. **E** is nearer, so we highlight it and the arc **EG**.



Now all the vertices have been selected and the minimum spanning tree is shown in green. In this case, it has weight 39.

U	Edge(u,v)	V \ U
{}		{A,B,C,D,E,F,G}
{D}	(D,A) = 5 **V** (D,B) = 9 (D,E) = 15 (D,F) = 6	{A,B,C,E,F,G}
{A,D}	(D,B) = 9 (D,E) = 15 (D,F) = 6 **V** (A,B) = 7	{B,C,E,F,G}
{A,D,F}	(D,B) = 9 (D,E) = 15 (A,B) = 7 **V** (F,E) = 8 (F,G) = 11	{B,C,E,G}
{A,B,D,F}	(B,C) = 8 (B,E) = 7 **V** (D,B) = 9 cycle (D,E) = 15 (F,E) = 8 (F,G) = 11	{C,E,G}

{A,B,D,E,F}	(B,C) = 8 (D,B) = 9 cycle (D,E) = 15 cycle (E,C) = 5 **V** (E,G) = 9 (F,E) = 8 cycle (F,G) = 11	{C,G}
{A,B,C,D,E,F}	(B,C) = 8 cycle (D,B) = 9 cycle (D,E) = 15 cycle (E,G) = 9 **V** (F,E) = 8 cycle (F,G) = 11	{G}
{A,B,C,D,E,F,G}	(B,C) = 8 cycle (D,B) = 9 cycle (D,E) = 15 cycle (F,E) = 8 cycle (F,G) = 11 cycle	{}

Max-flow min-cut theorem

In optimization theory, the **max-flow min-cut theorem** states that in a flow network, the maximum amount of flow passing from the source to the sink is equal to the minimum capacity which when removed in a specific way from the network causes the situation that no flow can pass from the source to the sink.

Definition

Let N = (V,E) be a network (directed graph) with s and t being the source and the sink of N respectively.

The **capacity** of an edge is a mapping c: $E \rightarrow R^+$, denoted by c_{uv} or c(u,v). It represents the maximum amount of flow that can pass through an edge.

A **flow** is a mapping f: $E \rightarrow R^+$, denoted by f_{uv} or f(u,v), subject to the following two constraints:

1. $f_{uv} \le c_{uv}$ for each $(u,v) \in E$ (capacity constraint)
2. $\sum_{u:\,(u,v)\in E} f_{uv} = \sum_{u:\,(v,u)\in E} f_{vu}$ for each $v \in V \setminus \{s,t\}$ (conservation of flows).

The **value of flow** is defined by $|f| = \sum_{v \in V} f_{sv}$, where s is the source of N. It represents the amount of flow passing from the source to the sink.

The maximum flow problem is to maximize | f |, that is, to route as much flow as possible from s to the t.

An **s-t cut** C = (S,T) is a partition of V such that $s \in S$ and $t \in T$. The **cut-set** of C is the set $\{(u,v) \in E \mid u \in S, v \in T\}$. Note that if the edges in the cut-set of C are removed, | f | = 0.

The **capacity** of an s-t cut is defined by $c(S,T) = \sum_{(u,v)\in S\times T} c_{uv}$.

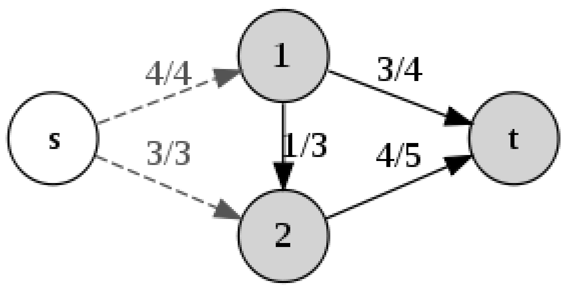
The minimum cut problem is to minimize c(S,T), that is, to determine S and T such that the capacity of the S-T cut is minimal.

Statement

The max-flow min-cut theorem states

The maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.

Example



A network with the value of flow equal to the capacity of an s-t cut

The figure above is a network having a value of flow of 7. The vertex in white and the vertices in grey form the subsets S and T of an s-t cut, whose cut-set contains the dashed edges. Since the capacity of the s-t cut is 7, which is equal to the value of flow, the max-flow min-cut theorem tells us that the value of flow and the capacity of the s-t cut are both optimal in this network.

Application

Generalized max-flow min-cut theorem

In addition to edge capacity, consider there is capacity at each vertex, that is, a mapping c: $V\to R^+$, denoted by c(v), such that the flow f has to satisfy not only the capacity constraint and the conservation of flows, but also the vertex capacity constraint

$\sum_{i\in V} f_{iv} \le c(v)$ for each $v \in V \setminus \{s,t\}$.

In other words, the amount of flow passing through a vertex cannot exceed its capacity. Define an s-t cut to be the set of vertices and edges such that for any path from s to t, the path contains a member of the cut. In this case, the capacity of the cut is the sum the capacity of each edge and vertex in it.

In this new definition, the **generalized max-flow min-cut theorem** states that the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut in the new sense.

Matching theory

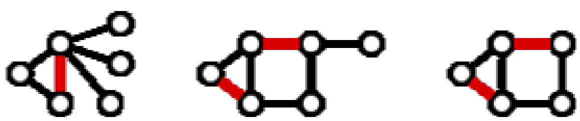
In the mathematical discipline of graph theory, a matching or independent edge set in a graph is a set of edges without common vertices. It may also be an entire graph consisting of edges without common vertices.

Definition

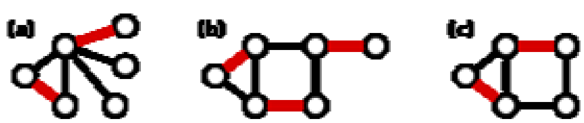
Given a graph G = (V,E), a **matching** M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

A vertex is **matched** (or **saturated**) if it is incident to an edge in the matching. Otherwise the vertex is **unmatched**.

A **maximal matching** is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching, that is, M is maximal if it is not a proper subset of any other matching in graph G. In other words, a matching M of a graph G is maximal if every edge in G has a non-empty intersection with at least one edge in M. The following figure shows examples of maximal matchings (red) in three graphs.



A **maximum matching** is a matching that contains the largest possible number of edges. There may be many maximum matchings. The **matching number** $\nu(G)$ of a graph G is the size of a maximum matching. Note that every maximum matching is maximal, but not every maximal matching is a maximum matching. The following figure shows examples of maximum matchings in three graphs.



A **perfect matching** (a.k.a. 1-factor) is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching. Figure (b) above is an example of a perfect matching. Every perfect matching is maximum and hence maximal. In some literature, the term **complete matching** is used. In the above figure, only part (b) shows a perfect matching. A perfect matching is also a minimum-size edge cover. Thus, $\nu(G) \leq \rho(G)$, that is, the size of a maximum matching is no larger than the size of a minimum edge cover.

A **near-perfect matching** is one in which exactly one vertex is unmatched. This can only occur when the graph has an odd number of vertices, and such a matching must be maximum. In the above figure, part (c) shows a near-perfect matching. If, for every vertex in a graph, there is a near-perfect matching that omits only that vertex, the graph is also called factor-critical.

Given a matching M,

- an **alternating path** is a path in which the edges belong alternatively to the matching and not to the matching.
- an **augmenting path** is an alternating path that starts from and ends on free (unmatched) vertices.

One can prove that a matching is maximum if and only if it does not have any augmenting path.

Properties

In any graph without isolated vertices, the sum of the matching number and the edge covering number equals the number of vertices.[1] If there is a perfect matching, then both the matching number and the edge cover number are |V|/2.

If A and B are two maximal matchings, then |A| ≤ 2|B| and |B| ≤ 2|A|. To see this, observe that each edge in A \ B can be adjacent to at most two edges in B \ A because B is a matching. Since each edge in B \ A is adjacent to an edge in A \ B by maximality, we see that

$$|A \setminus B| \leq 2|B \setminus A|.$$

Further we get that

$$|A| = |A \cap B| + |A \setminus B| \leq 2|B \cap A| + 2|B \setminus A| = 2|B|.$$

In particular, this shows that any maximal matching is a 2-approximation of a maximum matching and also a 2-approximation of a minimum maximal matching. This inequality is tight: for example, if G is a path with 3 edges and 4 nodes, the size of a minimum maximal matching is 1 and the size of a maximum matching is 2.

Matching polynomials

Main article: Matching polynomial

A generating function of the number of k-edge matchings in a graph is called a matching polynomial. Let G be a graph and m_k be the number of k-edge matchings. One matching polynomial of G is

$$\sum_{k\geq 0} m_k x^k.$$

Another definition gives the matching polynomial as

$$\sum_{k\geq 0} (-1)^k m_k x^{n-2k},$$

where n is the number of vertices in the graph. Each type has its uses; for more information see the article on matching polynomials.

Maximum matchings in bipartite graphs

Matching problems are often concerned with bipartite graphs. Finding a **maximum bipartite matching** (often called a **maximum cardinality bipartite matching**) in a bipartite graph G = (V = (X,Y),E) is perhaps the simplest problem. The **augmenting path algorithm** finds it by finding an augmenting path from each $x \in X$ to Y and adding it to the matching if it exists. As each path can be found in O(E) time, the running time is O(VE). This solution is equivalent to adding a super source s with edges to all vertices in X, and a super sink t with edges from all vertices in Y, and finding a maximal flow from s to t. All edges with flow from X to Y then constitute a maximum matching. An improvement over this is the Hopcroft-Karp algorithm, which runs in $O(\sqrt{V}E)$time. Another approach is based on the fast matrix multiplication algorithm and gives $O(V^{2.376})$ complexity, which is better in theory for sufficiently dense graphs, but in practice the algorithm is slower.

In a weighted bipartite graph, each edge has an associated value. A **maximum weighted bipartite matching**[2] is defined as a perfect matching where the sum of the values of the edges in the matching have a maximal value. If the graph is not complete bipartite, missing edges are inserted with value zero. Finding such a matching is known as the assignment problem. It can be solved by using a modified shortest path search in the augmenting path algorithm. If the Bellman-Ford algorithm is used, the running time becomes $O(V^2E)$, or the edge cost can be shifted with a potential to achieve $O(V^2\log(V) + VE)$ running time with the Dijkstra algorithm and Fibonacci heap. The remarkable Hungarian algorithm solves the assignment problem and it was one of the beginnings of combinatorial optimization algorithms. The original approach of this algorithm need $O(V^2E)$ running time, but it could be improved to $O(V^2\log(V) + VE)$ time with extensive use of priority queues.

Maximum matchings

There is a polynomial time algorithm to find a maximum matching or a maximum weight matching in a graph that is not bipartite; it is due to Jack Edmonds, is called the paths, trees, and flowers method or simply Edmonds's algorithm, and uses bidirected edges. A generalization of the same technique can also be used to find maximum independent sets in claw-free graphs. Edmonds' algorithm has subsequently been improved to run in time $O(\sqrt{V}E)$time, matching the time for bipartite maximum matching. Another algorithm by Mucha and Sankowski[3], based on the fast matrix multiplication algorithm, gives $O(V^{2.376})$ complexity.

Maximal matchings

A maximal matching can be found with a simple greedy algorithm. A maximum matching is also a maximal matching, and hence it is possible to find a largest maximal matching in polynomial time. However, no polynomial-time algorithm is known for finding a **minimum maximal matching**, that is, a maximal matching that contains the smallest possible number of edges. Note that a maximal matching with k edges is an edge dominating set with k edges. Conversely, if we are given a minimum edge dominating set with k edges, we can construct a maximal matching with k edges in polynomial time. Therefore the problem of finding a minimum maximal matching is essentially equal to the problem of finding a minimum edge dominating set. Both of these two optimisation problems are known to be NP-hard; the decision versions of these problems are classical examples of NP-complete problems.[6] Both problems can be approximated within factor 2 in polynomial time: simply find an arbitrary maximal matching M.

UNIT 5

Fundamental Principles of Counting

The Rules of Sum and Product

Our study of discrete and combinatorial mathematics beings with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic Principles. We want to develop the ability to "decompose" such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, Taking note of the principles being used. This is the approach we shall follow here.
Our first principle of counting can be stated as follows:

The Rule of Sum:

If a first task can be performed in ***m*** ways, while a second task can be performed in ***n*** ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any of ***m*** + ***n*** ways.

Note that when we say that a particular occurrence, such as a first task, can come about in ***m*** ways, these ***m*** ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

Example 1.1
A College library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among 40 + 50 = 90 textbooks in order to learn more about one or the other of these two subjects.

Example 1.2
The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

Example 1.3
The computer science instructor of Example 1.2 has two colleagues. One of three colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If ***n*** denotes the

maximum number of different books on this topic that this instructor can borrow from them, then $5 \le \boldsymbol{n} \le 8$, for here both colleagues may own copies of the same textbook(s).

Example 1.4

Suppose a university representative is to be chosen either from 200 teaching or 300 non-teaching employees, and then there are 200 + 300 = 500 possible ways to choose this representative.

Extension of Sum Rule:

If tasks $T_1, T_2, \ldots\ldots, T_m$ can be done in $n_1, n_2, \ldots\ldots, n_m$ ways respectively and no two of these tasks can be performed at the same time, then the number of ways to do ***one*** of these tasks is $n_1 + n_2 + \ldots. + n_m$.

Example 1.5

If a student can chose a project either 20 from mathematics or 35 from computer science or 15 from engineering, then the student can choose a project 20 + 35 + 15 = 70 ways.

The following example introduces our second principle of counting.

Example 1.6

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in 5 X 7 = 35 ways.

The rule of Product:

If a procedure can be broken down into first and second stages, and if there are ***m*** possible outcomes for the first stage and if, for each of these outcomes, there are ***n*** possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in ***mn*** ways.

Example 1.7

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in 6 **X** 8 = 48 ways.

Example 1.8

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are 26 **X** 25 **X** 10 **X** 9 **X** 8 **X** 7 = 3,276,000 different possible plates.

b) With repetitions of letters and digits allowed, 26 **X** 26 **X** 10 **X** 10 **X** 10 **X** 10 = 6,760,000 different license plates are possible.

c) If repetitions are allowed, as in part (**b**), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer)

Example 1.9

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a bit — that is, one of the binary digits 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its address. For some computer's, such as embedded microcontrollers (as found in the ignition system for an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a ***byte***. Using the rule of product, there are 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 = 2^8 = 256 such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.

A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" (such as the PICmicro microcontroller) contain thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there are 256 X 256 = 2^8 X 2^8 = 2^{16} = 65,536 available address that could be used to identifying cells in main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is presently used in the Pentium processor, where there are as many as 2^8 X 2^8 X 2^8 X 2^8 = 232 = 4,294,967,296 addresses for use in identifying the cells in main memory. When a programmer deals with the UltraSPARC or Itanium processors, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises 8 X 8 = 64 bits, and there are 2^{64} = 18,446,744,073,709,551,616 possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

Example 1.10

At times it is necessary to combine several different counting principles in the Solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

At the AWL Corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch — either a muffin and a hot beverage or a sandwich and a cold beverage.
By the rule of product, there are 6 X 2 = 12 ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are 8 X 3 = 24 possibilities for a sandwich and cold beverage. So by the rule of sum, there are 12 + 24 = 36 ways in which Carl can purchase Ms. Dodd's lunch.

Example 1.11

A tourist can travel from Hyderabad to Tirupati in four ways (by plane, train, bus or taxi). He can then travel from Tirupati to Tirumala hills in five ways (by RTC bus, taxi, rope way, motorcycle or walk). Then the tourist can travel from Hyderabad to Tirumala hills in 4 X 5 = 20 ways.

Extension of Product Rule: Suppose a procedure consists of performing tasks $T_1, T_2, \ldots, T_m$ in order. Suppose task Ti can be performed in ***ni*** ways after the tasks $T_1, T_2, \ldots, T_{i-1}$ are performed, then the number of ways the procedure can be executed in the designated order is $n1, n2, n3, \ldots, n_m$

Example 1.12

"Charmas" brand shirt available in 12 colors has a male and female version. It comes in four sizes for each sex, comes in three makes of economy, standard and luxury. Then the numbers of different types of shirts produced are 12 X 2 X 4 X 3 = 288.

Example 1.13

If there are 18 boys and 12 girls in a class, there are 18 + 12 = 30 ways of selecting 1 student (either a boy or a girl) as class representative.

Example 1.14

Suppose ***E*** is the event of selecting a prime number less than 10 and ***F*** is the event of selecting an even number less than 10. then ***E*** can happen in 4 ways. But, because 2 is an even prime, ***E*** and ***F*** can happen in only 4 + 4 – 1 = 7 ways.

Example 1.15

A bookshelf holds 6 different English books, 8 different French books, and 10 different German books. There are (***i***) (8) (9) (10) = 480 ways of selecting 3 books, 1 in each language; (***ii***) 6 + 8 + 10 = 24 ways of selecting 1 book in any one of languages.

Example 1.16

The scenario is as in Example 1.15. An English book and a French book can be selected in (6) (8) = 48 ways; an English book and a German book, in (6) (10) = 60 ways; a French book and a German book, in (8) (10) = 80 ways. Thus there are 48 + 60 + 80 = 188 ways of selecting 2 books in 2 languages.

Example 1.17

If each of the 8 questions in a multiple-choice examination has 3 answers (1 correct and 2 wrong), the number of ways of answering all questions is 38 = 6561.

Example 1.18

There are P(***6, 6***) = 720 6-letter “words” that can be made from the letters of word NUMBER, and there are P(6, 4) = 6!/2! = 360 4-letter “words”. An unordered selection of ***r*** out of the ***n*** elements of ***X*** is called an ***r-combination*** of ***X***. In other words, any subset of ***X*** with r elements is an ***r***-combination of ***X***. The number of ***r***-combinations or ***r***-subsets of a set of n distinct objects is denoted by ***C (n, r)*** *("n choose r").* For each ***r***-subset of ***X*** there is unique complementary (***n – r***)-subset, whence the important relation ***C (n, r) = C (n, n – r).*** To evaluate ***C (n, r)***, note that an ***r-****permutation* of an ***n***-set ***X*** is necessarily a permutation of some ***r***-subset of ***X***. Moreover distinct ***r***-subsets generate distinct ***r-***permutations. Hence, by the sum rule,

$$P(n, r) = P(r, r) + P(r, r) + \ldots + P(r, r)$$

The number of terms on the right is the number of ***r***-subset of ***X***; i.e. ***C (n, r).*** Thus
P (***n, r***)-subset, whence the important relation ***C (n, r) = C (n, n – r).***

Example 1.19

From a class consisting of 12 computer science majors, 10 mathematics majors, and 9 statistics majors, a committee of 4 computer science majors, 4 mathematics majors, and 3 statistics majors is to be formed. There are

$$C(12,4) = \frac{12!}{4!8!} = \frac{12.11.10.9}{4.3.2.1} = 11.5.9 = 495$$

Ways of choosing 4 computer science majors, C(10, 4) = 210 ways of choosing 4 mathematics majors, and C(9, 3)+ 84 ways of choosing 3 statistics majors. By the product rule, the number of ways of forming a committee is thus (495)(210)(84) = 8,731,800.

Example 1.20

Refer to Example 1.18 in how many ways can a committee consisting of 6 or 9 members be formed such that all 3 majors are equally represented?

A committee of 6(with 2 from each group) can be formed in ***C(12,2).C(10,2).C(9,2)*** = 106,920 ways. The number of ways of forming a committee of 9 (with 3 from each group) is ***C(12,3).C(10,3).C(9,3)*** = 2,217,600. Then, by the sum rule the number of ways of forming a committee is 106,920 + 2,217,600 = 2,324,520.

Example 1.21

There are 15 married couples in a party. Find the number of ways of choosing a woman and a man from the party such that the two are ***(a)*** married to each other, ***(b)*** not married to each other.

(a) A woman can be chosen in 15 ways. Once a woman is chosen, her husband automatically chosen. So the number of ways of choosing a married couple is 15.
(b) A woman can be chosen in 15 ways. Among the 15 men in the party, one is her husband. Out of the 14 other men, one can be chosen in 14 ways. The product rule the gives (15)(14) = 210 ways.

Example 1.22

Find the number of ***(a)*** 2-digit even numbers, ***(b) 2***-digit odd numbers, ***(c)*** 2-digit odd numbers with distinct digits, and ***(d)*** 2-digit even numbers with distinct digits.

Let ***E*** be the event of choosing a digit for the units' position, and ***F*** be the event choosing a digit for the tens' position.

(a) E can be done in 5 ways; ***F*** can be done in 9 ways. The number of ways of doing ***F*** does not depend upon how ***E*** is done; hence, the sequence {***E, F***} can be done in (5)(9) = 45 ways.

(b) The argument is as in ***(a)***: there are 45 2-digit odd numbers.

(c) If ***F*** is done first, the number of ways of doing ***E*** depends upon how ***F*** was done; so we cannot apply the product rule to the sequence {***F, E***}. But we can apply the product rule to the sequence {***E, F***}. There are 5 choices for the units' digit, and for each of these there are 8 choices for the tens' digit. So the sequence {***E, F***} can be done in 40 ways; i.e., there are 40 2-digit odd numbers with distinct digits.

(d) We distinguish two cases. If the units' digit is 0-which can be accomplished in 1 way-the tens' digit can be chosen in 9 ways. If 2,4,6, or 8 is chosen as units' digit, the tens' digit can be chosen in 8 ways. Thus the sum and product rules give a total of (1)(9)+(4)(8) = 41 ways.

Example 1.23

A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find

(a) the total number of passwords that can be created, and ***(b)*** the number of Passwords in which no digit repeats.

(a) The number of 4-charcter passwords is (26)(10)(10)(10), and the number of 5-charcter passwords is (26)(10)(10)(10)(10), by the product rule. So the total number of passwords is 26,000 + 260,000 = 286,000, by the sum rule.

(b) The number of 4-charcter passwords is (26)(10)(9)(8) = 18,720, the number of 5-charcter passwords is (26)(10)(9)(8)(7) = 131,040, for a total of 149,760.

Example 1.24

How many among the first 100,000 positive integers contain exactly one 3, one 4, and one 5 in their decimal representation?

It is clear that we may consider instead the 5-place numbers 00000 through 99999. The digit 3 can be in any one of the 5 places. Subsequently the digit 4 can be in any one of the remaining places. Then the digit 5 can be in one of 3 places. There are 2 places left, either of which may be filled by 7 digits. Thus there are (5)(4)(3)(7)(7) = 2940 integers in the desired category.

Example 1.25

Find the number of 3-digit even numbers with no repeated digits.

By problem 1.21***(d)***, the hundreds' and units' positions can be simultaneously filled in 41 ways. For each of these ways, the tens' position can be filled in 8 ways. Hence the desired number is (41)(8) = 328ways.

Example 1.26

A palindrome is a finite sequence of characters that reads the same forwards and backwards [GNUDUNG]. Find the numbers of 7-digit and 8-digit palindromes, under the restriction that no digit may appear more than twice.

By the mirror-symmetry of a palindrome (of length n), only the first $\lfloor (n+1)/2 \rfloor$ Positions need be considered. In our case this number is 4 for both lengths. Since the first digit may not be 0, there are 9 ways to fill the first position. There are then 10-1 = 9 ways to fill the second position; 10-2 = 8 ways for the third; 10-3 = 7 ways for the fourth. Thus there are (9)(9)(8)(7) = 4536 palindromic numbers of either length.

Example 1.27

In a binary palindrome the first digits is 1 and each succeeding digit may be 0 or 1. Count the binary palindromes of length ***n***.

See problem 1.25. Here we have $\lfloor (n+1)/2 \rfloor$ -1 = $\lfloor (n-1)/2 \rfloor$ free positions, so the desired number is

Example 1.28

Find the number of proper divisors of 441,000. (A proper divisor of positive integer ***n*** is any divisor other than 1 and ***n***)

Any integer can be uniquely expressed as product of powers of prime numbers; thus, 441,000 = $(2^3)(3^2)(5^3)(7^2)$. Any divisor, proper or improper, of given number must be of the form $(2^a)(3^b)(5^c)(7^d)$, where $0 \le a \le 3$, $0 \le b \le 2$, $0 \le c \le 3$, and $0 \le d \le 2$. in this paradigm the exponent a can be chosen in 4 ways; b in 3 ways; c in 4 ways; d in 3 ways. So, by the product rule, the total number of proper divisors will be (4)(3)(4)(3) – 2 = 142.

Example 1.29

In a binary sequence every element is 0 or 1. Let X be the set of all binary sequences of length ***n.*** A switching function (Boolean function) of ***n*** variables is A function from ***X*** to the set Y = {0, 1}. Find the number of distinct switching functions of ***n*** variables**.**

The cardinality of X is $r = 2^n$. So the number of switching functions is 2^r.

1.2 Permutations

Continuing to examine applications of rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called ***permutations*** when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

Example 1.14

In class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is arrangement, which designates the importance of order. If A, B, C, . . ., I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are there such different arrangements, even though the first two involve the same five students.

To answer this question, we consider the positions and possible numbers of students we can choose in order to fill each position. The filling of position is a stage of our procedure.

10	X	9	X	8	X	7	X	6
1st position		**2nd position**		**3rd position**		**4th position**		**5th position**

Each of the 10 students can occupy the 1st position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the 2nd position. Continuing in this way, we find only six students to select from in order to fill the 5th and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled from right to left namely, 6 X 7 X 8 X 9 X 10. if the 3rd position is filled first, the 1st position second, the 4th osition third, the 5th position fourth, and the 2nd position fifth then answer is 9 X 6 X 10 X 8 X 7, still the same value, 30,240.

Definition 1.1

As in Example 1.14, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

For an integer $n \geq 0$, n factorial (denoted ***n!***) is defined by

0! = 1
n! = (n)(n-1)(n-2)….(3)(2)(1), for $n \geq 1,$

One finds that 1! = 1, 2! = 2, 3! = 6, 4! = 24, and 5! = 120, in addition, for each $n \geq 0,$ (n + 1)! = (n + 1) (n!).

Before we proceed any further, let us try to get a somewhat better appreciation for how fast n! grows. We can calculate that 10! = 3,628,800, and it just so happens that this is exactly the number of seconds in six weeks, Consequently, 11! Exceeds the number of seconds in one year, 12! Exceeds the number in 12 years, and 13! Surpasses the number of seconds in century.

If we make use of the factorial notation, the answer in Example 1.14 can be Expressed in the following more compact form:

$$10\times 9\times 8\times 7\times 6 = 10\times 9\times 8\times 7\times 6\times \frac{5\times 4\times 3\times 2\times 1}{5\times 4\times 3\times 2\times 1} = \frac{10!}{5!}$$

Definition 1.2

Given a collection of *n* distinct objects. Any (linear) arrangement of these objects is called a permutation of the collection.

Starting with the letters a, b, c, there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size – 2 permutations: ab, ba, ac, ca, bc, cb.

If there are ***n*** distinct objects and *r* is an integer, with $1 \le r \le n$, then by the rule of product, the number of permutations of size r for the n objects are

P(n, r) =	n	X	(n - 1)	X	(n - 2)	X . . . X	(n - r + 1)
	1st position		**2nd position**		**3rd position**		**rth position**

$$= (n)(n-1)(n-2)...(n-r+1)\times \frac{(n-r)(n-r-1)...(1)(2)(3)}{(n-r)(n-r-1)...(1)(2)(3)}$$

$$= \frac{n!}{(n-r)!}$$

For r 0, P (n, 0) = 1 = n!/ (n- 0)!, so P (n, r) = n!/(n-r)! holds for all $0 \le r \le n$. A special case of this result is Example 1.14, where n = 10, r = 5, and P (10, 5) = 30, 240. When permuting all of the n objects in the collection, we have r = n and find that P (n, n) = n!/0!=n!.

Note, for example, that if $n \ge 2$, then P (n, 2) = n!/(n-2)! = n(n-1). When n>3 one finds that P(n,n-3) = n!/[n-(n-3)]!=n!/3!=(n)(n-1)(n_2)…(5)(4).

The number of permutations of size r, where $0 \le r \le n$, from a collection of n objects, is P (n, r) =n!/(n-r)! (Remember that P (n, r) counts (linear) arrangements in which the objects cannot be repeated.) However, if repetitions are allowed, then by the rule of product there are nr possible arrangements, with $r \ge 0$.

Example 1.15

The number of words of three distinct letters formed from the letters of word "JNTU" is P (4, 3) = 4!/(4 - 3)!=24. If repetitions are allowed, the number of possible six – letter sequence is 46 = 4096.

Example 1.16

In how many ways can eight men and eight women be seated in a row if (**a**) any person may sit next to any other (**b**) men and women must occupy alternate seats (**c**) generalize this result for n men and n women.

Here eight men and eight women are 16 indistinguishable objects.
a) The number of permutations 16 chosen form 16 objects is P (16, 16) = 16!
= 20922789890000.

b) Here men and women are distinct (different)

i)

M	W	M	W	M	W	M	W	M	W	M	W	M	W	M	W
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Man sitting first: the number of ways is 8! 8!

ii)

W	M	W	M	W	M	W	M	W	M	W	M	W	M	W	M
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Woman sitting first: 8! 8!
Thus the number of ways men and women occupy
Alternatively is 8! 8! + 8! 8! = $2(8!)^2$

c) Any person may sit: (2n)!
Men and women sit alternatively: $2(n!)^2$

Example 1.17

A committee of eight is to be formed from 16 men and 10 women. In how many ways can the committee be formed if (**a**) there are no restrictions (**b**) there must be 4 men and 4 women (**c**) there should be an even number of women (**d**) more women than men (**e**) at least 6 men.

a) No distinction between men and women. Problem is to choose 8 out of a set of 26 persons. So the number of ways 8 are chosen out of 26 is C(26, 8) = 26!/ 89! (18!) = 2.480721325 x 1017

b) First stage choose 4 men out of 16 given by C(16, 4). Second stage choose 4 women out of 10 in C(10, 4)ways. Using product rule, the number of ways in which the committee consisting of 4 men and women is C(16,4) C(10,4) = 1,820 x 210 = 382,200.

c) If $2i$ even number of women are chosen, then the remaining 8 -$2i$ members of the committee should be men. By product rule, C(10, $2i$)C(16, 8-$2i$). Then the total number of ways is

$$\sum_{i=1}^{4} \binom{10}{2i}\binom{16}{8-2i}$$

d) Since the strength of the committee is 8, there should be 5 or more women so that women outnumber men. Using product rule, the number of ways is.

$$\sum_{i=5}^{8} \binom{10}{i}\binom{16}{8-i}$$

e) When the number of men is 6 or more we get by a similar argument, the number of ways as

$$\sum_{i=5}^{8} \binom{16}{i}\binom{10}{8-i}$$

Example 1.18

The number of permutations of the letters in the word COMPUTER is 8!. If only five of the letters are used, the number of permutations (of size 5) is P(8, 5) = 8!/(8-5)! = 8!/3! = 6720. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} = 6.872 \times 10^{10}$.

A	B	L	L	A	B	L1	L2	A	B	L2	L1
A	L	B	L	A	L1	B	L2	A	L2	B	L1
A	L	L	B	A	L1	L2	B	A	L2	L1	B
B	A	L	L	B	A	L1	L2	B	A	L2	L1
B	L	A	L	B	L1	A	L2	B	L2	A	L1
B	L	L	A	B	L1	L2	A	B	L2	L1	A
L	A	L	L	L1	A	B	L2	L2	A	B	L1
L	A	A	B	L1	A	L2	B	L2	A	L1	B
L	B	L	L	L1	B	A	L2	L2	B	A	L1
L	B	A	A	L1	B	L2	A	L2	B	L1	A
L	L	A	B	L1	L2	A	B	L2	L1	A	B
L	L	B	A	L1	L2	B	A	L2	L1	B	A

(a) (b)

Example 1.19

Unlike example 1.18, the number of (linear) arrangements of the four letters in BALL is 12, not 4! (= 24), the reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in table 1.1(a).
If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have 4!=24 permutations. These are listed in Table 1.1(b). Table 1.1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a pair of permutations with distinct L's. Consequently,

2 X (Number of arrangements of the letters B, A, L, L)
= (Number of permutations of the symbols B, A, L1, L2),

And the answer to the original problem of finding all the arrangements of the four letters in BALL is 4!/2 = 12.

Example 1.20

Using the idea developed in Example 1.19, we now consider the arrangements of all nine letters in DATABASES.
There are 3! = 6 arrangements with the A's distinguished for each arrangements in which the A's are not distinguished. For example, DA_1 TA_2 BA_3 SES, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, and $DA_3TA_2BA_1SES$ all correspond to DATABASES, when we remove the

subscripts on A's. In addition, to the arrangement $DA_1TA_2BA_3SES$ there corresponds the pair of permutations $A_1TA_2BA_3S_1ES_2$ and $DA_1TA_2BA_3S_2ES_1$, when the S's are distinguished. Consequently,

(2!)(3!) (Number of arrangements of the letters in DATABASES) =
(Number of permutations of the symbols D, A_1, T, A_2, B, A_3, S_1, E, S_2)
So the number of arrangements of the nine letters in DATABASES is 9!/(2!3!)
= 30,240.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinational formulas.

If there are n objects with $\boldsymbol{n}_1$ indistinguishable objects of an $\boldsymbol{r}^{th}$ type, where $\boldsymbol{n}_1 + \ldots + \boldsymbol{n}_r = \boldsymbol{n}$, then there are $\frac{n!}{n_1!n_2!\ldots n_r!}$ (linear) arrangements of the given $\boldsymbol{n}$ objects

Example 1.21

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA. We find that there are

$$\frac{10!}{4!3!1!1!1!} = 25,200$$

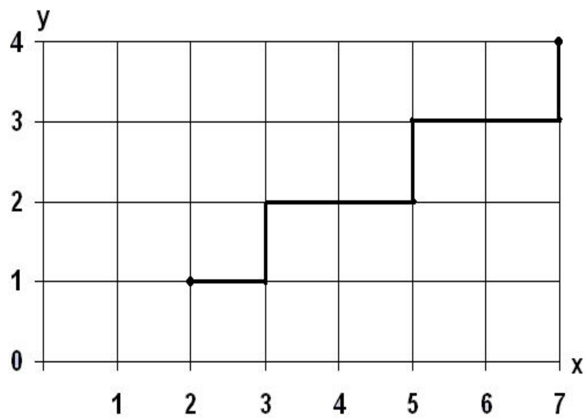
Possible arrangements. Among these are

$$\frac{7!}{3!1!1!1!1!} = 840$$

In which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

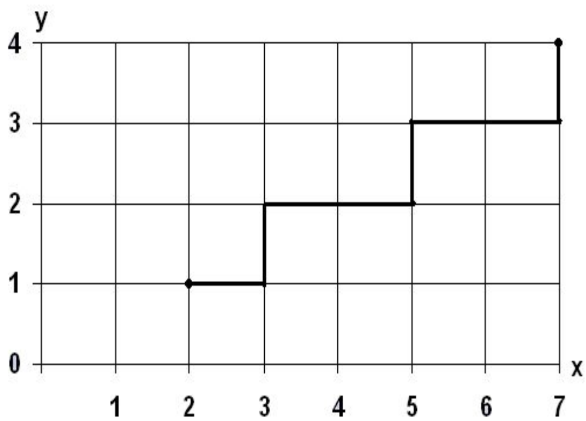
Example 1.22

Determine the number of (staircase) paths in the ***xy***-plane from (2, 1) to (7, 4), Where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these Paths.



(a) U,R,R,R,U,U,R,R

Figure 1.1



(a) U,R,R,R,U,U,R,R

Beneath each path in Fig. 1.1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. in general, the overall trip from (2, 1) to (7, 4) requires 7 – 2 = 5 horizontal moves to the right and 4 – 1 = 3 vertical moves upward. Consequently, each path corresponds to a list of five R's and U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is 8!/(5! 3!) = 56.

Example 1.23

We now do something a bit more abstract and prove that if ***n*** and ***k*** are positive integers with ***n*** = 2***k***, then n!/2k is an integer. Because our argument relies on Counting, it is an example of a ***combinatorial proof***.

Consider the ***n*** symbols ***x1, x1, x2 , x2***……………………., ***xk , xk***. The number of ways in which we can arrange all of these ***n*** =2***k*** symbols is an integer that equals

$$\underbrace{\frac{n!}{2!2!\ldots2!}}_{k \quad factors \quad of \quad 2!} = \frac{n!}{2^k}$$

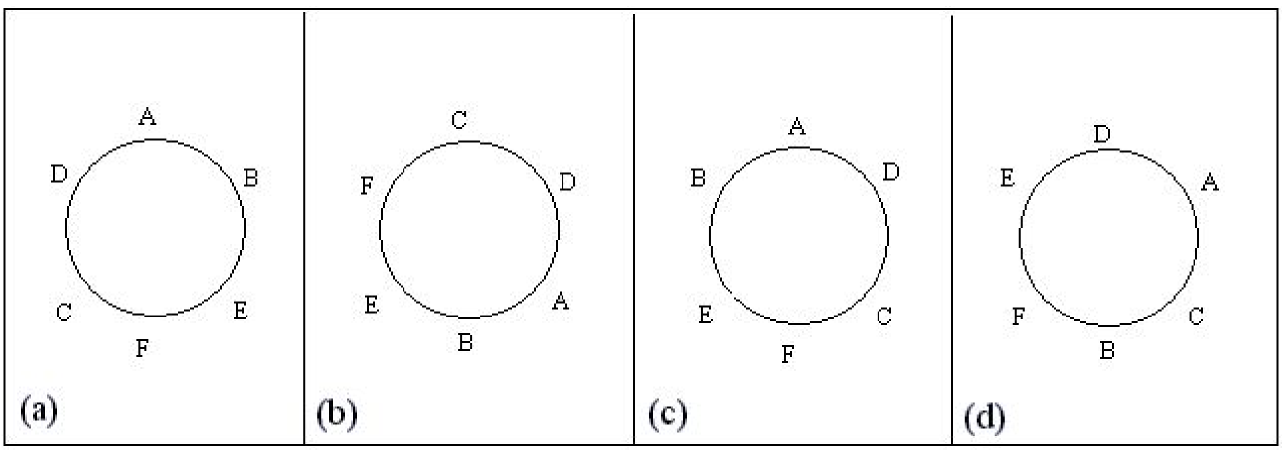


Figure 1.2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2 (a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangements. In addition to these two, four other linear arrangements – BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangements as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements,

We have 6 X (Number of circular arrangements of A, B, . . ., F) =

(Number of linear arrangements of A, B……F) = 6!.

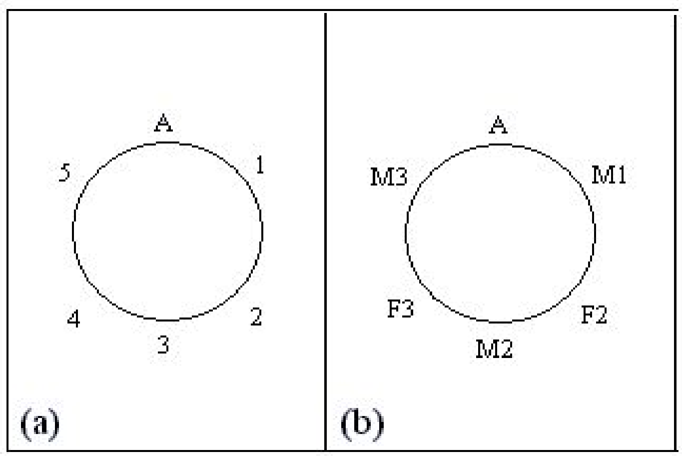
Consequently, There are 6!/6 = 5! = 120 arrangements of A, B,….F around the circular table.

Example 1.25

Suppose now that the six people of Example 1.24 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.24 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Using B, C,…,F to fill.

Figure 1.3



These five positions is the problem of permuting B, C, . . ., F in a linear manner, and this be done in 5! = 120 ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this Manner, by the rule of product, there are 3 x 2 x 2 x 1 x 1 = 12 ways in which these six people can be arranged with no two men or women seated next to each other.

1.3 Combinations: The Binomial Theorem

The standard Deck of playing Cards Consists of 52 cards comprising four suits: Clubs, diamond, hearts, and spades. Each suit has 13 cards: ace, 2, 3,…, 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52,3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine clubs), KD (King of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer AH-9C-KD, AH-KD-9C, 9C-AH-KD,9C-KD-AH, KD-9C-AH, and KD-AH-9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, with no reference to order, corresponds to 3! Permutations of three cards. In equation form this translates into

(3!) x (Number of selection of size 3 from a deck of 52)
= Number of permutations of size 3 for the 52 cards

Consequently, three cards can be drawn, without replacement, from a standard deck in 52!/(3! 49!) = 22,100 ways.

If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to r! Permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}, 0 \le r \le n.$$

In addition to C (n, r) the symbol $\binom{n}{r}$ is frequently used. Both C (n, r) and $\binom{n}{r}$ are sometimes read "n choose r." Note that for all n ≥ 0, C (n, r) = C (n, n) = 1. Further, for all n ≥ 1, C (n, 1) = C (n, n-1) = n. when 0 ≤ n < r, then C (n, r) $\binom{n}{r} = 0$

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem, when order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

Example 1.26

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in C (20, 11) = $\binom{20}{11}$ = 20!/(11! 9!) = 167, 960 ways. However, once the 11 arrive, how she arranges them around her rectangular dinning table is an arrangement problem. Unfortunately, no part of theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

Example 1.27

Lynn an Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the five numbers (between 1 and 49 inclusive). This she can do in $\binom{49}{5}$ ways (since matching does not involve order). Meanwhile Patti selects the powerball – here there are $\binom{42}{1}$ possibilities. Consequently, by the rule of product, Lynn and Patti can select the six numbers for their PowerBall ticket in $\binom{49}{5}\binom{42}{1}$ =80,089,128 ways.

Example 1.28

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!3!} = \frac{10\times 9\times 8}{3\times 2\times 1} = 12 ways$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3}$ = 10 ways, and the other four questions can be selected in $\binom{5}{4}$ = 5 ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4}$ = 10 X 5 = 50 ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

i) The student answers three of the first five questions and four of five: by the rule of product this can happen in = 10 X 5 = 50 ways, as in part (b). $\binom{5}{3}\binom{5}{4}$

ii) Four of the first five questions and three of the last five questions are selected by the student: this can come about in = 5 X 10 = 50 ways – again by the rule of product. $\binom{5}{4}\binom{5}{3}$

iii) The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that last case can occur in = 1 X 10 = 10 ways.

$$\binom{5}{5} \quad \binom{5}{2} \qquad \binom{5}{3}\binom{5}{4}+\binom{5}{4}\binom{5}{3}+\binom{5}{5}\binom{5}{2}$$

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make = 50 + 50 + 10 = 110 selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

$$\binom{5}{3}\binom{5}{4}+\binom{5}{4}\binom{5}{3}+\binom{5}{5}\binom{5}{2}$$

Example 1.29

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in = 4,431,613,550 ways. $\binom{53}{9}$

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in = 15,890,700 ways. $\binom{50}{6}$ $\binom{25}{5}$

c) For a certain tournament that team must comprise four juniors and five seniors. The teacher can select the four juniors in ways. For each of these selections she has ways to choose the five seniors. Consequently, by the rule of product, she can select her team in = 1,087,836,750 ways for this particular tournament.

$$\binom{28}{4}\binom{25}{5}$$

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following Example demonstrates this.

Example 1.30

The gym teacher of Example 1.29 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four terms? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in ways. For team B the selection process yields possibilities. This leaves and possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!27!}\right)\left(\frac{27!}{9!18!}\right)\left(\frac{18!}{9!9!}\right)\left(\frac{9!}{9!0!}\right) = \frac{36!}{9!9!9!9!} = 2.145\times 10^{19}\ ways$$

b) For an alternative solution, consider the 36 students lined up as follows:

$$\begin{matrix} 1st & 2nd & 3rd & & 35th & 36th \\ student & student & student & \cdots & student & student \end{matrix}$$

To select the four teams, we must distribute nine A's, nine B's, nine C's and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!9!9!9!},\ as\ in \quad part\ (a)$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

Example 1.31

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!2!2!2!1!1!} = 831,600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!2!2!1!1!} = 5040$$

Ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow E \uparrow E \uparrow S \uparrow T \uparrow L \uparrow L \uparrow S \uparrow H \uparrow$$

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are 5040 X 84 = 423,360 arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of list of $n + 1$ terms like $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called the Sigma Notation because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \ldots + a_{m+n} = \sum_{i=m}^{m+n} a_i.$$

Here, the letter i is called the index of the summation, and this index accounts for all integers starting with the *lower limit* m and Continuing on up to (and including) the *upper limit* $m + n$.

We may use this following notation

1) $\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j$ for there is nothing special about the letter i.

2) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2, because\, 0^2 = 0.$

3) $\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + \ldots + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$

4) $\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34)$

5) $\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$

6) $\sum_{i=1}^{5} a = a + a + a + a + a = 5a$

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 1.28 as

$$\binom{5}{3}\binom{5}{4}+\binom{5}{4}\binom{5}{3}+\binom{5}{5}\binom{5}{2}=\sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i}=\sum_{j=2}^{4}\binom{5}{7-j}\binom{5}{j}$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this book

Example 1.32

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called ***strings***, made up from a prescribed ***alphabet*** of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of length 2. Among the 27 strings of length 3 are 000,012, 202, and 110.

In general, if ***n*** is any positive integer, then by the rule of product there are 3″ strings of length ***n*** for the alphabet 0,1, and 2. If $x = x_1x_2x_3 \ldots x_n$ is one of these strings, we define the weight of ***x***, denoted wt(***x***), by wt(***x***) = $x_1 + x_2 + x_3 + \ldots + x_n$. For example, wt(12) = 3 and wt(22) = 4 for the case where n = 2; wt(10l) = 2, wt(210) = 3, and wt(222) = 6 for ***n*** = 3.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string ***x*** contains no 1's, then each of the 10 locations in ***x*** can be filled with either 0 or 2, and by the rule of product there are 210 such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}2^8$ strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table1.2.

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is

$$2^{10}+\binom{10}{2}2^{8}+\binom{10}{4}2^{6}+\binom{10}{6}2^{4}+\binom{10}{8}2^{2}+\binom{10}{10}=\sum\nolimits_{n=0}^{5}\binom{10}{2n}2^{10-2n}$$

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Table 1.2

Often we must be careful of *overcounting*—a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

Example 1.33

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

i) Ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.
ii) Five of spades, seven of spades, ten of spades, seven of diamonds, and me king of diamonds.
iii) Two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in me deck that are not clubs. Consequently, she can make her selection in ways. $\binom{39}{5}$

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were not counted in part (a). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2,598,960 - 575,757 = 2,023,203$$

of all five-card hands contain at least one club.

c) Can we obtain the result in part (b) in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1} \binom{51}{4} = 13 \text{ X } 249,900 = 3,248,700$$

Something here is definitely wrong! This answer is larger than that in part (b) by more than one million hands. Did we make a mistake in part (b)? Or is something wrong with our present reasoning?

For example, suppose that Ellen first selects
the three of clubs
and then selects
the five of clubs,
king of clubs,
seven of hearts, and
jack of spades.

If, however, she first selects
the five of clubs
and then selects
the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,

is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first
selects.

the king of clubs
and then follows this by selecting

the three of clubs,
five of clubs,
seven of hearts, and
jack of spades

is not different from the other two selections mentioned earlier.
Consequently, this approach is wrong because we are overcounting — by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 1.3. From the results in Table 1.3 we see, for example, that there are five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are such hands.

$$\binom{13}{2}\binom{39}{5}$$

Since no two of the cases in Table 1.3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{3}\binom{39}{2}$$

$$\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}$$

$$=\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$

$$=(13)(82{,}251)+(78)(9139)+(286)(741)+(715)(39)+(1287)(1)$$

$$=2{,}023{,}203$$

Table 1.3

Number of clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Non clubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

We shall close this section with three results related to the concept of combinations.

First we note that for integers ***n, r,*** with ***n* ≥ *r* ≥0,** $\binom{n}{r}=\binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size ***r*** from a collection of ***n*** distinct objects, the selection process leaves behind ***n* – *r*** objects. Consequently, $\binom{n}{r}=\binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size ***r*** (objects chosen) and the selections of size ***n* - *r*** (objects left behind). An example of this correspondence is shown in Table 1.4, where ***n*** = 5, ***r*** = 2, and the distinct objects are 1, 2, 3, 4, and 5.

This type of correspondence will be more formally defined in Chapter 5 and used in other counting situations.
Our second result is a theorem from our past experience in algebra.

Theorem 1.1
The Binomial Theorem. If ***x*** and ***y*** are variables and ***n*** is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

Before considering the general proof, we examine a special case. If ***n*** = 4, the coefficient of ***x2y2*** in the expansion of the product

(x+y)	***(x+y)***	***(x+y)***	***(x+y)***
1st	2nd	3rd	4th
factor	factor	factor	factor

is the number of ways in which we can select two ***x***'s from the four ***x***'s, one of which is available in each factor. (Although the ***x***'s are the same in appearance, we distinguish them as the ***x*** in the first factor, the ***x*** in the second factor, ... , and the ***x*** in the fourth factor.

Also, we note that when we select two ***x***'s, we use two factors, leaving us with two other factors from which we can select the two ***y***'s that are needed.) For example, among the possibilities, we can select (1) ***x*** from the first two factors and ***y*** from the last two or (2) ***X*** from the first and third factors and ***y*** from the second and fourth. Table 1.5 summarizes the six possible selections.

Consequently, the coefficient of x^2y^2 in the expansion of $(x + y)^4$ is $\binom{4}{2} = 6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Table 1.4

Selections of Size r = 2 (Objects Chosen)		Selections of Size n - r = 3 (Objects Left Behind)	
1. 1,2 2. 1,3 3. 1,4 4. 1,5 5. 2,3	6. 2,4 7. 2,5 8. 3,4 9. 3,5 10. 4,5	1. 3,4,5 2. 2,4,5 3. 2,3,5 4. 2,3,4 5. 1,4,5	6. 1,3,5 7. 1,3,4 8. 1,2,5 9. 1,2,4 10. 1,2,3

Table 1.5

Factors Selected for x	Factors Selected for y
1. 1,2	1. 3,4
2. 1,3	2. 2,4
3. 1,4	3. 2,3
4. 2,3	4. 1,4
5. 2,4	5. 1,3
6. 2,5	6. 1,2

Now we turn to the proof of the general case.

Proof: In the expansion of the product

(x+y) (x+y) (x+y) . . . . (x+y)
1st 2nd 3rd 4th
Factor Factor Factor Factor

The coefficient of $x^k y^{n-k}$, **where** $0 \le k \le n$, is the number of different ways in which we can select **k** *x*'s [and consequently **(*n* - *k*) *y*'s]** from the *n* available factors. (One way, for example, is to choose ***x*** from the first ***k*** factors and ***y*** from the last ***n* – *k*** factors) The total number of such selections of size ***k*** form a collection of size ***n*** is C (***n***, ***k***) = $\binom{n,}{k}$ and from this the binomial theorem follows.

Example 1.34

In view of this theorem, is often referred to as a binomial coefficient. Notice that it is also possible to express the result of Theorem 1.1 as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}.$$

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x + y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$, replace 2***a*** by ***x*** and 3***b*** by ***y***. From the binomial theorem the coefficient of x^5y^2 in $(x + y)^7$ is and

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5\,(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2.$$

Corollary 1.1

For each integer n > 0,

$$a)\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n, and$$

$$b)\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0.$$

Proof: Part (a) follows from the binomial theorem when we set x = y = 1. When x = -1 and y = 1, part (b) results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem*.

Theorem 1.2

For positive integers ***n, t,*** the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\ldots x_t^{n_t}$ in the expansion of $(x^1 + x^2 + x^3 + \ldots + x^t)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_t!}$$

Where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

$$x_1^{n_1}x_2^{n_2}x_3^{n_3}\ldots x_t^{n_t}$$

Proof: As in the proof of the binomial theorem, the coefficient of
is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, …., and x_t from nt of the last $n - n_1 - n_2 - n_3 - \ldots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.30, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\ldots-n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this product is equal to

$$\frac{n!}{n_1!n_2!n_3!\ldots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient*. (When t = 2 this reduces to a binomial coefficient)

Example 1.35

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!2!3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!0!4!} = 35$.

b) Suppose we need to know the coefficient of **a2b3c2d5** in the expansion of ***(a + 2b – 3c + 2d + 5)16***. If we replace ***a*** by ***v***, ***2b*** by ***w***, - ***3c*** by ***x***, 2d by ***y***, and 5 by ***z***, then we can apply the multinomial Theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But

$$\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4$$

$$= \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4\left(a^2b^3c^2d^5\right)$$

$$= 435{,}891{,}456{,}000{,}000\, a^2b^3c^2d^5$$

1.4 Combinations with Repetition

When repetitions are allowed, we have seen that for ***n*** distinct objects an arrangement of size ***r*** of these objects can be obtained in n^r ways, for an integer $r \geq 0$. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

Example 1.36

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively. Here we are concerned with how many of each item are purchased, not with the order in which they are purchased, so the problem is one of selections, or combinations, with repetition.

In Table 1.6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

Table 1.6

(a)	(b)
1. c, c, h, h, t, t, f	**8. x x | x x | x x | x**
2. c, c, c, c, h, t, f	**9. x x x x | x | x | x**
3. c, c, c, c, c, c, f	**10. x x x x x x | | | x**
4. h, t, t, f, f, f, f	**11. | x | x x | x x x x**
5. t, t, t, t, t, f, f	**12. | | x x x x x | x x**
6. t, t, t, t, t, t, t	**13. | | x x x x x x x |**
7. f, f, f, f, f, f, f	**14. | | | x x x x x x x**

For a purchase in column (b) of Table 1.6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t's, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in column (b) of Table 1.6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so by our correspondence the number of different purchases for column (a) is.

$$\frac{10!}{7!3!} = \binom{10}{7}$$

In this example we note that the seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the 3+1= 4 possible food items that can be chosen.

When we wish to select, with repetition, ***r*** of ***n*** distinct objects, we find (as in Table 1.6) that we are considering all arrangements of ***r*** x's and ***n*** - 1 |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

Consequently, the number of combinations of n objects taken r at a time, *with repetition*, is C (***n* + *r* – *1*, *r***).
(In Example 1.36, ***n*** = 4, ***r*** = 7, so it is possible for ***r*** to exceed ***n*** when repetitions are allowed)

Example 1.37

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in C (20 + 12 - 1, 12) = C(31, 12) = 141,120,525 ways. (Here ***n*** = 20, ***r*** = 12.)

Example 1.38

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them $1000 in Christmas bonus checks, where each check will be written for a multiple of $100.

a) Allowing the situation in which one or more of the vice presidents get nothing, President Helen is making a selection of size 10 (one for each unit of \$100) from a collection of size 4 (four vice presidents), with repetition. This can be done in C(4 + 10 - l, 10) = C(13, 10) = 286 ways.
b) If there are to be no hard feelings, each vice president should receive at least \$ 100. With this restriction, President Helen is now faced with making a selection of size 6 (the remaining six units of \$100) from the same collection of size 4, and the choices now number C(4+6- 1,6) = C(9, 6) = 84. [For example, here the selection 2, 3, 3,4, 4, 4 is interpreted as follows: Betty does not get anything extra—for there is no 1 in the selection. The one 2 in the selection indicates that Goldie gets an additional \$100. Mary Lou receives an additional \$200 (\$100 for each of the two 3's in the selection). Due to the three 4's, Mona's bonus check will total \$100 + 3(\$100) = \$400.]

c) If, each vice president must get at least \$100 and Mona, as executive vice president, gets at least \$500, then the number of ways President Helen can distribute the bonus checks is

$$\underbrace{C(3+2-1,2)}_{\text{Mona gets exactly \$500}}+\underbrace{C(3+1-1,1)}_{\text{Mona gets exactly \$600}}+\underbrace{C(3+0-1,0)}_{\text{Mona gets exactly \$700}}=10=\underbrace{C(4+2-1,2)}_{\text{Using the technique in part (b)}}$$

Having worked examples utilizing combinations with repetition, we now consider two examples involving other counting principles as well.

Example 1.39

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

After giving each child one banana, consider the number of ways the remaining three bananas can be distributed among these four children. Table 1.7 shows four of the distributions we are considering here. For example, the second distribution in part (a) of Table 1.7—namely, 1, 3, 3—indicates that we have given the first child (designated by 1) one additional banana and the third child (designated by 3) two additional bananas. The corresponding arrangement in part (b) of Table 1.7 represents this distribution in terms of three b's and three bars.

These six symbols—three of one type (the b's) and three others of a second type (the bars)—can be arranged in 6!/(3! 3!) = C(6, 3) = C(4+3 - 1, 3) =20 ways. [Here n = 4,

r = 3.] Consequently, there are 20 ways in which we can distribute the three additional
bananas among these four children. Table 1.8 provides the comparable situation for distributing the six oranges. In this case we are arranging nine symbols—six of one type (the o's) and three of a second type (the bars). So now we learn that the number of ways we can distribute the six oranges among these four children is 9!/(6! 3!) = C(9, 6) = C(4+6 - 1, 6) = 84 ways. [Here n = 4, r = 6.] Therefore, by the rule of product, there are 20 X 84 = 1680 ways to distribute the fruit under the stated conditions.

Table 1.7

(a)	(b)
1. 1, 2, 3	**5. b | b | b |**
2. 1, 3, 3	**6. b | | b b |**
3. 3, 4, 4	**7. | | b | b b**
4. **4, 4, 4**	**8. | | | b b b**

Table 1.8

(a)	(b)
1. 1, 2, 2, 3, 3, 4	**5. 0 | 0 0 | 0 0 | 0**
2. 1, 2, 2, 4, 4, 4	**6. 0 | 0 0 | | 0 0 0**
3. 2, 2, 2, 3, 3, 3	**7. | 0 0 0 | 0 0 0 |**
4. 4, 4, 4, 4, 4, 4	**8. | | | 0 0 0 0 0 0**

Example 1.40

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in C (11 +12—1, 12) ==646,646 ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{20}{12} = 3.097 \times 10^{14}$ ways.

In the next example an idea is introduced that appears to have more to do with number theory than with combinations or arrangements. Nonetheless, the solution of this example will turn out to be equivalent to counting combinations with repetitions.

Example 1.41

Determine all integer solutions to the equation

$x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$.

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the identical pennies) from a collection of size 4 (the distinct children), so there are C(4 +7 - 1,7) = 120 solutions.

At this point it is crucial that we recognize the equivalence of me following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + ... + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In terms of distributions, part (c) is valid only when the r objects being distributed are identical and the n containers are distinct. When both the r objects and the n containers are distinct, we can select any of the n containers for each one of the objects and get n^r distributions by the rule of product.

When the objects are distinct but the containers are identical, we shall solve the problem using the Stirling numbers of the second kind (Chapter 5). For the final case, in which both objects and containers are identical, the theory of partitions of integers (Chapter 9) will provide some necessary results.

Example 1.42

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \ldots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is C(6 + 10 — 1, 10) = 3003.

We now examine two other examples related to the theme of this Section.

Our next two examples provide applications from the area of computer science. Furthermore, the second example will lead to an important summation formula that we shall use in many later chapters.

Example 1.43

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j : = 1 to i do
    for k : = 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Among the possible choices for i, j, and k (in the order i-first, j-second, k-third) that will lead to execution of the **print** statement, we list (1) 1, 1, 1; (2) 2, 1, 1; (3) 15, 10, 1; and (4) 15, 10, 7. We note that i = 10, j == 12, k = 5 is not one of the selections to be considered, because j = 12 > 10 = i; this violates the condition set forth in the second **for** loop. Each of the above four selections where the **print** statement is executed satisfies the condition $1 \leq k \leq j \leq i \leq 20$. In fact, any selection a, b, c ($a \leq b \leq c$) of size 3, with repetitions allowed, from the list 1, 2, 3,.... 20 results in one of the correct selections: here, k = a, j = b, i = c. Consequently the **print** statement is executed

$$\binom{20+3-1}{3} = \binom{22}{3} = 1540\,times$$

If there had been r (≥ 1) **for** loops instead of three, the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

Example 1.44

Here we use a program segment to derive a summation formula. In this program segment, the variables ***i, j, n,*** and counter are integer variables. Furthermore, we assume that the value of ***n*** has been set prior to this segment.

```
counter : = 0
 for i : = 1 to n do
   for j : = 1 to i do
    counter : = counter +1
```

From the results in Example 1.43, after this segment is executed the value of (the variable) *counter* will be $\binom{n+2-1}{2} = \binom{n+1}{2}$.

(This is also the number of times that the statement

(*) counter : = counter +1

is executed.)

This result can also be obtained as follows: when i := 1, then j varies from 1 to 1 and (*) is executed once; when i is assigned the value 2, then j varies from 1 to 2 and (*) is executed twice; j varies from 1 to 3 when i is assigned the value 3, and (*) is executed three times; in general, for 1 ≤ k ≤ n, when i := k, then j varies from 1 to k and (*) is executed k times. In total, the variable counter is incremented [and the statement (*) is executed] 1+2+3+…+n times.

Consequently,

$$\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

The derivation of this summation formula, obtained by counting the same result in two different ways, constitutes a combinatorial proof.

UNIT 6

PRINCIPLE OF INCLUSION AND EXCLUSION

The Principle of Inclusion and Exclusion

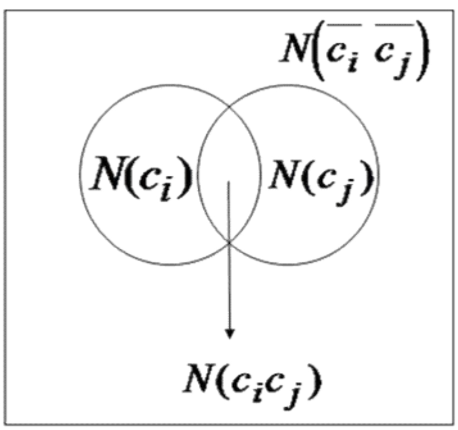
Let S be a set with $|S| = N$, and let $c_1, c_2, \ldots, c_t$ be a collection of conditions or properties satisfied by some, or all, of the elements of S. Some elements of S may satisfy more than one of the conditions, whereas others may not satisfy any of them.

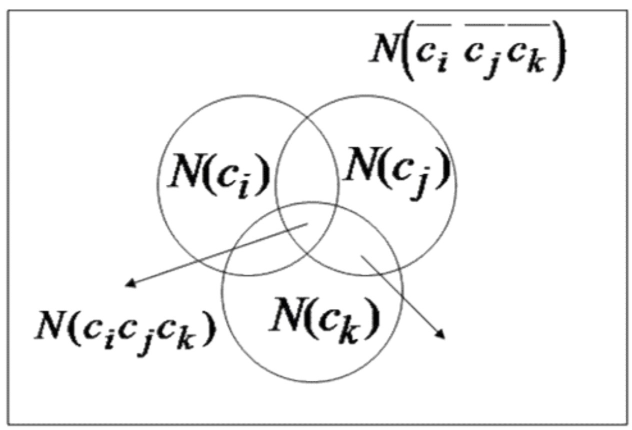
$N(c_i)$: the number of elements in S that satisfy condition c_i

$N(c_i c_j)$: the number of elements in S that satisfy both of the conditions c_i, c_j, and perhaps some others

$N(\overline{c_i}) = N - N(c_i)$

$N(\overline{c_i}\,\overline{c_j})$: the number of elements in S that do not satisfy either of the conditions c_i or c_j ($\neq N(\overline{c_i c_j})$)





$N(\overline{c_i}\ \overline{c_j}) = N - [N(c_i) + N(c_j)]$

Corollary 8.1 The number of elements in S that satisfy at least one of the conditions is N - $\overline{N}$.

Notations

$S_0 = N, S_1 = \sum N(c_i), S_2 = \sum N(c_i c_j),$

$, S_k = \sum N(c_{i_1} c_{i_2} \quad c_{i_k}), 1 \le k \le t.$

Ex. 8.1 Determine the number of positive integer n where $1 \le n \le$ 100 and n is not divisible by 2,3, or 5.

Here $S = \{1,2, \quad ,100\}$, $N = 100$, c_1 : divisible by 2, c_2 : divisible by 3, c_3 : divisible by 5.

$$\therefore N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\right) = S_0 - S_1 + S_2 - S_3 = 100 - \left(\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor\right)$$

$$+\left(\left\lfloor \frac{100}{2\times 3} \right\rfloor + \left\lfloor \frac{100}{2\times 5} \right\rfloor + \left\lfloor \frac{100}{3\times 5} \right\rfloor\right) - \left\lfloor \frac{100}{2\times 3\times 5} \right\rfloor = 26$$

$$\overline{N} = N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_i c_j) - \sum_{1\le i<j<k\le t} N(c_i c_j c_k)$$

$$+ \quad + (-1)^t N(c_1 c_2 \quad c_t)$$

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms. Consequently, x contributed a count of 1 to each side. The other possibility is that x satisfies exactly r of the conditions, $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right - hand side, x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \quad + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times.}$$

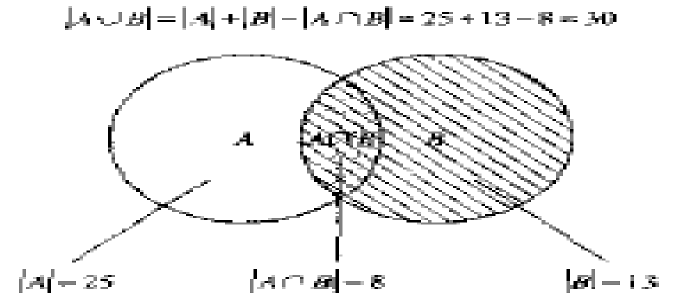


FIGURE 1 The Set of Students in a Discrete Mathematics Class.

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. From Example 2 of Section 2.3, we know that among the positive integers not exceeding 1000 there are $\lfloor 1000/7 \rfloor$ integers divisible by 7 and $\lfloor 1000/11 \rfloor$ divisible by 11. Since 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by $7 \cdot 11$. Consequently, there are $\lfloor 1000/(11 \cdot 7) \rfloor$ positive integers not exceeding 1000 that are divisible by both 7 and 11. It follows that there are

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor \\ &= 142 + 90 - 12 \\ &= 220 \end{aligned}$$

positive integers not exceeding 1000 that are divisible by either 7 or 11. This computation is illustrated in Figure 2. ■

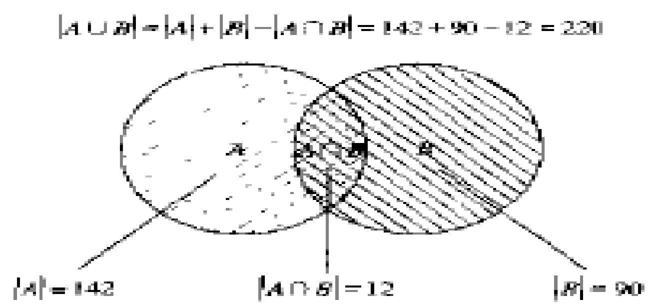


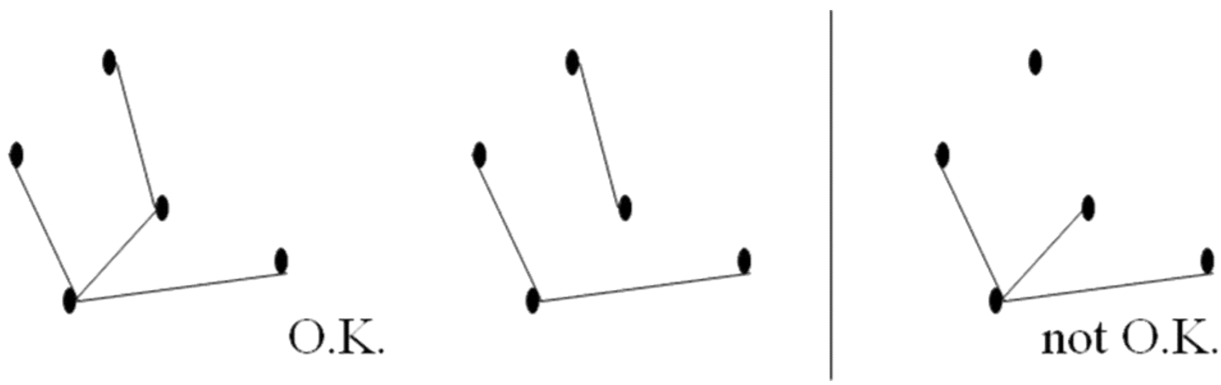
FIGURE 2 The Set of Positive Integers Not Exceeding 1000 Divisible by Either 7 or 11.

s

In general, $\Phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$, where the product is taken over all primes p dividing n. When $n = p$, a prime,

$$\Phi(n) = \Phi(p) = p\left(1 - \frac{1}{p}\right) = p - 1.$$

Ex. Construct roads for 5 villages such that no village will be isolated. In how many ways can we do this?



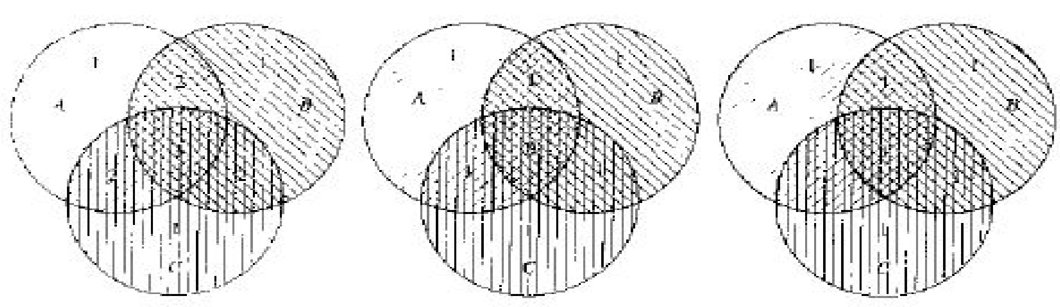
Ex. 8.6 Construct roads for 5 villages such that no village is isolated.

$N = 2^{\binom{5}{2}} = 2^{10}$. c_i : village i is isolated for $1 \le i \le 5$.

$$N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}\right) = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$

$$= 2^{10} - \binom{5}{1}2^{\binom{4}{2}} + \binom{5}{2}2^{\binom{3}{2}} - \binom{5}{3}2^{\binom{2}{2}} + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768.$$

Finding a Formula for the Number of Elements in the Union of Three Sets.



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C|$

(c) Count of elements by $|A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$$|S| = 1232, \qquad |F| = 879, \qquad |R| = 114,$$
$$|S \cap F| = 103, \qquad |S \cap R| = 23, \qquad |F \cap R| = 14,$$

and

$$|S \cup F \cup R| = 2092.$$

Inserting these quantities into the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

gives

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

Solving for $|S \cap F \cap R|$ shows that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in Spanish, French, and Russian. This is illustrated in Figure 5. ■

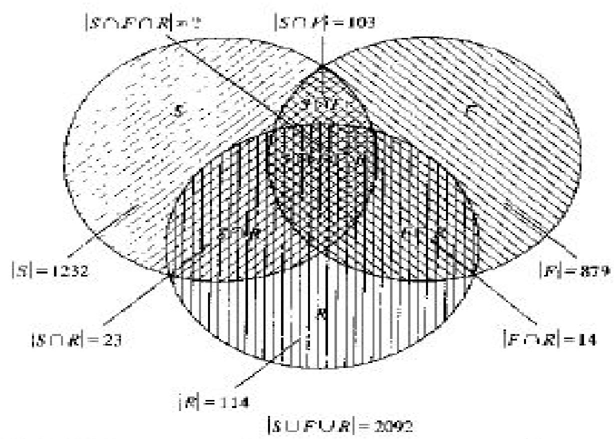


FIGURE 5 The Set of Students Who Have Taken Courses in Spanish, French, and Russian.

Give a formula for the number of elements in the union of four sets.

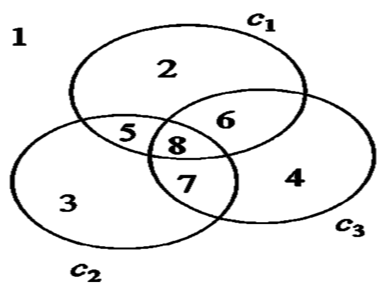
Solution: The inclusion–exclusion principle shows that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\ &\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\ &\quad - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$. ■

Generalizations of the Principle

If $m \in Z^{+}$ and $1 \le m \le t$, we now want to determine E_m, which denotes the number of elements in S that satisfy exactly m of the t conditions. (At present, we can obtain E_0)



E_1: regions 2,3,4
E_2: regions 5,6,7

$$E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$$

$$E_2 = S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$$

$$E_3 = S_3$$

Theorem 8.2 For each $1 \le m \le t$, the number of elements in S that satisfy exactly m of the conditions $c_1, c_2, \quad , c_t$ is given by

$$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \quad + (-1)^{t-m}\binom{t}{t-m}S_t$$

Proof : Let $x \in S$, consider the following three cases :

(a) x satisfies fewer than m conditions : it contributes 0 to both side

(b) x satisfies exactly m of the conditions : it contributes 1 to both side (E_m and S_m)

(c) x satisfies r of the conditions, where $m < r \le t$. Then x contributes nothing to E_m. For the right side, x is counted

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \quad +$$

$$(-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$$ times. For $0 \le k \le r - m$,

$$\binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{k!m!} \cdot \frac{r!}{(m+k)!(r-m-k)!} =$$

$$\frac{r!}{m!} \cdot \frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!} \cdot \frac{(r-m)!}{k!(r-m-k)!} = \binom{r}{m}\binom{r-m}{k}$$

Consequently, on the right hand side, x is counted

$$\binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \quad + (-1)^{r-m}\binom{r-m}{r-m}\right] =$$

$$\binom{r}{m}(1-1)^{r-m} = 0$$ times.

Let L_m denote the number of elements in S that satisfy at least m of the t conditions. Then we have :

Corollary 8.2 $L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \quad + (-1)^{t-m} \binom{t-1}{m-1} S_t.$

When $m = 1$, $L_1 = S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 - \quad + (-1)^{t-1} \binom{t-1}{0} S_t$

$= S_1 - S_2 + S_3 - \quad + (-1)^{t-1} S_t = N - \overline{N}$

How many solutions does

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers with $x_1 \le 3$, $x_2 \le 4$, and $x_3 \le 6$?

Solution: To apply the principle of inclusion–exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \le 3$, $x_2 \le 4$, and $x_3 \le 6$ is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3).$$

Using the same techniques as in Example 6 of Section 4.6, it follows that

- N = total number of solutions $= C(3 + 11 - 1, 11) = 78$,
- $N(P_1)$ = (number of solutions with $x_1 \ge 4$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$,
- $N(P_2)$ = (number of solutions with $x_2 \ge 5$) $= C(3 + 6 - 1, 6) = C(8, 6) = 28$,
- $N(P_3)$ = (number of solutions with $x_3 \ge 7$) $= C(3 + 4 - 1, 4) = C(6, 4) = 15$,
- $N(P_1P_2)$ = (number of solutions with $x_1 \ge 4$ and $x_2 \ge 5$) $= C(3 + 2 - 1, 2) = C(4, 2) = 6$,
- $N(P_1P_3)$ = (number of solutions with $x_1 \ge 4$ and $x_3 \ge 7$) $= C(3 + 0 - 1, 0) = 1$,
- $N(P_2P_3)$ = (number of solutions with $x_2 \ge 5$ and $x_3 \ge 7$) $= 0$,
- $N(P_1P_2P_3)$ = (number of solutions with $x_1 \ge 4$, $x_2 \ge 5$, and $x_3 \ge 7$) $= 0$.

Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ shows that the number of solutions with $x_1 \le 3$, $x_2 \le 4$, and $x_3 \le 6$ equals

$$N(P_1'P_2'P_3') = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6.$$ ∎

Derangements: Nothing Is in Its Right Place

A **derangement** is a permutation of objects that leaves no object in its original position.

Theorem:

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof: Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$, or

$$D_n = N(P'_1 P'_2 \cdots P'_n).$$

Using the principle of inclusion–exclusion, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \cdots P_n),$$

where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fix at least three elements, and so on. All the quantities that occur on the right-hand side of this equation will now be found.

First, note that $N = n!$, since N is simply the total number of permutations of n elements. Also, $N(P_i) = (n-1)!$. This follows from the product rule, since $N(P_i)$ is the number of permutations that fix element i, so that the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_i P_j) = (n-2)!,$$

since this is the number of permutations that fix elements i and j, but where the other $n-2$ elements can be arranged arbitrarily. In general, note that

$$N(P_{i_1} P_{i_2} \cdots P_{i_m}) = (n-m)!,$$

because this is the number of permutations that fix elements $i_1, i_2, \ldots, i_m$, but where the other $n-m$ elements can be arranged arbitrarily. Because there are $C(n, m)$ ways to choose m elements from n, it follows that

$$\sum_{1 \le i \le n} N(P_i) = C(n, 1)(n-1)!,$$

$$\sum_{1 \le i < j \le n} N(P_i P_j) = C(n, 2)(n-2)!,$$

and in general,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} N(P_{i_1} P_{i_2} \cdots P_{i_m}) = C(n, m)(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$D_n = n! - C(n, 1)(n-1)! + C(n, 2)(n-2)! - \cdots + (-1)^n C(n, n)(n-n)!$$
$$= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!.$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right].$$ □

Ex. Find the number of permutations such that 1 is not in the first place, 2 is not in the second place, ..., and 10 is not in the tenth place. (derangements of 1,2,3,...,10)

c_i : i is in the ith place for $1 \le i \le 10$

$$d_{10} = N(\overline{c_1}\,\overline{c_2} \cdots \overline{c_{10}}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \cdots + \binom{10}{10}0!$$

$$= 10!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\right) \approx 10!\, e^{-1} \text{ since}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex. Assign 7 books to 7 reviewers two times such that everyone gets a different book the second times.
Ans: first time 7!, second time d_7

therefore, 7! d_7

$$n! = \binom{n}{0} d_0 + \binom{n}{1} d_1 + \binom{n}{2} d_2 + \quad + \binom{n}{n} d_n = \sum_{k=0}^{n} \binom{n}{k} d_k$$

$$d_k = \text{the number of derangements of } 1,2, \quad ,k;\ d_0 = 1$$

Rook Polynomials

In combinatorial mathematics, a **rook polynomial** is a generating polynomial of the number of ways to place non-attacking rooks on a **board** that looks like a checkerboard; that is, no two rooks may be in the same row or column. The board is any subset of the squares of a rectangular board with *m* rows and *n* columns; we think of it as the squares in which one is allowed to put a rook. The board is the ordinary chessboard if all squares are allowed and $m = n = 8$ and a chessboard of any size if all squares are allowed and $m = n$. The coefficient of x^k in the rook polynomial $R_B(x)$ is the number of ways k rooks, none of which attacks another, can be arranged in the squares of B. The rooks are arranged in such a way that there is no pair of rooks in the same row or column. In this sense, an arrangement is the positioning of rooks on a static, immovable board; the arrangement will (usually) be different if the board is rotated or reflected.

The term "rook polynomial" was coined by John Riordan. Despite the name's derivation from chess, the impetus for studying rook polynomials is their connection with counting permutations with restricted position. A board B that is a subset of the $n \times n$ chessboard corresponds to permutations of n objects, which we may take to be the numbers 1, 2, ..., n, such that the number a_j in the j-th position in the permutation must be the column number of an allowed square in row j of B. Famous examples include the number of ways to place n non-attacking rooks on:

- an entire $n \times n$ chessboard, which is an elementary combinatorial problem;
- the same board with its diagonal squares forbidden; this is the derangement or "hat-check" problem;
- the same board without the squares on its diagonal and immediately above its diagonal (and without the bottom left square), which is essential in the solution of the problème des ménages.

Interest in rook placements, i.e., in permutations with restricted position, arises from pure and applied combinatorics, group theory, number theory, and statistical physics. The particular value of rook polynomials comes from the utility of the generating function approach, and also from the fact that the zeroes of the rook polynomial of a board provide valuable information about its coefficients, i.e., the number of non-attacking placements of k rooks.

Definition

The **rook polynomial** of a board B, $R_B(x)$, is the generating function for the numbers of arrangements of non-attacking rooks:

$$R_B(x) = \sum_{k=0}^{\infty} r_k(B) x^k$$

where r_k is the number of ways to place k non-attacking rooks on the board. Despite the notation, this is a finite sum, since the board is finite so there is a maximum number of non-attacking rooks it can hold; indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.

i. Complete boards

The first few rook polynomials on square $n \times n$ boards are (with $R_n = R_B$):

$$R_1(x) = x + 1$$
$$R_2(x) = 2x^2 + 4x + 1$$
$$R_3(x) = 6x^3 + 18x^2 + 9x + 1$$
$$R_4(x) = 24x^4 + 96x^3 + 72x^2 + 16x + 1.$$

In words, this means that on a 1 × 1 board, 1 rook can be arranged in 1 way, and zero rooks can also be arranged in 1 way (empty board); on a complete 2 × 2 board, 2 rooks can be arranged in 2 ways (on the diagonals), 1 rook can be arranged in 4 ways, and zero rooks can be arranged in 1 way; and so forth for larger boards.

For complete $m \times n$ rectangular boards $B_{m,n}$ we write $R_{m,n} := R_{B_{m,n}}$. The smaller of m and n can be taken as an upper limit for k, since obviously $r_k = 0$ if $k > \min(m,n)$. This is also shown in the formula for $R_{m,n}(x)$.

The rook polynomial of a square chessboard is closely related to the generalized Laguerre polynomial $L_n^{\alpha}(x)$ by the identity:

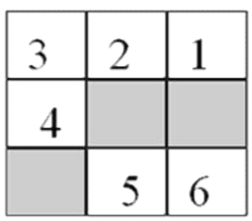
$$R_{m,n}(x) = n! x^n L_n^{(m-n)}(-x^{-1}).$$

ii. Matching polynomials

A rook polynomial is a special case of one kind of matching polynomial, which is the generating function of the number of k-edge matchings in a graph.

The rook polynomial $R_{m,n}(x)$ corresponds to the complete bipartite graph $K_{m,n}$. The rook polynomial of a general board B ⊆ $B_{m,n}$ corresponds to the bipartite graph with left vertices v_1, v_2, ..., v_m and right vertices w_1, w_2, ..., w_n and an edge v_iw_j whenever the square (i, j) is allowed, i.e., belongs to B. Thus, the theory of rook polynomials is, in a sense, contained in that of matching polynomials.

We deduce an important fact about the coefficients r_k, which we recall give the number of non-attacking placements of k rooks in B: these numbers are unimodular, i.e., they increase to a maximum and then decrease. This follows (by a standard argument) from the theorem of Heilmann and Lieb about the zeroes of a matching polynomial (a different one from that which corresponds to a rook polynomial, but equivalent to it under a change of variables), which implies that all the zeroes of a rook polynomial are negative real numbers.



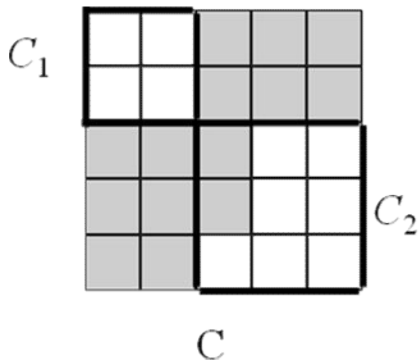
Determine the number of ways in which k rooks can be placed on the chessboard so that no two of them can take each other, i.e., no two of them are in the same row or column of the chessboard. Denote this number by r_k or $r_k(C)$.

$$r_1 = 6, r_2 = 8, r_3 = 2, r_k = 0, \text{ for } k \geq 4$$

With $r_0 = 1$, the rook polynomial

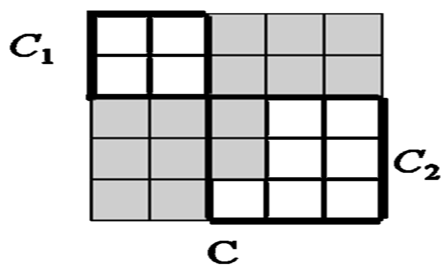
$$r(C,x) = 1 + 6x + 8x^2 + 2x^3$$

idea: break up a large board into smaller subboards



Did this occur by luck or is something happening here that we should examine more closely?

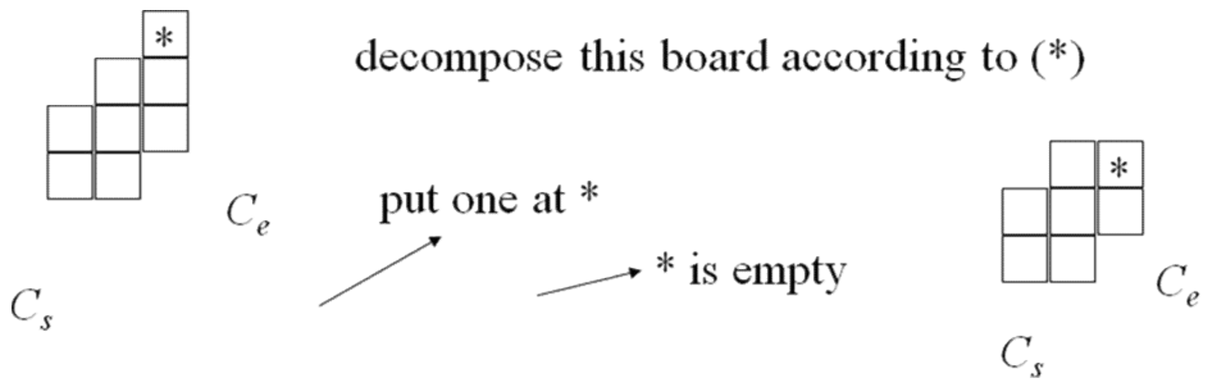
$$r(C_1, x) = 1 + 4x + 2x^2, r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$
$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x)$$



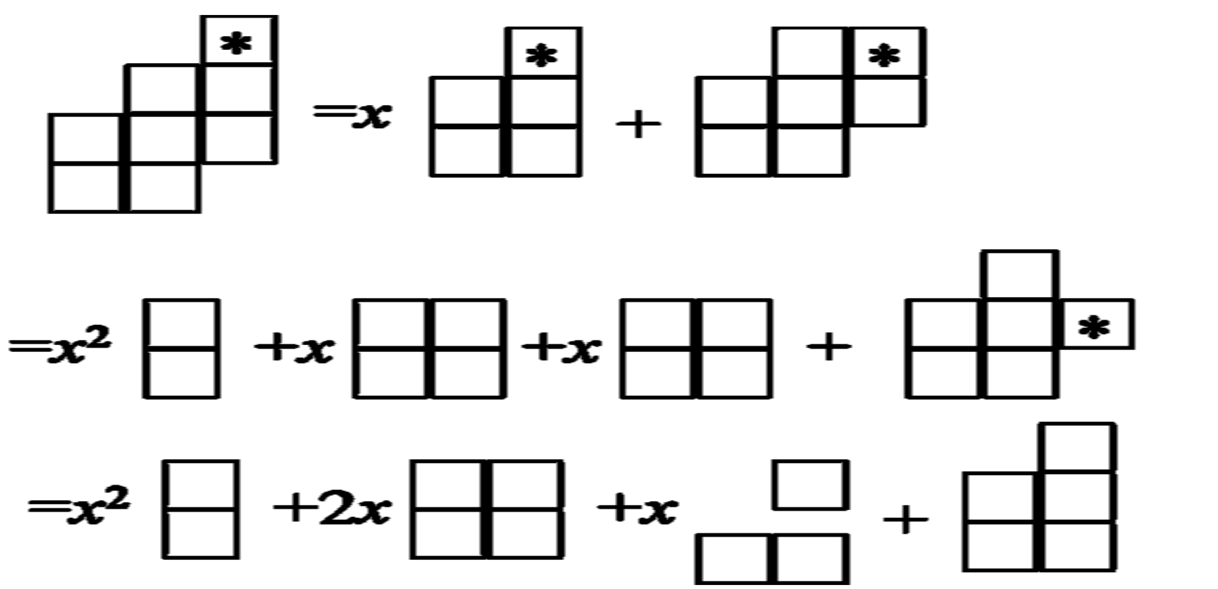
To obtain r_3 for C:
(a) All three rooks are on C_2:(2)(1)=2 ways
(b) Two on C_2 and one on C_1:(10)(4)=40
(c) One on C_2 and two on C_1:(7)(2)=14
total=(2)(1)+(10)(4)+(7)(2)=56

In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2, \quad , C_n$, then $r(C, x) = r(C_1, x) r(C_2, x) \quad r(C_n, x)$.

$$r(C_1, x) = 1 + 4x + 2x^2, r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$
$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x)$$



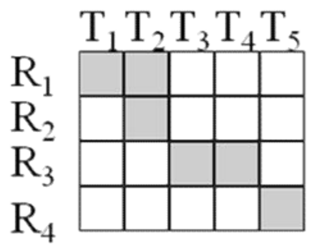
$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$
$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k$$
$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k$$
$$1 + \sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k + 1$$
$$r(C,x) = x \cdot r(C_s,x) + r(C_e,x)$$



$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2) +$$
$$[x(1+2x) + (1+4x+2x^2)] = 1 + 8x + 16x^2 + 7x^3$$

Arrangements with Forbidden Positions

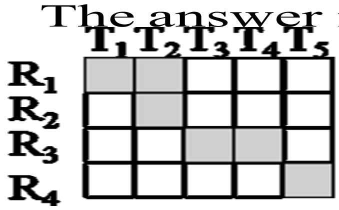
Ex. Arrange 4 persons to sit at five tables such that each one sits at a different table and with the following conditions satisfied:
(a) R_1 will not sit at T_1 or T_2 (b) R_2 will not sit at T_2
(c) R_3 will not sit at T_3 or T_4 (b) R_4 will not sit at T_4 or T_5



condition c_i: R_i is in a forbidden position

It would be easier to work with the shaded area since it is less than the unshaded one.

The answer is $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = S_0 - S_1 + S_2 - S_3 + S_4$.



condition c_i: R_i is in a forbidden position

condition c_i: R_i is in a forbidden position

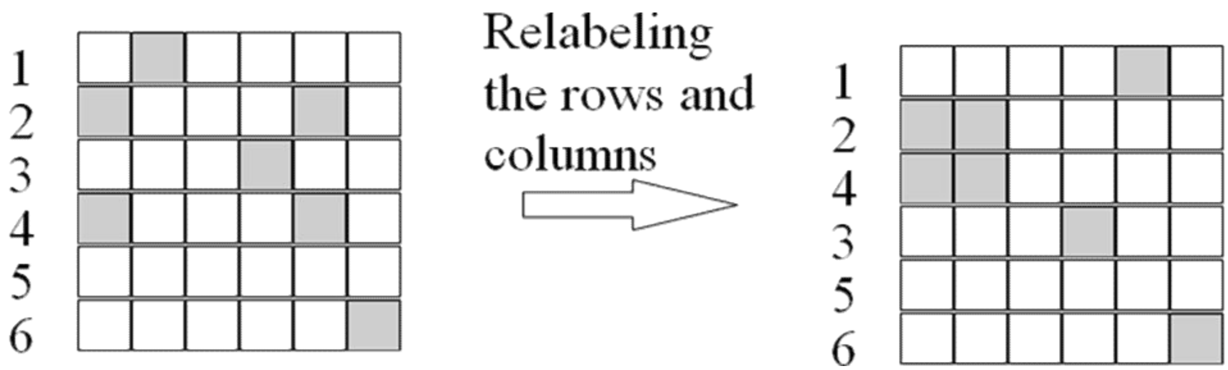
The answer is $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = S_0 - S_1 + S_2 - S_3 + S_4$.

$S_0 = P(5,4) = 5!$, $S_i = r_i(5-i)!$, where r_i is the number of ways in which it is possible to place i nontaking rooks on the shaded chessboard.

$$r(C,x) = (1+3x+x^2)(1+4x+3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

So $N\left(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\right) = \sum_{i=0}^{4} (-1)^i r_i (5-i)! = 25$

Ex. We have a pair of dice; one is red, the other green. We roll these dice six times. What is the *probability* that we obtain all six values on both the red dice and the green die if we know that the ordered pairs (1,2), (2,1),(2,5),(3,4),(4,1),(4,5), and (6,6) did not occur? [(x,y) indicates x on the red die and y on the green.]



For chessboard C of seven shaded squares,

$$r(C,x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$$

c_i: the condition where, having rolled the dice six times, we find that all six values occur on both the red die and the green die, *but i on the red die is paired with one of the forbidden numbers on the green die*

Then the number of ordered sequences of the six rolls of the dice for the event we are interested in is:

$$6!\,N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}\,\overline{c_5}\,\overline{c_6}) = 6!\sum_{i=0}^{6}(-1)^i S_i =$$

$$6!\sum_{i=0}^{6}(-1)^i r_i (6-i)! = 138{,}240$$

The probability is $138240/(29)^6 \approx 0.00023$

UNIT 7

GENERATING FUNCTIONS

Consider the Problem.

Mildred buys 12 oranges for her children Grace, Mary, and Frank. In how many ways she can distribute oranges so that Grace gets at least four, Mary and Frank gets at least two, but Frank gets no more than five?

The following table lists all possible distributions.

G	M	F	G	M	F	G	M	F
4	3	5	5	3	4	6	4	2
5	4	4	5	4	3	7	2	3
6	5	3	5	5	2	7	3	2
7	6	2	6	2	4	8	2	3
5	2	5	6	3	3			

We see that we have all the integer solutions to the equation

Considering the first two cases in this table, we find the solutions

$4 + 3 + 5 = 12$ *and* $4 + 4 + 4 = 12$.

When we multiply three polynomials

$$\left(x^4 + x^5 + x^6 + x^7 + x^8\right)\left(x^2 + x^3 + x^4 + x^5 + x^6\right)$$
$$\left(x^2 + x^3 + x^4 + x^5\right) \ldots\ldots\ (1)$$

Two of the ways to obtain x^{12} are as follows;

1.From the product $x^4x^3x^5$, where x^4 is taken from $(x^4 + x^5 + x^6 + x^7 + x^8)$ and x^3 is taken from $(x^2 + x^3 + x^4 + x^5 + x^6)$ *and* x^3 *from* $(x^2 + x^3 + x^4 + x^5)$.

2. From the product $x^4x^4x^4$,where first x^4 is found in first polynomial, the second x^4 is found in second and third x^4 in third polynomial.

Examining the eqn(1) in previous slide more closely, we realise that we obtain the product $x^ix^jx^k$ for every triplet (i, j, k) that appears in the table of possible solutions. Consequently the coefficient of x^{12} in the $f(x)$ counts the number of distributions which is 14.

$$f(x) = \left(x^4 + x^5 + x^6 + x^7 + x^8\right)\left(x^2 + x^3 + x^4 + x^5 + x^6\right)$$
$$\left(x^2 + x^3 + x^4 + x^5\right) \ldots\ldots\ (2)$$

[*The function f(x) is called generating function for distribution*]

The factor $(x^4 + x^5 + x^6 + x^7 + x^8)$ indicates that we can give 4 or 5 or 6 or 7 or 8 of the oranges to Grace. The coefficient of each x is one because oranges are identical objects

and there is only one way to distribute four oranges to Grace and one to give five oranges and so on. Since Mary and Frank must get at least two oranges each, the other terms *($x^2 + x^3 + x^4 + x^5 + x^6$) and ($x^2 + x^3 + x^4 + x^5$)* start with x^2 and for frank we stop at x^5 so that he does not receive more than f oranges.
The same can be modeled as under also.
Find the number of integer solutions to

$$c_1 : x^4 + x^5 + x^6 + x^7 + x^8 = c_1(x)$$
$$c_2 : x^2 + x^3 + x^4 + x^5 + x^6 = c_2(x)$$
$$c_3 : x^2 + x^3 + x^4 + x^5 = c_3(x)$$

The coefficient of x^{12} in $f(x) = c_1(x)c_2(x)c_3(x)$,
which is 14, is the answer.
$f(x)$ is called a *generating function* for the distributions.

Example:

If there are at least 24 number of red, green, white and black jelly colors beans, in how many ways can Douglas select 24 of these candies so that he has even number of white beans and at least six black ones?

The polynomials associated with colors are as following :

1. red : $1 + x + x^2 + ... + x^{24}$, where leading 1 is for $1x^0$, because one possibility for the red is that none is selected.
2. Green : $1 + x + x^2 + ... + x^{24}$, where leading 1 is for $1x^0$, because one possibility for the green is that none is selected.
3. white : $(1 + x^2 + x^4 + x^6 ...... + x^{24})$
4. *black* : $(x^6 + x^7 + x^8 + ... + x^{24})$

One such selection is five red, three green, eight white and eight black jelly. This arises from x^5 in the first factor, x^3 in the second factor, x^8 in the third factor and X^8 in the forth factor.

Example: How many integer solutions are there for the equation?

$$c_1 + c_2 + c_3 + c_4 = 25, \quad 0 \le c_i, 1 \le i \le 4?$$

For each c_i, the possibility can be described by
$1 + x + x^2 + \quad + x^{25}$. Then the answer is the coefficient of x^{25} in the generating function :

$$f(x) = \left(1 + x + x^2 + \quad + x^{25}\right)^4 \text{ or}$$

$$g(x) = \left(1 + x + x^2 + \quad + x^{25} + x^{26} + \quad\right)^4 = \frac{1}{(1-x)^4} = (1-x)^{-4}$$

Example:

Determine the generating function for the n-combinations of apples, bananas,Oranges and pears where in each n-combination the number of apples is Even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear. The problem is finding the number of nonnegative integral solutions of $e_1+e_2+e_3+e_4=n$ where e_1 is even that counts number of apples, e_2 is odd that counts number of bananas, $0 \le e_3 \le 4$ that counts number of oranges, and $e_4 \ge 1$ that counts number of pears. Create one factor for each type of fruit where the exponents are allowable number's in the n-combinations for that type of fruit.

$$g(x) = (1 + x^2 + x^4 + ...)(x + x^3 + x^5 + ...)(1 + x + x^2 + x^3 + x^4 ...)$$
$$(x + x^2 + x^4 + ...).$$

Where the first factor corresponds to apples, second for bananas, third for oranges and fourth for pears and

$$1 + x^2 + x^4 + ... = \frac{1}{1 - x^2}$$

$$x + x^3 + x^5 + ... = x(1 + x^2 + x^4 + ...) = \frac{x}{1 - x^2}$$

$$1 + x + x^2 + x^3 + x^4 ... = \frac{1 - x^5}{1 - x}$$

$$x + x^2 + x^4 + ... = \frac{x}{1 - x}$$

Thus,

$$g(x) = \frac{1}{1 - x^2} . \frac{x}{1 - x^2} . \frac{1 - x^5}{1 - x} . \frac{x}{1 - x}$$
$$= \frac{x^2(1 - x^5)}{(1 - x^2)^2 (1 - x)^2}$$

Hence the coefficients in the Taylor series for this rational function count the number of combinations of the type considered.

Example:

If e_k represents the number of ways to make change for k rupees, using Rs.1, Rs.2, Rs.5, Rs.10, and Rs.100, find the generating function for e_k.

$$f(x) = (\text{Rs}\,1\text{ factor})(\text{Rs}\,2\text{ factor})(\text{Rs}\,5\text{ factor})(\text{Rs}\,10\text{ factor})(\text{Rs}\,100\text{ factor})$$
$$= (1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + ...)$$
$$(1 + x^5 + x^{10} + x^{15} + ...)(1 + x^{10} + x^{20} + ...)$$
$$(1 + x^{100} + x^{200} + x^{300} + ...)$$
$$= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{10}}\right)\left(\frac{1}{1-x^{100}}\right)$$

Definition.

Let $a_0, a_1, a_2,$ be a sequence of real numbers. Thefunction

$$f(x) = a_0 + a_1x + a_2x^2 + \quad = \sum_{i=0}^{\infty} a_i x^i$$

is called the generating function for the given sequence.

For any $n \in Z^+$,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{n}x^n$$

so $(1+x)^n$ is the generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \quad , \binom{n}{n}, 0, 0, 0, \quad .$$

(a) For $n \in Z^+$, $(1 - x^{n+1}) = (1-x)(1 + x + \quad + x^n)$.

So $\dfrac{1-x^{n+1}}{1-x}$ is the generating function for $\underbrace{1,1, \quad 1}_{n+1\,1's},0,0, \quad .$

(b) If $n \to \infty$ and $|x| < 1$, $1 = (1-x)(1 + x + x^2 + x^3 + \quad)$.

So $\dfrac{1}{1-x}$ is the generating function for $1,1,1, \quad .$

(c) with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{i=0}^{\infty} x^i$$

taking the derivative,

$$\frac{d}{dx}\frac{1}{1-x} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$= \frac{d}{dx}(1 + x + x^2 + x^3 + \quad) = 1 + 2x + 3x^2 + 4x^3 +$$

Consequently, $\frac{1}{(1-x)^2}$ is the generating function of 1,2,3, , while

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + ...$$

is the generating function for the sequence 0, 1, 2, 3, 4, ...

(d) Continuing from part c,

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}\left(0 + x + 2x^2 + 3x^3 + 4x^4 + ...\right),$$

or

$$\frac{x+1}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + ...$$

hence

$$\frac{x+1}{(1-x)^3} \quad \text{generates } 1^2, 2^2, 3^2, 4^2, ...$$

and

$$\frac{x(x+1)}{(1-x)^3} \quad \text{generates } 0^2, 1^2, 2^2, 3^2, 4^2, ...$$

(b) Find the generating function for 0,2,6,12,20,30,42,

$a_0 = 0 = 0^2 + 0,$ $a_1 = 2 = 1^2 + 1,$

$a_2 = 6 = 2^2 + 2,$ $a_3 = 12 = 3^2 + 3,$

$a_4 = 20 = 4^2 + 4,$.

In general, we have $a_n = n^2 + n$, for each $n \geq 0$.

Therefore, the generating function is

$$= \frac{x(1+x)}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{x(x+1) + x(1-x)}{(1-x)^3}$$

$$= \frac{2x}{(1-x)^3}$$

Extension of binomial coefficient :

For each n belongs to Z^+, the binomial theorem tells that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{n}x^n$$

We want to extend this idea where a) n<0 and b) n is not necessarily integer

With $n, r \in Z^+$ and $n \geq r > 0$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2) \quad (n-r+1)}{r!}.$$

If $n \in R$, we use $\frac{n(n-1)(n-2) \quad (n-r+1)}{r!}$ as the definition

of $\binom{n}{r}$. For example, if $n \in Z^+$, we have $\binom{-n}{r} =$

$$\frac{(-n)(-n-1)(-n-2) \quad (-n-r+1)}{r!} = \frac{(-1)^r (n)(n+1) \quad (n+r-1)}{r!}$$

$= (-1)^r \binom{n+r-1}{r}$. And for any real n, define $\binom{n}{0} = 1$.

Ex. For $n \in Z^+, (1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r$

$$= \sum_{r=0}^{\infty} \binom{-n}{r} x^r$$

Ex. Find the coefficient of x^5 in $(1-2x)^{-7}$.

$$\binom{-7}{5}(-2)^5 = (-1)^5 \binom{7+5-1}{5}(-32) = 14,784.$$

Ex. 9.10 Find the coefficient of x^{15} in $f(x) = (x^2 + x^3 + \quad)^4$.

$f(x) = \left[x^2(1+x+x^2+ \quad)\right]^4 = \frac{x^8}{(1-x)^4}$. The coefficient of x^7 in

$\frac{1}{(1-x)^4}$ is $\binom{-4}{7}(-1)^7 = 120$

Ex. In how many ways can we select, with repetitions allowed, r objects from n distinct objects?

For each of the n distinct objects, the geometric series $1+x+x^2+x^3+\ \ $ represents the possible choices for the object. Considering all of the n objects, the generating functions is $f(x)=(1+x+x^2+x^3+\ \ )^n$, and the required answer is the coefficient of x^r in $f(x)$. $f(x)=(1-x)^{-n}=\sum_{i=0}^{\infty}\binom{-n}{i}x^i=$ $\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i$. So the answer is $\binom{n+r-1}{r}$.

Example:

In how many ways can a police captain distribute 24 riffle shells to four police officers so each police officer gets at three shells but not more than eight shells?

The choices for the number of shells each officer receives are given by

$x^3+x^4+...+x^8$

There are four officers, so the resulting generating function is,

$f(x)=(x^3+x^4+...+x^8)^4$.

We seek the coefficient of x^{24} in *f(x)*. with

$$f(x)=(x^3+x^4+...+x^8)^4.$$

$$=x^{12}(1+x+x^2+...+x^5)^4$$

$$=x^{12}\left(\frac{1-x^6}{1-x}\right)^4,$$

the answer is the coefficient of x^{12} in

$(1-x^6)^4.(1-x)^4$ is

$$=\left[1-\binom{4}{1}x^6+\binom{4}{2}x^{12}-\binom{4}{3}x^{18}+x^{24}\right]\left[\binom{-4}{0}+\binom{-4}{1}(-x)+\binom{-4}{2}(-x)^2+...\right]$$

$$=125$$

Example:

Verify that for all $n \in Z^+$, $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.

Since $(1+x)^{2n} = [(1+x)^n]^2$, by comparison of coefficients,

the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$, must equal the

coefficient of x^n in $\left[\binom{n}{0} + \binom{n}{1}x + \quad + \binom{n}{n}x^n\right]^2$, and that is

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \quad + \binom{n}{n}\binom{n}{0}$. With $\binom{n}{r} = \binom{n}{n-r}$, the result follows.

Ex. Determine the coefficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$.

since $\frac{1}{(x-a)} = \left(\frac{-1}{a}\right)\left(\frac{1}{\left(1-\frac{x}{a}\right)}\right) = \left(\frac{-1}{a}\right)\left[1+\frac{x}{a}+\left(\frac{x}{a}\right)^2+....\right]$ for any $a \neq 0$,

we could solve this problem by finding the co-eff of x^8 in

$\frac{1}{[(x-3)(x-2)^2]}$ expressed as

$$\left(\frac{-1}{3}\right)\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+....\right]\left(\frac{1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)+\binom{-2}{2}\left(\frac{-x}{2}\right)^2+....\right].$$

Example:

Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$.

partial fraction decomposition :

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \text{ or}$$

$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3)$. By comparing coefficients, $A = 1$, $B = -1$, and $C = -1$. Hence,

$$\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} + \frac{-1}{x-2} + \frac{-1}{(x-2)^2} = \left(\frac{-1}{3}\right)\frac{1}{1-(x/3)}$$
$$+\left(\frac{1}{2}\right)\frac{1}{1-(x/2)} + \left(\frac{-1}{4}\right)\frac{1}{(1-(x/2))^2}$$

Example:

Use generating functions to determine how many four-element subsets of S={1,2,3,...,15} contain no consecutive integers.

Consider one such subset {1,3,7,10}, and write 1≤1<3<7<10≤15. We see that this set of inequalities determines the differences 1-1=0, 3-1=2, 7-3=4, 10-7=3 and 15-10=5 and these differences sum to 14.

Consider another subset {2,5,11,15}, we write 1≤2<5<11<15 ≤15; these inequalities also yield the differences 1,3,6,4 and 0, which will sum to 14.

These examples suggest us a one-to-one correspondence between four element subsets to be counted and integer solutions to *c1+c2+c3+c4 +c5 =14* where 0 ≤c1, c5 and 2 ≤c2 ,c3 , c5.. The answer is the co-eff of x14 in

$$f(x) = (1+x+x^2+x^3+....)(x^2+x^3+x^4....)^3(1+x+x^2+x^3+....)$$
$$= x^6(1-x)^{-5}.$$

This then is the co-eff of x^8 in $(1-x)^{-5}$ which is

$$\binom{-5}{8}(-1)^8 = \binom{5+8-1}{8}$$
$$= \binom{12}{8}$$
$$= 495$$

Example:

Use generating functions to determine how many four-element subsets of S={1,2,3,...,15} contain no consecutive integers.

Let $\{a_1, a_2, a_3, a_4\}$ be one such subset with $1 \le a_1 < a_2 < a_3 < a_4 \le 15$. Let $c_1 = a_1 - 1, c_i = a_i - a_{i-1}$ for $2 \le i \le 4$, and $c_5 = 15 - a_4$. Then $\sum_{i=1}^{5} c_i = 14$ with $0 \le c_1, c_5$ and $2 \le c_2, c_3, c_4$. Therefore, the answer is the coefficient of x^{14} in $f(x) = (1 + x + x^2 + \quad)^2 (x^2 + x^3 + \quad)^3 = x^6 (1-x)^{-5}$, which is $\binom{-5}{8}(-1)^8 = 495$.

Example:

$f(x) = \dfrac{x}{(1-x)^2}$ generates $0,1,2, \quad (a_0, a_1, a_2, \quad)$

and $g(x) = \dfrac{x(x+1)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, \quad (b_0, b_1, b_2, \quad)$

Then $h(x) = f(x)g(x) = \sum_{k=0}^{\infty} c_k x^k$, where

$$c_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \quad + a_{k-2} b_2 + a_{k-1} b_1 + a_k b_0$$
$$= \sum_{i=0}^{k} i(k-i)^2 = \sum_{i=0}^{k} i(k^2 - 2ki + i^2) = k^2 \sum_{i=0}^{k} i$$
$$- 2k \sum_{i=0}^{k} i^2 + \sum_{i=0}^{k} i^3 = k^2 \cdot \frac{k(k+1)}{2} - 2k \cdot \frac{k(k+1)(2k+1)}{6}$$
$$+ \left[\frac{k(k+1)}{2}\right]^2$$

. Convolution of $a_0, a_1, a_2, \quad$ and $b_0, b_1, b_2, \quad$.

Example:

Find the co-eff of x^{60} in $(x^8 + x^9 + x^{10} + ...)^7$

$$(x^8 + x^9 + x^{10} + ...)^7 = (x^8)^7 (1 + x + x^2 + x^3 + ...)^7$$
$$= x^{56} \left(\frac{1}{1-x}\right)^7$$
$$= x^{56} (1-x)^{-7}$$

so, co-eff of x^{60} in $(x^8 + x^9 + x^{10} + ...)^7$ is the co-eff of x^4 in $(1-x)^{-7}$

which is $\binom{-7}{4}(-1)^4 = (-1)^4 \binom{7+4-1}{4}(-1)^4$

$$= \binom{10}{4}$$

$$5)\ \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{i=0}^{\infty} x^i$$

$$6)\ \frac{1}{1-ax} = 1 + (ax) + (ax)^2 + (ax)^3 + ...$$

$$= \sum_{i=0}^{\infty} (ax)^i = \sum_{i=0}^{\infty} a^i x^i$$

$$= 1 + ax + a^2x^2 + a^3x^3 + ...$$

$$7)\ \frac{1}{(1+x)^n} = \binom{-n}{0} + \binom{-n}{1} x + \binom{-n}{2} x^{-2} + ... = \sum_{i=0}^{\infty} \binom{-n}{i} x^i$$

$$= 1 + (\text{-}1)\binom{n+1-1}{1} x + (-1)^2 \binom{n+2-1}{2} x^2 + ... = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$$

$$8)\ \frac{1}{(1-x)^n} = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + ...$$

$$= \sum_{\text{i}=0}^{\infty} \binom{-n}{i} (-x)^i$$

$$= 1 + (\text{-}1)\binom{n+1-1}{1}(-x) + (-1)^2 \binom{n+2-1}{2}(-x)^2 + ...$$

$$= \sum_{\text{i}=0}^{\infty} \binom{n+i-1}{i} (-x)^i$$

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$$= \mathbf{x^{12}}\left(\frac{1-x^6}{1-x}\right)^4,$$

the answer is the coefficient of x^{12} in $(1 - x^6)^4.(1-x)^{-4}$ is

$$= \left[1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + x^{24}\right].$$
$$\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + ...\right]$$
$$= 125$$

Example:

In how many ways can we select, with repetitions allowed, r objects from n distinct objects?

For each of the n distinct objects, the geometric series $1 + x + x^2 + x^3 +$ represents the possible choices for the object. Considering all of the n objects, the generating function is $f(x) = (1 + x + x^2 + x^3 + \quad)^n$, and the required answer is the coefficient of x^r in $f(x)$.

$$f(x) = (1 - x)^{-n} = \sum_{i=0}^{\infty}\binom{-n}{i}x^i =$$

$\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i$. So the answer is $\binom{n+r-1}{r}$.

Ex. Verify that for all $n \in Z^+$, $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$.

Since $(1+x)^{2n} = [(1+x)^n]^2$, by comparison of coefficients, the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$, must equal the coefficient of x^n in $\left[\binom{n}{0} + \binom{n}{1}x + \quad + \binom{n}{n}x^n\right]^2$, and that is $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \quad + \binom{n}{n}\binom{n}{0}$. With $\binom{n}{r} = \binom{n}{n-r}$, the result follows.

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partial fraction decomposition :

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \qquad \text{or}$$

$$1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3). \qquad \text{or}$$

$$0.x^2 + 0.x + 1 = (A+B)x^2 + (-4A+5B+C)x + (4A+6B-3C).$$

By comparing coefficients(for x^2, x, and 1 respectively),

$A + B = 0,\ -4A + 5B + C = 0,\ 4A + 6B - 3C = 1$

solving these equations we get,

$A = 1, B = -1,$ and $C = -1$.

Hence, $\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} + \frac{-1}{x-2} + \frac{-1}{(x-2)^2}$

$$= \left(\frac{-1}{3}\right)\frac{1}{1-(x/3)} + \left(\frac{1}{2}\right)\frac{1}{1-(x/2)} + \left(\frac{-1}{4}\right)\frac{1}{(1-(x/2))^2}$$

$$= \left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i + \left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^i$$

$$+ \left(\frac{-1}{4}\right)\left[\binom{2}{0} + \binom{-2}{1}\left(\frac{-x}{2}\right) + \binom{-2}{2}\left(\frac{-x}{2}\right)^2 + \ldots\right]$$

The coeff of x^8 is

$$\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^8 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^8 + \left(\frac{-1}{4}\right)\binom{-2}{8}\left(\frac{-1}{2}\right)^8$$

$$= -\left[\left(\frac{1}{3}\right)^9 + 7\left(\frac{1}{2}\right)^{10}\right].$$

Another solution is,

$$\text{Since } \frac{1}{\text{x - a}} = \frac{-1}{a}.\left(\frac{1}{\left(1-\frac{x}{a}\right)}\right) = \left(\frac{\text{-1}}{\text{a}}\right)\left[1+\left(\frac{x}{a}\right)+\left(\frac{x}{a}\right)^2+...\right]$$

for a $\neq$ 0, we could slove this by finding

the coeff of x^8 in $\frac{1}{(\text{x - 3})(\text{x - 2})^2}$ expressed as

$$\left(\frac{-1}{3}\right)\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+...\right].\left(\frac{1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)+\binom{-2}{2}\left(\frac{-x}{2}\right)^2+...\right]$$

Example:

. Use generating functions to determine how many four-element subsets of *S*={1,2,3,...,15} contain no consecutive integers.

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$$(1+x+x^2+x^3+....)$$
$$= \mathbf{x^6}(1-x)^{-5}.$$

This then is the co - eff of x^8 in $(1-x)^{-5}$ which is

$$\binom{-5}{8}(-1)^8 = \binom{5+8-1}{8} = \binom{12}{8} = 495$$

Let $h(x) = \frac{x}{(1-x)^2}$. This is the gen. fn for $a_0, a_1, a_2...$

where $a_k = k$ for all $k \in N$.

Let $g(x) = \frac{x(x+1)}{(1-x)^3}$. This is the gen. fn for $b_0, b_1, b_2...$

where $b_k = k^2$ for all $k \in N$.

and the function,

$h(x) = f(x)g(x)$ gives us

$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_2)x^2 + ...$

which is the generating function for $c_0, c_1, c_2,...$ where

$c_k = a_0b_k + a_1b_{k-1} + a_2b_{k-2} + ...a_{k-2}b_2 + a_{k-1}b_1 + a_kb_0.$

The sequence $c_0, c_1, c_2,...$ is called convolution of sequences

$a_0, a_1, a_2...$ and $b_0, b_1, b_2...$

Example. Find the convolution of the sequences 1, 1, 1, 1, ….. and 1,-1,1,-1,1,-1,….

let $f(x) = \frac{1}{(1-x)}$

$= 1 + x + x^2 + x^3 + ...$

= generates the sequence 1,1,1,1,...

let $g(x) = \frac{1}{(1+x)}$

$= 1 - x + x^2 - x^3 + ...$

= generates the sequence 1,-1,1,-1,...

Then

$f(x)g(x) = \frac{1}{(1-x)} . \frac{1}{(1+x)}$

$= \frac{1}{(1-x^2)}$

$= 1 + x^2 + x^4 + x^6 .....$

= generates the sequence 1,0,1,0,1....

= which is convolution of 1,1,1,1,... and 1,-1,1,-1,...

Example : Find the co - eff of $\mathbf{x^{60}}$ in $\left(\mathbf{x^8} + x^9 + x^{10} + ...\right)^7$

$$\left(\mathbf{x^8} + x^9 + x^{10} + ...\right)^7 = (x^8)^7\left(1 + x + x^2 + x^3 + ...\right)^7$$

$$= \mathrm{x}^{56}\left(\frac{\mathbf{1}}{\mathbf{1-x}}\right)^7$$

$$= \mathrm{x}^{56}\left(1 - x\right)^{-7}$$

so, co - eff of x^{60} in $\left(\mathbf{x^8} + x^9 + x^{10} + ...\right)^7$ is the co - eff of x^4 in $\left(1 - \mathrm{x}\right)^{-7}$

which is $\binom{\mathbf{-7}}{\mathbf{4}}(-\mathbf{1})^{\mathbf{4}} = (-\mathbf{1})^{\mathbf{4}}\binom{\mathbf{7+4-1}}{\mathbf{4}}(-\mathbf{1})^{\mathbf{4}}$

$$= \binom{\mathbf{10}}{\mathbf{4}}$$

Example: Determine the co- eff of x^0 in $(4x^3 - 5/x)^{16}$

The term x^k in the binomial expansion $(\mathrm{x} + \mathrm{y})^k$

is $\binom{\mathrm{n}}{\mathrm{n-k}} x^k y^{\mathrm{n-k}}$,

Replace x by $4\mathrm{x}^3$ and y by $\left(\frac{-5}{\mathrm{x}}\right)$ and $\mathrm{n} = 16$, we get,

$$\binom{16}{16-k}\left(4x^3\right)^k\left(\frac{-5}{x}\right)^{16-k}$$

$$= \binom{16}{16-k}(4)^k.(-5)^{16-k}.x^{3k}.x^{16-k}$$

For constant term (with x^0) we must have,

$$\mathrm{x}^{3\mathrm{k}}.x^{16-k} = x^0$$

Therefore, $3\mathrm{k} + 16 - \mathrm{k} = 0$,

Thus $\mathrm{k} = 4$.

The constant term is,

$$\binom{16}{16\text{-}4}4^4.(-5)^{16-4}$$

$$= \binom{16}{12}4^4.5^{12}$$

Example: Determine the sequence generated by $(1 - 4x)^{-1/2}$

We know that,

$$(1-4x)^{-1/2} = \binom{-1/2}{0} + \binom{-1/2}{1}(-4x) + \binom{-1/2}{2}(-4x)^2 + \ldots$$

The coeff of x^n in $\binom{-1/2}{n}(-4)^n$

$$= \frac{((-1/2)-n+1)((-1/2)-n+2)\ldots((-1/2)-1)(-1/2)}{n!}.(-4)^n$$

$$= \frac{(1+2n-2)(1+2n-4)\ldots..(1+2)(1)}{n!}.2^n$$

$$= \frac{(2n-1)(2n-3)\ldots..5.3.1.2^n}{n!}.n! = \frac{(2n)!}{n!n!} = \binom{2n}{n} \text{ where } n \in N.$$

Example: Determine the number of ways to color squares of a 1 x n chess board using the colors, red, white, and blue, if an even number of squares are to be colored red.

Let a_n be the number of such colorings, with $a_0 = 1$.

Let a_n equals the number of n-permutations a multilist of three colors (red, white and blue), each with an infinite repetition number in which red occurs an even number of times. Thus the exponential generating sequence $a_0, a_1, a_2, .. a_n..$ is the product of red, white and blue factors:

$$= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \frac{1}{2}\left(e^x + e^{-x}\right).e^x.e^x$$

$$= \frac{1}{2}\left(e^{3x} + e^x\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} (3^n + 1).\frac{x^n}{n!}\right)$$

Hence $a_n = \frac{(3^n + 1)}{2}$ is the number of ways to color the chess board.

Partitions of Integers

Partition a positive integer *n* into positive summands and seeking the number of such partitions, without regard to order.This number is denoted by p(n).

For example, $p(1)=1$: 1

$p(2)=2$: 2=1+1

$p(3)=3$: 3=2+1=1+1+1
$p(4)=5$: 4=3+1=2+2=2+1+1=1+1+1+1
$p(5)=7$: 5=4+1=3+2=3+1+1=2+2+1=2+1+1+1
=1+1+1+1+1

We should like to obtain $p(n)$ for a given n without having to list all the partitions. We need a tool to keep track of the numbers of 1's, 2's, ..., n's that are used as summands for n.

keep track of 1's : $1+x+x^2+x^3+$

keep track of 2's : $1+x^2+x^4+x^6+$

keep track of k's : $1+x^k+x^{2k}+x^{3k}+$

For example, p(10) is the coefficient of x^{10} in

$$f(x)=(1+x+x^2+\quad)(1+x^2+x^4+\quad)\quad(1+x^{10}+\quad)$$

$$=\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\quad\frac{1}{(1-x^{10})}=\prod_{i=1}^{10}\frac{1}{(1-x^i)}.$$

In general, $P(x)=\prod_{i=1}^{\infty}\frac{1}{1-x^i}$ generate the sequence $p(0), p(1),$

Exampe: Find the generating function for the number of ways an advertising agent can purchase n minutes of air time if time slots for commercials come in blocks of 30, 60, or 120 seconds.

Let 30 seconds represent one time unit. Then the answer is the number of integer solutions to the equation $a+2b+4c=2n$ with $0\le a,b,c$. The associated generating function is

$$f(x)=(1+x+x^2+\quad)(1+x^2+x^4+\quad)(1+x^4+x^8+\quad)$$

$$=\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}$$ and the coefficient of x^{2n} is the answer.

Example: Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into distinct summands.

Let us consider 11 partitions of 6:

1)1+1+1+1+1+1	2)1+1+1+1+2	3)1+1+1+3
4)1+1+4	5)1+1+2+2	6)1+5
7)1+2+3	8)2+2+2	9)2+4
10)3+3	11)6	

Partitions 6,7,9 and 11 have distinct summands, so $P_d(6)=4$

For any $k \in Z^+$, There are two posiibilities either k is not used as a summand or it is. This can be accounted for by the polynomial $1 + x^k$. Consequently, the generating function is

$$P_d(x) = (1+x)(1+x^2)(1+x^3) \quad = \prod_{i=1}^{\infty}(1+x^i)$$

for each $n \in Z^+$, $p_d(n)$ is the coeff of x^n in $(1+x)(1+x^2)\ldots\ldots(1+x^n)$. and $p_d(0) = 1$.

when n = 6, the coeff of x^6 *in* $(1+x)(1+x^2)\ldots\ldots(1+x^6)$ is 4.

Considering the partitions, we see that there are four partitions of 6 into odd summands, namely 1, 3, 6 and 10 in the previous example. We also have $p_d(6) = 4$. Let $p_0(n)$ denote the number of partitions of n into odd summands, when $n \geq 1$. We define $p_0(0) = 1$. The generating function for the sequence $p_0(0), p_0(1), p_0(2), \ldots$ is given by

$$P_0(x) = (1+x+x^2+x^3+\ldots..)(1+x^3+x^6+\ldots.)(1+x^5+x^{10}+\ldots)(1+x^7+x^{14}+\ldots)$$

$$= \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}.\frac{1}{1-x^7}\ldots..$$

Now because,

$$1+x = \frac{1-x^2}{1-x}, \quad 1+x^2 = \frac{1-x^4}{1-x^2}, \quad 1+x^3 = \frac{1-x^6}{1-x^3}, \quad \ldots..$$

we have,

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\ldots.$$

$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\ldots.$$

$$= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\ldots.$$

$$= P_0(x)$$

From equality of generating functions,

$$p_d(n) = p_0(n), \text{ for all } n \geq 0.$$

Example: Partition into odd summands but each such odd summands must occur an odd number of times-or not at all. Here, for example, there is one such partition of integer 1, namely 1, there are no partitions of 2, there two such partitions for integer 3, namely 3 and 1+1+1. one partition for integer 4 namely 3+1. The generating function for the partitions described is given by

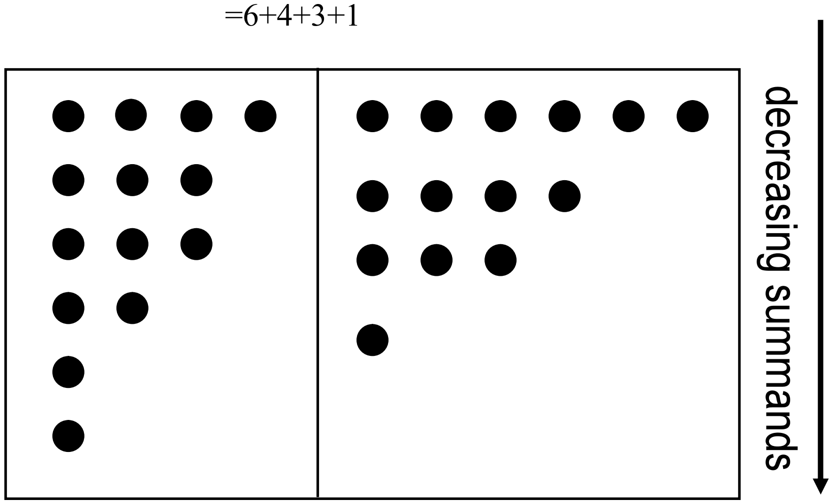
$$f(x) = (1+x+x^3+x^5+ \quad)(1+x^3+x^9+x^{15}+ \quad)(1+x^5+x^{15}+$$

$$) \quad = \prod_{k=0}^{\infty}\left(1+\sum_{i=0}^{\infty}x^{(2k+1)(2i+1)}\right).$$

Ferrer's graphHere we see a partition of 14 into summands, where 4 is the largest summand, and a second partition into exactly four summands

The number of partitions of an integer ***n* into *m* summands** is equal to the number of partitions where ***m* is the largest summands**.

14=4+3+3+2+1+1
=6+4+3+1



Counting the compositions of a positive integer n using Generating Functions

Start with,

$$\frac{1}{1-x} = x + x^2 + x^3 + x^4 + ...$$

Where , for example, the co-eff of x^4 is 1, for one summand composition of 4 namely, 4.

To obtain number of compositions of n, we need the co-eff of x^n in

$$\left(x + x^2 + x^3 + ...\right)^2 = \left[\frac{x}{(1-x)}\right]^2 = \frac{x^2}{(1-x)^2}$$

Here for instance we obtain x^4 in $(x+x^2+x^3+x^4+\dots)^2$ from products $(x^1.x^3)$, $(x^2.x^2)$, and $(x^3.x^1)$. So co-eff of x^4 in $x^2/(1-x)^2$ is 3, which is number of two summand compositions of 4), 1+3, 2+2, 3+1.

Continuing with the three summand compositions we now examine

$$\left(x + x^2 + x^3 + x^4 + ...\right)^3 = \left[\frac{x}{(1-x)}\right]^3 = \frac{x^3}{(1-x)^3}$$

Once again we look at the ways x^4 comes about – namely, from products $(x^1.x^1.x^2)$, $(x^1.x^2.x^1)$, and $(x^2.x^1.x^1)$. So here co-eff of x^4 is 3, which accounts for the three summand compositions 1+1+2, 1+2+1, and 2+1+1 (of 4).

Finally the co-eff of x^4 in below function is 1,

$$\left(x + x^2 + x^3 + x^4 + \dots\right)^4 = \left[\frac{x}{(1-x)}\right]^4 = \frac{x^4}{(1-x)^4}$$ for one four summand composition

1+1+1+1 (of 4).

These result tell us that the co-eff of x^4 in

$\sum_{i=1}^{4}\left[\frac{x}{(1-x)}\right]^i$ is 1+3+3+1 = 8 (=2^3), the number of compositions of 4. In fact this is

also the co-eff of x^4 in the above equn.

Generalizing the situation we find that the number of compositions of a positive integer n is the co-eff of x^n in the generating function

$$f(x) = \sum_{i=1}^{\infty}\left[\frac{x}{(1-x)}\right]^i \quad \text{.........(1).}$$

But if we set y=x/(1-x), then it follows that

$$f(x) = \sum_{i=1}^{\infty} y^i = y\sum_{i=0}^{\infty} y^i = y\left(\frac{1}{1-y}\right) = \left(\frac{\mathrm{x}}{(1\text{-}\mathrm{x})}\right)\left(\frac{1}{1-\left(\frac{x}{1-x}\right)}\right)$$

$$= \left(\frac{\mathrm{x}}{(1\text{-}\mathrm{x})}\right)\left(\frac{1}{\frac{1-x-x}{1-x}}\right)$$

$$= \frac{x}{(1-2x)} = \mathrm{x}\left[1 + (2\mathrm{x}) + (2x)^2 + (2x)^3 + \dots\right]$$

$$= 2^0 x + 2^1 x^2 + 2^2 x^3 + 2^3 x^4 + \dots.$$

So the number of integer compositions of a positive integer n is the co-eff of x^n in f(x) and this is 2^{n-1} as derived in the equation in previous slide.

Let us examine the identity

$\left(\frac{1-\mathrm{x}^{n+1}}{1-\mathrm{x}}\right) = 1 + x + x^2 + x^3 + \dots\dots + x^n$ When x is replaced by 2 in this the result tells

that for all n belonging to Z^+,

$$1 + 2 + 2^2 + 2^3 + \dots\dots + 2^n = \left(\frac{1-2^{n+1}}{1-2}\right) = 2^{n+1} - 1.$$ Where do we use this?

Consider the special compositions of integers 6 and 7, that read same left to right as right to left.

	6				**7**	
	1+4+1				**1+5+1**	

	2+2+2				**2+3+2**	
	1+1+2+1+1				**1+1+3+1+1**	
	3+3				**3+1+3**	
	1+2+2+1				**1+2+1+2+1**	
	2+1+1+2				**2+1+1+1+2**	
	1+1+1+1+1+1				**1+1+1+1+1+1+1**	

These are palindromes for 6 and 7. We find that for 7 there are $1+(1+2+4) = 1+(1+2^1+2^2) = 1+(2^3-1) = 2^3$ palindromes. There is one palindrome with one summand, 7. There is also one palindrome where center summand is 5 and where we place one composition of 1 on either side of this summand (palindrome 2).

For the center summand 3 we place one of the two compositions of 2 on the right and then match it on the left, with same composition, in reverse order. (palindromes 3 and 4) finally when the center summand is 1, we put a given composition of 3 on the right side f this 1 and match on left side with same composition, in reverse order. There are $2^{3-1} = 4$ compositions of 3 (palindromes 5,6,7,8).

The situation is same for palindromes of 6 except case where + sign appears as center. So for n=6,

i)Center summand 6 1 palindrome

ii)Center summand 4 $1(=2^{1-1})$ palindrome

iii) Center summand 2 $2(=2^{2-1})$ palindrome

iv) + sign at Center $4(=2^{3-1})$ palindrome

So there are $1+(1+2^1+2^2) = 1+(2^3-1)=2^3$ palindromes for 6.

Now we look at the general situation. For n=1 there is one palindrome. If n = 2k+1, for k belonging to Z^+, then there is one palindrome with center summand n. for $1 \le t \le k$, there are 2^{t-1} palindromes of n with center summand n-2t. Hence the total number of palindromes of n is

$$1+(1+2^2+2^3+\ldots\ldots+2^{k-1}) = 1+(2^k-1) = 2^k = 2^{(n-1)/2}$$

Now consider n even, say n = 2k for k belonging to Z^+.

Here there is one palindrome with center summand n-2s (one palindrome for each of 2^{s-1} compositions of s). In addition there are 2^{k-1} palindromes where a + sign is at the center (one palindrome for each of the 2^{k-1} compositions of k).In total, n has

$$1+(1+2^1+2^2+2^3+\ldots\ldots+ 2^{k-2}+2^{k-1}) = 1+(2^k-1) = 2^k = 2^{n/2}$$

Observe that for $n \in Z^+$, n has $2^{\lfloor n/2 \rfloor}$palindromes.

Partitions of Integers Partition a positive integer *n* into positive summands and seeking the number of such partitions, without regard to order. This number is denoted by p(n).

For example, $p(1)$=1: 1

$p(2)$=2: 2=1+1

$p(3)$=3: 3=2+1=1+1+1

$p(4)$=5: 4=3+1=2+2=2+1+1=1+1+1+1

$p(5)$=7: 5=4+1=3+2=3+1+1=2+2+1=2+1+1+1

=1+1+1+1+1

We should like to obtain $p(n)$ for a given n without having to list all the partitions. We need a tool to keep track of the numbers of 1's, 2's, ..., n's that are used as summands for n.

keep track of 1's : $1+x+x^2+x^3+$

keep track of 2's : $1+x^2+x^4+x^6+$

keep track of k's : $1+x^k+x^{2k}+x^{3k}+$

For example, p(10) is the coefficient of x^{10} in

$$f(x)=(1+x+x^2+\quad)(1+x^2+x^4+\quad)\quad(1+x^{10}+\quad)$$

$$=\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\quad\frac{1}{(1-x^{10})}=\prod_{i=1}^{10}\frac{1}{(1-x^i)}.$$

In general, $P(x)=\prod_{i=1}^{\infty}\frac{1}{1-x^i}$ generate the sequence $p(0), p(1),$

Example: Find the generating function for the number of ways an advertising agent can purchase n minutes of air time if time slots for commercials come in blocks of 30, 60, or 120 seconds.

Let 30 seconds represent one time unit. Then the answer is the number of integer solutions to the equation $a+2b+4c=2n$ with $0\le a,b,c$. The associated generating function is

$$f(x)=(1+x+x^2+\quad)(1+x^2+x^4+\quad)(1+x^4+x^8+\quad)=$$

$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}$ and the coefficient of x^{2n} is the answer.

Example: Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into distinct summands.

Let us consider 11 partitions of 6:

1)1+1+1+1+1+1	2)1+1+1+1+2	3)1+1+1+3
4)1+1+4	5)1+1+2+2	6)1+5
7)1+2+3	8)2+2+2	9)2+4
10)3+3	11)6	

Partitions 6,7,9 and 11 have distinct summands, so $P_d(6)=4$

For any $k \in Z^+$, There are two posiibilities
either k is not used as a summand or it is.
This can be accounted for by the polynomial $1+x^k$.
Consequently, the generating function is

$$P_d(x) = (1+x)(1+x^2)(1+x^3) \quad = \prod_{i=1}^{\infty}(1+x^i)$$

for each $n \in Z^+$, $p_d(n)$ is the coeff of x^n in
$(1+x)(1+x^2).......(1+x^n)$. and $p_d(0)=1$.
when $n = 6$, the coeff of x^6 in $(1+x)(1+x^2).......(1+x^6)$ is 4.

Considering the partitions, we see that there are four
partitions of 6 into odd summands, namely 1, 3, 6 and 10
in the previous example. We also have $p_d(6) = 4$.
let $p_0(n)$ denote the number of partitions of n into odd
summands, when $n \geq 1$. We define $p_0(0) = 1$. The generating
function for the sequence $p_0(0), p_0(1), p_0(2),....$ is given by

$$P_0(x) = (1+x+x^2+x^3+.....)(1+x^3+x^6+....)(1+x^5+x^{10}+...).$$

$$(1+x^7+x^{14}+...) = \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}.\frac{1}{1-x^7}.....$$

Now because,

$$1+x = \frac{1-x^2}{1-x}, \quad 1+x^2 = \frac{1-x^4}{1-x^2}, \quad 1+x^3 = \frac{1-x^6}{1-x^3}, \quad .....$$

we have,

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)....$$

$$= \frac{1-x^2}{1-x}.\frac{1-x^4}{1-x^2}.\frac{1-x^6}{1-x^3}.\frac{1-x^8}{1-x^4}....$$

$$= \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}....$$

$$= P_0(x)$$

From equality of generating functions,

$$p_d(n) = p_0(n), \text{ for all } n \geq 0.$$

Example: Partition into odd summands but each such odd summands must occur an odd umber of times-or not at all. Here, for example, there is one such partition of integer 1, namely, there are no partitions of 2, there are two such partitions for integer 3, namely 3 and 1+1+1. one partition for integer 4 namely 3+1. The generating function for the partitions described is given by

$$f(x) = (1 + x + x^3 + x^5 + \quad)(1 + x^3 + x^9 + x^{15} + \quad)(1 + x^5 + x^{15} + \quad) = \prod_{k=0}^{\infty}\left(1 + \sum_{i=0}^{\infty} x^{(2k+1)(2i+1)}\right).$$

Using Generating functions, we will also be able to deal with a sample space that is discrete but not finite.

Example: Suppose that Brianna takes an examination until she passes it. Further, suppose the probability that she passes the examinations on any given attempt is 0.8 and the result of each attempt, after the first, is independent of any previous attempt. If we let P denote "pass" and F denote "fail", for any given attempt, then our sample space may be expressed as

= {P, FP, FFP, FFFP,….}Where, for example, Pr(FFP) is the probability that she fails the exams is twice before she passes it, which is given by $(0.2)^2(0.8)$. In addition, the sum of probabilities for the outcomes in is Now suppose we want to know the probability she passes the exam on an even numbered attempt. That is we want Pr(A) where A is the event {FP, FFFP, ….}.

At this point we introduce the discrete random variable Y where Y counts the number of attempts up to and including the one where she passes the exam. Then the probability distribution for Y is given by $\Pr(Y=y) = (0.2)^{y-1}(0.8)$, $y \geq 1$.

So Pr(A) can be determined as follows:

$$\begin{aligned}
\Pr(A) = \sum_{i=1}^{\infty}\Pr(y=2i) = \sum_{i=1}^{\infty}(0.2)^{2i-1}(0.8) &= (0.8)\sum_{i=1}^{\infty}(0.2)^{2i-1} \\
&= (0.8)\left[(0.2) + (0.2)^3 + (0.2)^5 + \ldots.\right] \\
&= (0.8)(0.2)\left[1 + (0.2)^2 + (0.2)^3 + (0.2)^4 + \ldots.\right] \\
&= (0.8)(0.2)\frac{1}{1-(0.2)^2} \\
&= \frac{(0.8)(0.2)}{0.96} = \frac{1}{6}
\end{aligned}$$

Continuing with Y, now we'd like to find E(Y), the number of time she expects to take exam before she passes it. To determine E(Y) we'll start with the formula,

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + ...$$

taking the derivative both sides, we find that

$$(-1)(1-t)^{-2}(-2) = \frac{1}{(1-t)^2} = \frac{d}{dt}\left[\frac{1}{1-t}\right]$$

$$= 1 + 2t + 3t^2 + 4t^3 + ...$$

where this series converges for $|t| < 1$.

Therefore,

$$E(Y) = \sum_{y=1}^{\infty} y.Pr(Y = y) = \sum_{y=1}^{\infty} y(0.2)^{y-1}(0.8)$$

$$= (0.8)\sum_{y=1}^{\infty} y(0.2)^{y-1}$$

$$= (0.8)\left[1 + 2(0.2) + 3(0.2)^2 + +4(0.2)^3 + ...\right]$$

$$= (0.8)\frac{1}{(1-0.2)^2} = \frac{0.8}{(0.8)^2}$$

$$= \frac{1}{0.8} = 1.25$$

so she expects to take exam 1.25 times before she passes it**.**

Finally, to determine Var(Y), we find first $E(Y^2)$.

To do so miltiply by t the differentiated previous result.

then,

$$\frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + 4t^4 + ...$$

Differentiate both sides, now we get,

$$\frac{(1-t)^2(1) - t(2)(1-t)(-1)}{(1-t)^4} = \frac{1+t}{(1-t)^3} = \frac{d}{dt}\left[\frac{t}{(1-t)^2}\right]$$

$$= 1^2 + 2^2 t + 3^2 t^2 + 4^2 t^3 + ...$$

and this aslo converges for $|t| < 1$.

So now we have,

$$E(Y^2) = \sum_{y=1}^{\infty} y^2 \Pr(Y = y) = \sum_{y=1}^{\infty} y^2 (0.2)^{y-1}(0.8)$$

$$= (0.8)\sum_{y=1}^{\infty} y^2 (0.2)^{y-1}$$

$$= (0.8)\left[1^2 + 2^2 + (0.2) + 3^2(0.2)^2 + 4^2(0.2)^3 + ...\right]$$

$$= (0.8)\left[\frac{1+0.2}{(1-0.2)^3}\right]$$

$$= \frac{1.2}{(0.8)^2} = \frac{15}{8}$$

Consequently,

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{15}{8} - \left(\frac{5}{4}\right)^2$$

$$= \frac{30-25}{16}$$

$$= \frac{5}{16}$$

Exponential Generating Functions:

The generating functions we have dealt now are called ordinary Generating functions, which arose in selection problems where order was irrelevant. Now let us turn to the problems where order is relevant and crucial. We seek a tool. To find such a tool let us consider the binomial theorem. For each n belongs to Z^+,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + ... + \binom{n}{n}x^n,$$

so $(1+x)^n$ is the ordinary generating function for the sequence, When dealing with

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, ..., \binom{n}{n}, 0,0,0...$$

this we wrote that C(n,r) represented the number of combinations of n objects taken r at a time with $0 \le r \le n$. Consequently $(1+x)^n$ generated the sequence C(n,0), C(n,1), C(n,2), C(n,3),….., C(n,n)

Now for all $0 \le r \le n$,

$$C(n,r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right)P(n,r),$$

where $P(n,r)$ denotes the permutations of n objects taken r at a time.So,

$$(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + ... + C(n,n)x^n$$

$$= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + ... + P(n,n)\frac{x^n}{n!}.$$

On the basis of this observation We have the following definition.

For a sequence $a_0, a_1, a_2, a_3, a_4, a_5, ...$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + ... = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the exponential generating function for the given sequence.

Eg : The Maclaurian series expansion for e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

so e^x is the exponential generating function for the sequence 1,1,1,1,1,.......

The function e^x is the ordinary generating function for the sequence,

$$1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, ...$$

Example: In how many ways can four letters of ENGINE be arranged?

The following table shows list of possible selections of size 4 from the letters E,N,G,I,N,E, along with number of arrangements those 4 letters determine.

E E N N	4!/(2!2!)	E G N N	4!/2!
E E G N	4!/2!	E I N N	4!/2!
E E I N	4!/2!	G I N N	4!/2!
E E G I	4!/2!	E I G N	4!

Let us obtain the solution by using exponential gen. fun.

For the letter E we use $[1+x+(x^2/2!)]$, because there are 0, 1 or 2 E's to arrange. The number of distinct ways to arrange two E's is 1 (co-eff of the term $x^2/2$).For the letter N we use $[1+x+(x^2/2!)]$, because There are 0, 1 or 2 N's to arrange. The number of distinct ways to arrange two N's is 1 (co-eff of the term $x^2/2$).The arrangements for each of the letters G and I are represented by (1+x).Consequently, the exponential generating function is,

$f(x) = \left[1 + x + \left(x^2/2!\right)\right]^2 (1+x)^2$ the answer is co - eff of $x^4/4!$Consider two of the eight ways in which the term $x^4/4!$ arises in the expansion of

$$f(x) = \left[1 + x + \left(x^2/2!\right)\right]\left[1 + x + \left(x^2/2!\right)\right](1+x)(1+x)$$

$$\left(x^2/2!\right)\left(x^2/2!\right)(1)(1)$$

1) From the product where $\left(x^2/2!\right)$ is taken from first two factors and 1 is taken from last two factors.

Then

$$\left(x^2/2!\right)\left(x^2/2!\right)(1)(1)$$
$$= \left(x^4/2!2!\right)(1)(1)$$
$$= (4!/2!2!).\left(x^4/4!\right)$$

And the co-eff of $x^4/4!$ is 4!/(2!2!) which is the number of ways one can arrange four letters E, E, N, N.

2) From the product

$$\left(x^2/2!\right)(1)(x)(x)$$

where $\left(x^2/2!\right)$ is taken from first factor, 1 is taken from second factor and x is taken from last two factors.

Here

$(x^4/2!)(1)(x)(x) = x^4/2! = (4!/2!)(x^4/4!)$

So the co-eff of $x^4/4!$ is 4!/2! Which is the number of ways the four letters E, E, G, I can be arranged.In the complete expansion of the f(x), the term involving x^4

$$\left(\frac{x^4}{2!2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + x^4\right)$$

and consequently $x^4/4!$, is

$$= \left[\left(\frac{4!}{2!2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + 4!\right]\left(\frac{x^4}{4!}\right)$$

Where the co-eff of $x^4/4!$ Is the answer (102 arrangements) produced by the eight results in the table

Example: Consider the Maclaurian series expansions of e^x and e^{-x}

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

add these series together we get,

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

subtract the series we get

$$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

These results help us in following examples

Example: A ship carries 48 flags, 12 each of the colors red, blue, white and black. 12 of these flags are placed on a vertical pole in order to communicate a signal to other ships.

a) How many of these signals use an even number of blue flags and an odd number of black flags?

Exponential generating function,

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

considers all signal made up of n flags, $n \geq 1$. The last two factors restrict to even no. of blue and odd no. of black flags.

Since,

$$f(x) = \left(e^x\right)^2 \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{1}{4}\right)\left(e^{2x}\right)\left(e^{2x} - e^{-2x}\right) = \frac{1}{4}\left(e^{4x} - 1\right)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \frac{1}{4}\left(\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}\right)$$

The co-eff of $x^{12}/12!$ in f(x) yields $(1/4)(4^{12})=4^{11}$ signals made up of 12 flags with even no. blue & odd no. black flags

b) How many of the signals have at least 3 white flags, or no white flags at all?

Exponential generating function,

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)\left(e^x\right)^2 = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right)$$

$$= e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2 e^{3x}$$

$$= \sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty}\frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}$$

Here the factor,

$$\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right) = e^x - x - \frac{x^2}{2!}$$

restricts the signals to those that contain three or more of the 12 white flags, or none at all.

The answer for the no. signals here is the co - eff of $x^{12}/12!$ in f(x). As we consider each summand, we find

i) $\sum_{i=0}^{\infty}\frac{(4x)^i}{i!}$, here we have a term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$,

so the co - eff of $\frac{x^{12}}{12!}$ is 4^{12}.

ii) $x\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$. In this, in order to consider the term $x^{12}/12!$,

we need to consider the term

$$x\left[\frac{(3x)^{11}}{11!}\right] = 3^{11}\left[\frac{(x)^{12}}{11!}\right] = (12)(3)^{11}\left[\frac{(x)^{12}}{12!}\right]$$

and here the co - eff of $\left[\frac{(x)^{12}}{12!}\right]$ is $(12)(3)^{11}$.

iii) $\left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$, for this last summand, in order to get term $\frac{x^{12}}{12!}$,

we need to consider the term

$$\left(\frac{x^2}{2}\right)\left[\frac{(3x)^{10}}{10!}\right]=\left(\frac{1}{2}\right)(3x)^{10}\left(\frac{x^{12}}{10!}\right)=\left(\frac{1}{2}\right)(12)(11)(3)^{10}\left(\frac{x^{12}}{12!}\right),$$

where the co - eff of $\frac{x^{12}}{12!}$ is $\left(\frac{1}{2}\right)(12)(11)(3)^{10}$.

consequently, the number of 12 flag signals with at least

3 white flags, or none at all, is

Result of i + Result of ii + Result of iii

$$4^{12}+12\left(3^{11}\right)+\left(\frac{1}{2}\right)(12)(11)(3)^{10}=10{,}754{,}218.$$

Example: Company hires 11 new employees, each of whom is to be assigned to one of the four subdivisions. Each subdivision will get at least one new employee. In how many ways can these assignments be made?

Calling the subdivisions A, B, C and D, we can equivalently count the 11 letter sequences in which there is at least one occurrence of each letters A, B, C, and D. The exponential generating function for these arrangement is:

$$f(x)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)^4$$

$$f(x)=\left(e^x-1\right)^4$$

$$=e^{4x}-4e^{3x}+6e^{2x}-4e^x+1$$

the answer is the co - eff of $\frac{x^{11}}{11!}$ in f(x) :

$$4^{11}-4(3)^{11}+6(2)^{11}-4(1)^{11}$$

$$=\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^{11}$$

Example: Determine the sequences generated by following exponential generating functions.

a) $f(x) = 5e^{5x}$.

$\text{soln}: f(x) = 5e^{5x} = 5\sum_{n=0}^{\infty}\frac{(5x)^n}{n!}$

this produces the sequence $5, 5^2, 5^3, 5^4, ...$

b) $f(x) = 7e^{8x} - 4e^{3x}$

$\text{soln}: f(x) = 7e^{8x} - 4e^{3x} = 7\sum_{n=0}^{\infty}\frac{(8x)^n}{n!} - 4\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$

the sequence is $7(8)^n - 4(3)^n$ with $n = 0,1,2,3....$

i.e $3, 44, 412, 3476, ...$

c) $f(x) = 2e^x + 3x^2$

$\text{soln}: f(x) = 2e^x + 3x^2 = 2\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) + 3x^2$

so the sequence is $2, 2, (2+3), 2, 2, 2, ...$

d) $f(x) = e^{3x} - 28x^3 + 6x^2 + 9x$

$\text{soln}: f(x) = e^{3x} - 28x^3 + 6x^2 + 9x$

$= \left(\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}\right) - 28x^3 + 6x^2 + 9x$

so sequence is $3^0, (3^1 + 9), (3^2 + 6), (3^3 - 28), 3^4, ...$

which is 3,12,3,-1,91,....

Summation Operator

In this section we introduce a technique that helps us to go from ordinary generating function for sequence $a_0, a_1, a_2, a_3, ….$ to generating function for the sequence $a_0, a_0+a_1, a_0+a_1+a_2, a_0+a_1+a_2+a_3, ….$

for $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...,$ consider, the function $\frac{f(x)}{(1-x)}$.

$$\frac{f(x)}{(1-x)} = f(x).\frac{1}{(1-x)}$$
$$= \left[a_0 + a_1x + a_2x^2 + a_3x^3 + ...\right]\left[1 + x + x^2 + x^3 + ...\right]$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + ...$$

so, $\frac{f(x)}{(1-x)}$ generates the sequence $a_0, (a_0 + a_1), (a_0 + a_1 + a_2), ...$

Thus we refer to $\frac{1}{(1-x)}$ as summation operator.

We know that $\frac{1}{1-x}$ is the gen. fun. for the sequence $1,1,1,...$

Apply the summation operator $\frac{1}{1-x}$, we get,

$\frac{1}{1-x}.\frac{1}{1-x}$ is the gen. fun. for the sequence $1, 1+1, 1+1+1, ...$

$\frac{1}{1-x}.\frac{1}{1-x} = \frac{1}{(1-x)^2}$ is the gen. fun. for the sequence $1,2,3,4....$

Consider the gen.fun. $x + x^2$, for the sequence $0,1,1,0,0,0...$

Apply the summation operator, we get,

$(x + x^2)\left(\frac{1}{1-x}\right) = \frac{x + x^2}{1-x}$ which is gen. fun for $0, 0+1, 0+1+1, 0+1+1+1, ...$

i.e the sequence $0,1,2,3,4,...$

Apply again the summation operator, we get,

$\left(\frac{x + x^2}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{x + x^2}{(1-x)^2}$ which is gen.fun. for the sequennce

$0, 0+1, 0+1+2, 0+1+2+2, ...$ ie, $0,1,3,5,....$

Apply again the summation function, we get,

$$\frac{x+x^2}{(1-x)^2}\left(\frac{1}{1-x}\right)=\frac{x+x^2}{(1-x)^3}$$ which is the gen. fun. for

the sequence $0, 0+1, 0+1+3, 0+1+3+5, ...$

i.e $0, 1, 4, 9,.....$

This suggests that for $n \ge 1$, $\sum_{k=1}^{n}(2k-1) = n^2$

Example: Find a formula to express $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2$ as a function of n.

We start with

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... \quad \text{then,}$$

$$(-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} = \frac{dg(x)}{dx} = 1 + 2x + 3x^2 + 4x^3 + ...$$

so $\frac{x}{(1-x)^2}$ is the gen. fun. for the sequence $0, 1, 2, 3, 4,...$

Repeating this technique we find that,

$$x\frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = \frac{x(1+x)}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + 4^2x^4 + ...$$

so $\frac{x(1+x)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2, ...$

Apply summation operator to this, we get,

$$\frac{x(1+x)}{(1-x)^3}\cdot\frac{1}{(1-x)} = \frac{x(1+x)}{(1-x)^4}$$

this genearates $0^2, 0^2+1^2, 0^2+1^2+2^2, 0^2+1^2+2^2+3^2, ...$

Hence co - eff of x^n in $\frac{x(1+x)}{(1-x)^4}$ is $\sum_{i=0}^{n} i^2$

But this co - eff can also be calcualted as,

$$\frac{x(1+x)}{(1-x)^4} = \left(x + x^2\right)(1-x)^{-4}$$

$$= \left(x + x^2\right)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + ...\right]$$

so the co - eff of x^n is,

$$\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$$

$$= (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2}$$

$$= \binom{n+2}{n-1} + \binom{n+2}{n-2}$$

$$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!}$$

$$= \frac{1}{6}\left[(n+2)(n+1)(n) + (n+1)(n)(n-1)\right]$$

$$= \frac{1}{6}(n)(n+1)\left[(n+2) + (n-1)\right]$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Example: Find a formula for the sum of first n natural numbers using the generating function for the sequence 0, 1, 3, 6, 10, 15, ….

We know that,

$$\frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

for n = 3, we have, $\frac{1}{(1-x)^3} = \sum_{i=0}^{\infty} \binom{i+2}{i} x^i$

Thus the function $\frac{1}{(1-x)^3}$ genrates 1,3,6,10,15,...

Then $\frac{x}{(1-x)^3}$ generates 0,1,3,6,10,15,....

Now,

$$\sum_{k=0}^{n} k = \text{co-eff of } x^n \text{ in } \frac{x}{(1-x)^3}$$

$$= \text{co-eff of } x^n \text{ in } (1-x)^{-3}$$

$$= \text{co-eff of } x^{n-1} \text{ in } (1-x)^{-3}$$

$$= \binom{-3}{n-1}(-1)^{n-1} = (-1)^{n-1}\binom{3+(n-1)-1}{n-1}(-1)^{n-1}$$

$$= \binom{n+3}{n-1} = \frac{1}{2}(n)(n+1)$$

Summaries (*m* objects, *n* containers)

Objects Are Distinct	Containers Are Distinct	Some Containers May Be Empty	Number of Distributions
Yes	Yes	Yes	n^m
Yes	Yes	No	$n!S(m,n)$
Yes	No	Yes	$S(m,1)+S(m,2)+...+S(m,n)$
Yes	No	No	$S(m,n)$
No	Yes	Yes	$\binom{n+m-1}{m}$
No	Yes	No	$\binom{n+(m-n)-1}{m-n}$
No	No	Yes	(1) $p(m)$, for $n=m$
No	No	No	(2) $p(m,1)+p(m,2)+...+p(m,n)$, $n<m$ $p(m,n)$

$p(m.n)$:number of partitions of m into exactly n summands

UNIT 8

Sequences and Recurrence Relations

EXAMPLE 8.1.2

Consider the following two sequences:

$$S_1: \ 3, 5, 7, 9, \ldots$$
$$S_2: \ 3, 9, 27, 81, \ldots$$

We can find a formula for the nth term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1: \ 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \ldots$$
$$S_2: \ 3^1, 3^2, 3^3, 3^4, \ldots$$

For S_1, $a_n = 2n + 1$ for $n \geq 1$, and for S_2, $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

EXAMPLE 8.1.3

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\begin{aligned}
\text{3rd term} &= 2 = 1 + 1 = \text{1st term} + \text{2nd term}\\
\text{4th term} &= 3 = 1 + 2 = \text{2nd term} + \text{3rd term}\\
\text{5th term} &= 5 = 2 + 3 = \text{3rd term} + \text{4th term}\\
\text{6th term} &= 8 = 3 + 5 = \text{4th term} + \text{5th term}\\
\text{7th term} &= 13 = 5 + 8 = \text{5th term} + \text{6th term}
\end{aligned}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \qquad (8.1)$$

for all $n \geq 3$ and

$$\left.\begin{aligned} f_1 &= 1,\\ f_2 &= 1. \end{aligned}\right. \qquad (8.2)$$

EXAMPLE 8.1.4

Consider the function $f : \mathbb{N}^0 \to \mathbb{Z}^+$ defined by

$$f(0) = 1,$$
$$f(n) = nf(n-1) \quad \text{for all } n \geq 1.$$

Then

$$f(0) = 1 = 0!,$$
$$f(1) = 1 \cdot f(0) = 1 = 1!,$$
$$f(2) = 2 \cdot f(1) = 2 \cdot 1 = 2 = 2!,$$
$$f(3) = 3 \cdot f(2) = 3 \cdot 2 \cdot 1 = 6 = 3!,$$

and so on. Here $f(n) = nf(n-1)$ for all $n \geq 1$ is the recurrence relation, and $f(0) = 1$ is the initial condition for the function f. Notice that the function f is nothing but the factorial function, i.e., $f(n) = n!$ for all $n \geq 0$.

Sequences and Recurrence Relations

Let us consider the function f as given in (8.3). If we write $a_n = f(n)$, then (8.3) translates into the following equation:

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2.$$

That is, a_n is defined in terms of a_{n-1} and a_{n-2}. As remarked previously, such an equation is called a recurrence relation. Moreover, (8.4) translates into $a_0 = 5$ and $a_1 = 7$. These are called the initial conditions for the recurrence relation.

DEFINITION 8.1.5

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ is an equation that relates a_n to some of the terms $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for all integers n with $n \geq k$, where k is a nonnegative integer. The **initial conditions** for the recurrence relation are a set of values that explicitly define some of the members of $a_0, a_1, a_2, \ldots, a_{k-1}$.

The equation

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2,$$

as defined above, relates a_n to a_{n-1} and a_{n-2}. Here $k = 2$. So this is a recurrence relation with initial conditions $a_0 = 5$ and $a_1 = 7$.

EXAMPLE 8.1.9

Number of subsets of a finite set. Let s_n denote the number of subsets of a set A with n elements, $n \geq 0$. In Worked-Out Exercise 9 (Chapter 2, page 144), we proved that

$$s_0 = 1,$$
$$s_n = 2s_{n-1}, \quad \text{if } n > 0$$

Hence, a recurrence relation for the sequence $s_0, s_1, s_2, s_3, s_4, \ldots$ is

$$s_n = 2s_{n-1}, \quad n \geq 1$$

and an initial condition is $s_0 = 1$.

EXAMPLE 8.1.10

Compound Interest. Sam received a yearly bonus and deposited \$10,000 in a local bank yielding 7% interest compounded annually. Sam wants to know the total amount accumulated after n years. Let A_n denote the total amount accumulated after n years. Let us determine a recurrence relation and initial conditions for the sequence $A_0, A_1, A_2, A_3, \ldots$.

The amount accumulated after one year is the initial amount plus the interest on the initial amount. Now A_{n-1} is the amount accumulated after $n - 1$ years. This implies that the amount at the beginning of nth year is A_{n-1}. It follows that the total amount accumulated after n years is the amount at the beginning of the nth year plus the interest on this amount. Because the interest rate is 7%, the interest earned during the nth year is $(0.07)A_{n-1}$. Hence,

$$A_n = A_{n-1} + (0.07)A_{n-1}$$
$$= 1.07A_{n-1}, \quad n \geq 1,$$
$$A_0 = 10000.$$

- **Tower of Hanoi**
- In the nineteenth century, a game called the Tower of Hanoi became popular in Europe. This game represents work that is under way in the temple of Brahma.
- There are three pegs, with one peg containing 64 golden disks. Each golden disk is slightly smaller than the disk below it.
- The task is to move all 64 disks from the first peg to the third peg.

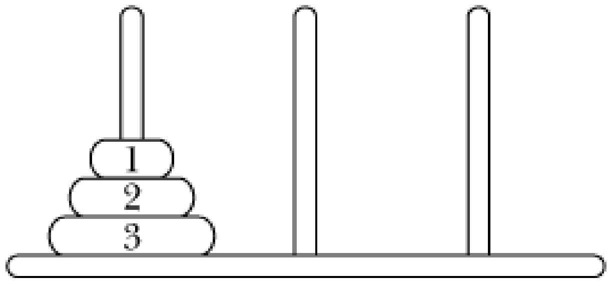


FIGURE 8.1 Tower of Hanoi problem with three disks

- The rules for moving the disks are as follows:
    1. Only one disk can be moved at a time.
    2. The removed disk must be placed on one of the pegs.
    3. A larger disk cannot be placed on top of a smaller disk.

- The objective is to determine the minimum number of moves required to transfer the disks from the first peg to the third peg.

- First consider the case in which the first peg contains only one disk.
    – The disk can be moved directly from peg 1 to peg 3.

- Consider the case in which the first peg contains two disks.
    – First move the first disk from peg 1 to peg 2.
    – Then move the second disk from peg 1 to peg 3.
    – Finally, move the first disk from peg 2 to peg 3.

- Consider the case in which the first peg contains three disks and then generalize this to the case of 64 disks (in fact, to an arbitrary number of disks).
    – Suppose that peg 1 contains three disks. To move disk number 3 to peg 3, the top two disks must first be moved to peg 2. Disk number 3 can then be moved from peg 1 to peg 3. To move the top two disks from peg 2 to peg 3, use the same strategy as before. This time use peg 1 as the intermediate peg.
    – Figure 8.2 shows a solution to the Tower of Hanoi problem with three disks.

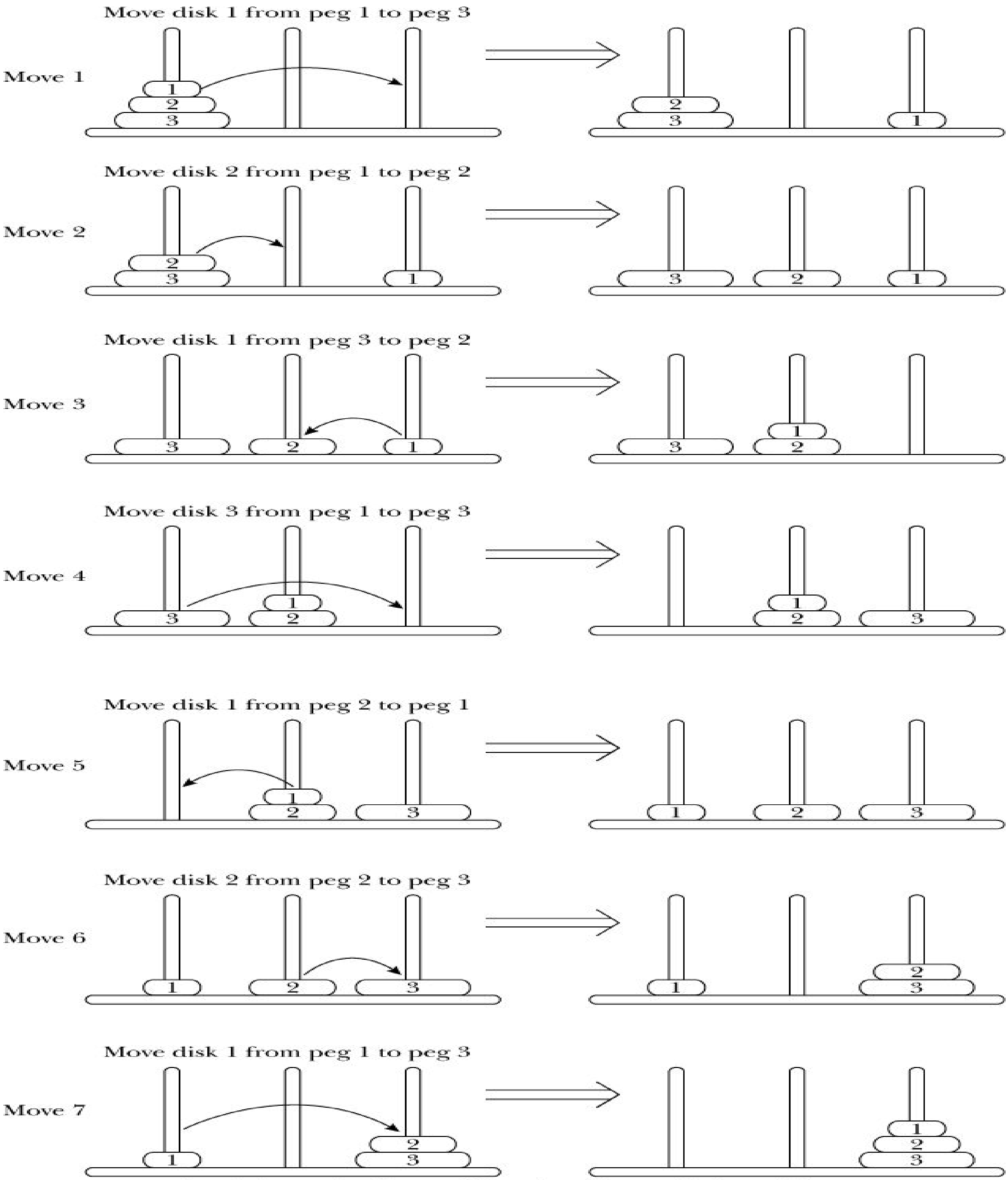


FIGURE 8.2 A solution to the Tower of Hanoi problem with three disks

- Generalize this problem to the case of 64 disks. To begin, the first peg contains all 64 disks. Disk number 64 cannot be moved from peg 1 to peg 3 unless the top 63 disks are on the second peg. So first move the top 63 disks from peg 1 to peg 2, and then move disk number 64 from peg 1 to peg 3. Now the top 63 disks are all on peg 2.

- To move disk number 63 from peg 2 to peg 3, first move the top 62 disks from peg 2 to peg 1, and then move disk number 63 from peg 2 to peg 3. To move the remaining 62 disks, follow a similar procedure.

- In general, let peg 1 contain $n \geq 1$ disks.
  1. Move the top $n - 1$ disks from peg 1 to peg 2 using peg 3 as the intermediate peg.
  2. Move disk number n from peg 1 to peg 3.
  3. Move the top $n - 1$ disks from peg 2 to peg 3 using peg 1

Let c_n denote the number of moves required to move n disks, $n \geq 0$, from peg 1 to peg 3. Step (1) requires us to move the top $n - 1$ disks from peg 1 to peg 2, which requires c_{n-1} moves. Step (2) requires us to move the nth disk from peg 1 to peg 3, which requires 1 move. Step (3) requires us to move $n - 1$ disks from peg 2 to peg 3, which requires c_{n-1} moves. Thus, it follows that

$$c_n = 2c_{n-1} + 1, \quad \text{if } n > 1, \tag{8.5}$$

and

$$c_1 = 1. \tag{8.6}$$

Now (8.5) is a recurrence relation for the sequence $\{c_n\}_{n=1}^{\infty}$ with the initial condition given by (8.6).

Suppose a recurrence relation for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$, is given. By a *solution of the recurrence relation* we mean to obtain an explicit formula for a_n, i.e., to find an expression for a_n that does not involve any other a_i.

Let S be the sequence $\{a_n\}_{n=0}^{\infty}$, where

$$a_n = 7a_{n-1} - 6a_{n-2} \quad \text{for all } n \geq 2. \tag{8.8}$$

Because a_n is defined in terms of the preceding terms a_{n-1} and a_{n-2}, Equation (8.8) is a recurrence relation.

Let us show that $a_n = 5 = 5 + 0 \cdot n$ is a solution of Equation (8.8). Here $a_0 = 5$, $a_1 = 5$, $a_2 = 5, \ldots, a_n = 5$, and so on. Let us evaluate the right side of Equation (8.8), i.e.,

$$7a_{n-1} - 6a_{n-2} = 7 \cdot 5 - 6 \cdot 5 = 35 - 30 = 5 = a_n.$$

Hence, $a_n = 5, n \geq 0$ is a solution of the recurrence relation (8.8).

Now let $a_n = 6^n$. Here $a_0 = 6^0 = 1$, $a_1 = 6^1 = 6$, $a_2 = 6^2 = 36, \ldots, a_{n-2} = 6^{n-2}$, $a_{n-1} = 6^{n-1}$, $a_n = 6^n$, and so on. Let us evaluate the right side of Equation (8.8), using the terms of this sequence. We have

$$\begin{aligned} 7a_{n-1} - 6a_{n-2} &= 7 \cdot 6^{n-1} - 6 \cdot 6^{n-2} \\ &= 7 \cdot 6^{n-1} - 6^{n-1} \\ &= (7-1) \cdot 6^{n-1} \\ &= 6 \cdot 6^{n-1} \\ &= 6^n \\ &= a_n. \end{aligned}$$

Therefore, $a_n = 6^n, n \geq 0$ is also a solution of the recurrence relation (8.8).

Note that the expression $a_n = 2^n, n \geq 0$ is not a solution of Equation (8.8).

Linear Homogenous Recurrence Relations

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers. A **linear homogeneous recurrence relation** of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \tag{8.31}$$

where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots$, and c_k are constants.

Linear Homogenous Recurrence Relations

Consider the following recurrence relations.

(i) $a_n = 3a_{n-1} + a_{n-2}$
(ii) $a_n = 3a_{n-1} + 5$
(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$
(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$
(v) $a_n = 3a_{n-1} + na_{n-2}$

Recurrence relations (i), (ii), (iii), and (iv) are recurrence relations with constant coefficients. Recurrence relation (v), $a_n = 3a_{n-1} + na_{n-2}$, is not a relation with constant coefficients. Notice that (i) is a linear homogeneous recurrence

Linear Homogenous Recurrence Relations

A sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is said to **satisfy** a linear homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k}, \quad c_k \neq 0 \qquad (8.32)$$

of order k with constant coefficients if $s_n = c_1s_{n-1} + c_2s_{n-2} + c_3s_{n-3} + \cdots + c_ks_{n-k}$.

If a sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is also called a **solution** of that recurrence relation.

Consider the recurrence relation $a_n = 3a_{n-1}$. This is a linear homogeneous recurrence relation of order 1. Let t be a nonzero number and suppose $a_n = t^n$ for all $n \geq 0$. Then $a_n = 3a_{n-1}$ implies that $t^n = 3t^{n-1}$. Therefore, $t = 3$. Thus, we find that $a_n = 3^n$. Hence, the sequence $1, 3, 3^2, 3^3, \ldots 3^n, \ldots$ is a solution of the recurrence relation $a_n = 3a_{n-1}$.

Theorem 8.2.7: Let

$$a_n = c_1a_{n-1} + c_2a_{n-2}, \quad c_2 \neq 0, \quad n > 1 \qquad (8.34)$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ satisfies the above recurrence relation if and only if

$$t^2 - c_1t - c_2 = 0. \qquad (8.35)$$

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $c_2 \neq 0$, $n > 1$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^2 - c_1 t - c_2 = 0$$

is called the **characteristic equation** of the recurrence relation.

Theorem 8.2.9: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n > 1 \tag{8.37}$$

be a linear homogeneous recurrence relation of order 2, where c_1 and c_2 are constants and $c_2 \neq 0$

(i) If the sequences $\{s_n\}$ and $\{p_n\}$ satisfy (8.37), then for any constants b and d, the sequence $\{bs_n + dp_n\}$ satisfies (8.37).

(ii) Let r be a root of the characteristic equation

$$t^2 - c_1 t - c_2 = 0 \tag{8.38}$$

of (8.37). Then the sequence $\{r^n\}$ is a solution of (8.37).

Theorem 8.2.10: Suppose that a sequence $\{d_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the distinct roots of the characteristic equation (8.38), then there exist constants b and d, which

Corollary 8.2.11: Suppose that

$$a_0 = d_0, \qquad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38). If $r_1 \neq r_2$, then there exist constants b and d, which are to be determined by initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br_1^n + dr_2^n, \quad n = 0, 1, \ldots.$$

EXAMPLE 8.2.12

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 7a_{n-1} - 10a_{n-2} \tag{8.41}$$

with initial conditions

$$a_0 = 1$$
$$a_1 = 8.$$

The characteristic equation of the given recurrence relation is:

$$t^2 - 7t + 10 = 0.$$

Next, we find the roots of this equation. Now,

$$t^2 - 7t + 10 = (t-5)(t-2)$$

and so

$$(t-5)(t-2) = 0.$$

This implies that the roots of the characteristic equation are $t = 5$, and $t = 2$. The roots are distinct. By Theorem 8.2.10, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 5^n + c_2 2^n, \quad n \geq 0.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1 + c_2,$$
$$a_1 = 5c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 + c_2 = 1,$$
$$5c_1 + 2c_2 = 8.$$

Solving these equations for c_1 and c_2, we get $c_1 = 2$ and $c_2 = -1$. Hence,

$$a_n = 2 \cdot 5^n - 2^n, \quad n \geq 0.$$

Hence, the sequence $\{2 \cdot 5^n - 2^n\}$ is the solution.

Theorem 8.2.13: Suppose that a sequence $\{s_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the roots of the characteristic equation (8.38) such that $r_1 = r_2 = r$, then there exist constants b and d, which are to be determined, such that the solution of the recurrence relation (8.37) is

$$s_n = br^n + dnr^n, \quad n = 0, 1, \ldots.$$

Corollary 8.2.14: Suppose that

$$a_0 = d_0, \qquad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38) such that $r_1 = r_2 = r$. Then there exist constants b and d, which are to be determined from initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br^n + dnr^n, \quad n = 0, 1, \ldots.$$

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial conditions

$$a_0 = 4$$
$$a_1 = 12.$$

The characteristic equation of this recurrence relation is the quadratic equation

$$t^2 - 4t + 4 = 0.$$

We find the roots of this equation. Now,

$$t^2 - 4t + 4 = (t - 2)(t - 2)$$

and so

$$(t - 2)(t - 2) = 0.$$

This implies that the roots of the characteristic equation are $t = 2$, and $t = 2$. The roots are not distinct. Therefore, by Theorem 8.2.13, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 2^n + c_2 n 2^n, \quad n = 0, 1, \ldots.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1$$
$$a_1 = 2c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 = 4,$$
$$2c_1 + 2c_2 = 12.$$

Solving these equations for c_1 and c_2, we get $c_1 = 4$ and $c_2 = 2$. Hence,

$$a_n = 4 \cdot 2^n + 2 \cdot n \cdot 2^n = 2 \cdot 2^{n+1} + n2^{n+1} = (2 + n)2^{n+1} = (n + 2)2^{n+1}, \quad n \geq 0.$$

Thus, we find that the sequence $\{(n + 2)2^{n+1}\}$ is the solution.

Theorem 8.2.16: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \qquad (8.42)$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ is a solution of the above recurrence relation if and only if

$$t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} = 0.$$

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$$

is called the **characteristic equation** of this linear homogeneous recurrence relation.

To obtain the characteristic equation of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$, substitute $a_n = t^n, t \neq 0$, to get

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k}.$$

Thus,

$$\begin{aligned} t^n &= c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k} \\ \Rightarrow t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} &= 0 \\ \Rightarrow t^{n-k}(t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k) &= 0. \end{aligned}$$

Because $t \neq 0$, we have, $t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$, which is the characteristic equation.

Theorem 8.2.19: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \qquad (8.44)$$

be a linear homogeneous recurrence relation of order k, where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots,$ and c_k are constants. Let

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$$

be the characteristic equation of (8.44).

(i) If the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ are solutions of (8.44), then for any constants b and d, the sequence $\{bs_n + dp_n\}_{n=0}^{\infty}$ is a solution of (8.44).

(ii) If r is a root of the characteristic equation, then the sequence $1, r, r^2, \ldots, r^n, \ldots$ is a solution of (8.44).

(iii) If $r_1, r_2, \ldots, r_k$ are distinct roots of the characteristic equations, then there exist constants $b_1, b_2, b_3, \ldots, b_k$, which are to be determined from initial conditions, such that a solution of (8.44) is given by

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \cdots + b_k r_k^n,$$

(iv) If r is a root, of multiplicity m, of the characteristic equation, then $a_n = r^n$, $a_n = nr^n$, $a_n = n^2r^n, \ldots$, and $a_n = n^{m-1}r^n$ are solutions of (8.44).

(v) Suppose that

$$a_0 = d_0, a_1 = d_1, \ldots, a_{n-1} = d_{n-1}$$

are the initial conditions for the recurrence relation (8.44), where $d_0, d_1, \ldots,$ and d_{n-1} are constants. If $r_1, r_2, \ldots,$ and r_t are t distinct roots of the characteristic equation with multiplicities $m_1, m_2, \ldots, m_t$ and $m_1 + m_2 + \cdots + m_t = k$, then there exist constants c_{ij}, which are to be determined from the initial conditions, such that the solution of the recurrence relation (8.44) is

$$\begin{aligned} a_n = (c_{00} + c_{01}n + \cdots + c_{0m_1}n^{m_1-1})r_1^n \\ + (c_{10} + c_{11}n + \cdots + c_{1m_2}n^{m_2-1})r_2^n \\ + \cdots + (c_{t0} + c_{t1}n + \cdots + c_{tm_t}n^{m_t-1})r_t^n, \quad n = 0, 1, \ldots. \end{aligned}$$

A **linear nonhomogeneous recurrence relation** with constants coefficients is a recurrence relation of the form

$$a_n + c_1a_{n-1} + \cdots + c_ka_{n-k} = f(n), \tag{8.55}$$

where $c_i, i = 1, 2, \ldots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function.

If $f(n) = 0$, then (8.55) is a linear homogeneous equation (which we discussed in the previous section). There is no known general method for solving nonhomogeneous linear recurrence equations. However, we can develop a method for solving the special case

$$c_0a_n + c_1a_{n-1} + \cdots + c_ka_{n-k} = b^np(n), \tag{8.56}$$

where b is a constant and $p(n)$ is a polynomial in n.

Linear Nonhomogenous Recurrence Relations

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n.$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = 1$.

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n(n^2 + 6n + 5).$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = n^2 + 6n + 5$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.5: Let

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n) \tag{8.62}$$

be a nonhomogeneous recurrence relation, where c_i, $i = 1, 2, \ldots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function. Suppose $\{r_n\}$ is a particular solution of (8.62). Then $\{u_n\}$ is a solution of (8.62) if and only if $u_n = r_n + s_n$, for all n, and $\{s_n\}$ is a solution of the associated homogeneous part, $a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = 0$.

Theorem 8.3.6: Let

$$a_n - da_{n-1} = b^n u, \quad n \geq 1 \tag{8.67}$$

be a nonhomogeneous linear recurrence relation, with the initial condition

$$a_0 = e_0, \tag{8.68}$$

where d, b, u, and e_0 are constants, and b and u are nonzero. This nonhomogeneous linear recurrence relation can be transformed into the following linear homogeneous recurrence relation:

$$a_n - (b + d)a_{n-1} + bda_{n-2} = 0, \quad n \geq 2$$

with the initial conditions $a_0 = e_0$ and $a_1 = de_0 + bu$.

Moreover,

(i) if $b \neq d$, then there exists a constant c_0, which is to be determined from the initial condition, such that

$$a_n = c_0 d^n + \left(\frac{bu}{b-d}\right) b^n.$$

(ii) if $b = d$, then there exists a constant c_0, which to be is determined from the initial condition, such that

$$a_n = c_0 b^n + unb^n.$$

In this example, we use Theorem 8.3.6 to solve the recurrence relation

$$a_n - 4a_{n-1} = 8^n, \quad n \geq 1,$$

with the initial condition

$$a_0 = 1.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n u,$$

Let $\{r_n\}$ be a solution of (8.83).

(i) Suppose $b \neq d$. Then r_n is of the form

$$r_n = c_0 d^n + c_1 b^n + c_2 n b^n,$$

where c_0, c_1, and c_2 are some constants.

(ii) Suppose $b = d$. Then $\{r_n\}$ is of the form

$$r_n = c_0 b^n + c_1 n b^n + c_2 n^2 b^n,$$

where c_0, c_1, and c_2 are some constants.

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n+3), \quad n > 1 \tag{8.94}$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un+v).$$

Here $d = 3, b = 2, u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n+3), \quad n > 1 \tag{8.94}$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un+v).$$

Here $d = 3, b = 2, u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Put $n = 2$ in (8.92) to get

$$a_2 - 3a_1 = 2^2(4 \cdot 2 + 3) = 44.$$

Because $a_1 = 14$, we get

$$a_2 = 3 \cdot 14 + 44 = 86.$$

Thus,

$$\begin{aligned} a_0 &= c_0 + c_1 = 0 \\ a_1 &= c_0 \cdot 3 + c_1 \cdot 2 + c_2 \cdot 2 = 14 \\ a_2 &= c_0 \cdot 3^2 + c_1 \cdot 2^2 + c_2 \cdot 2 \cdot 2^2 = 86 \end{aligned}$$

This implies that

$$\begin{aligned} c_0 + c_1 &= 0 \\ 3c_0 + 2c_1 + 2c_2 &= 14 \\ 9c_0 + 4c_1 + 8c_2 &= 86 \end{aligned}$$

We solve these equations for c_0, c_1, and c_2 to obtain $c_0 = 30$, $c_1 = -30$, and $c_2 = -8$. Thus, we find that

$$a_n = 30(3^n) - 30(2^n) - n2^{n+3}, \quad n \geq 0. \tag{8.95}$$

Theorem 8.3.13: Let

$$a_n + d_1 a_{n-1} + \cdots + d_k a_{n-k} = b^n p(n) \tag{8.96}$$

be a nonhomogeneous linear recurrence relation, where $p(n)$ is a polynomial of degree m. Then from this nonhomogeneous linear recurrence relation we can obtain a linear homogeneous recurrence that has following characteristic equation:

$$(t^k + d_1 t^{k-1} + \cdots + d_k)(t - b)^{m+1} = 0. \tag{8.97}$$

Moreover, a solution of (8.96) is also a solution of the linear homogeneous recurrence whose characteristic equation is given by (8.97).

Linear Recurrences

There is a class of recurrence relations which *can* be solved analytically in general. These are called *linear recurrences* and include the Fibonacci recurrence.
Begin by showing how to solve Fibonacci:

Solving Fibonacci

Recipe solution has 3 basic steps:

1) Assume solution of the form $an = r\ n$
2) Find all possible r's that seem to make this work. Call these1 $r1$ and $r2$. Modify assumed solution to ***general solution*** $an = Ar1n + Br2n$ where A,B are constants.
3) Use initial conditions to find A,B and obtain specific solution.

Solving Fibonacci

1) Assume exponential solution of the form $an = r\ n$:

Plug this into $a_n = a_{n-1} + a_{n-2}$:

$$r^n = r^{n-1} + r^{n-2}$$

Notice that all three terms have a common r^{n-2} factor, so divide this out:

$r^n / r^{n-2} = (r^{n-1} + r^{n-2})/r^{n-2} \Rightarrow r^2 = r + 1$

This equation is called the ***characteristic equation*** of the recurrence relation.

2) Find all possible r's that solve characteristic

$$r^2 = r + 1$$

Call these r_1 and r_2.[1] General solution is

$a_n = Ar_1^n + Br_2^n$ where A,B are constants.

Quadratic formula2 gives:

$$r = (1 \pm \sqrt{5})/2$$

So $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$

General solution:

$a_n = A\,[(1+\sqrt{5})/2]^n + B\,[(1-\sqrt{5})/2]^n$

Solving Fibonacci

◆

Use initial conditions $a_0 = 0$, $a_1 = 1$ to find A,B and obtain specific solution.

$0=a_0 = A\,[(1+\sqrt{5})/2]^0 + B\,[(1-\sqrt{5})/2]^0 = A + B$

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 \qquad = A(1+\sqrt{5})/2 + B\,(1-\sqrt{5})/2$

$= (A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2nd:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

(CHECK IT!)

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Recurrences with Constant Coefficients

Previous method generalizes to solving "*linear recurrence relations with constant coefficients*":

DEF: A recurrence relation is said to be ***linear*** if *an* is a linear combination of the previous terms plus a function of *n*. I.e. no squares, cubes or other complicated function of the previous *ai* can occur. If in addition all the coefficients are constants then the recurrence relation is said to have ***constant coefficients***.

Linear Recurrences with Constant Coefficients

Q: Which of the following are linear with constant coefficients?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$
3. $a_n = a_{n-1}^{\,2}$
4. Partition function:

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES

2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: YES

3. $a_n = a_{n-1}^2$: NO. Squaring is not a linear operation. Similarly $a_n = a_{n-1}a_{n-2}$ and $a_n = \cos(a_{n-2})$ are non-linear.

4. Partition function: $p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$ NO.

   This is linear, but coefficients are not constant as $C(n-1, n-1-i)$ is a non-constant function of n.

Homogeneous Linear Recurrences

To solve such recurrences we must first know how to solve an easier type of recurrence relation:

DEF: A linear recurrence relation is said to be ***homogeneous*** if it is a linear combination of the previous terms of the recurrence *without* an additional function of n.

Q: Which of the following are homogeneous?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2_{n-3} - a_{n-3}$
3. Partition function: $p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: No. There's an extra term $f(n) = 2^{n-3}$
3. Partition function:

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

YES. No terms appear not involving the previous p_i

Homogeneous Linear Recurrences with Const. Coeff.'s

The 3-step process used for the Fibonacci recurrence works well for general homogeneous linear recurrence relations with constant coefficients. There are a few instances where some modification is necessary.

Homogeneous – Complications

1) *Repeating roots* in characteristic equation. Repeating roots imply that don't learn anything new from second root, so may not have enough information to solve formula with given initial conditions. We'll see how to deal with this on next slide.
2) *Non-real number roots* in characteristic equation. If the sequence has periodic behavior, may get complex roots (for example $a_n = -a_{n-2}$)[1]. We won't worry about this case (in principle, same method works as before, except use complex arithmetic).

Complication: Repeating Roots

EG: Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$
Find characteristic equation by plugging in $a_n = r^n$:

$$r^2 - 2r + 1 = 0$$

Since $r^2 - 2r + 1 = (r-1)^2$ the root $r = 1$ repeats.
If we tried to solve by using general solution

$$a_n = Ar_1^n + Br_2^n = A1^n + B1^n = A + B$$

which forces a_n to be a constant function (→←).
SOLUTION: Multiply second solution by n so general solution looks like:

$$a_n = Ar_1^n + Bnr_1^n$$

Complication: Repeating Roots

Solve $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 1$, $a_1 = 2$
General solution: $a_n = A1^n + Bn1^n = A + Bn$

Plug into initial conditions
$1 = a_0 = A + B \cdot 0 \cdot 1^0 = A$
$2 = a_0 = A \cdot 1^1 + B \cdot 1 \cdot 1^1 = A + B$
Plugging first equation $A = 1$ into second: $2 = 1 + B$ implies $B = 1$.
Final answer: $a_n = 1 + n$

The Nonhomogeneous Case

Consider the Tower of Hanoi recurrence (see Rosen p. 311-313)

$a_n = 2a_{n-1}+1$.

Could solve using telescoping. Instead let's solve it methodically. Rewrite:

$a_n - 2a_{n-1} = 1$

1) Solve with the RHS set to 0, i.e. solve the homogeneous case.
2) Add a particular solution to get general solution. I.e. use rule:

General Nonhomogeneous = General homogeneous + Particular Nonhomogeneous

The Nonhomogeneous Case

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve

$$a_n - 2a_{n-1} = 0$$

Characteristic equation: $r - 2 = 0$

so unique root is $r = 2$. General solution to homogeneous equation is

$$a_n = A\cdot 2^n$$

The Nonhomogeneous Case

2) Add a particular solution to get general solution for $a_n - 2a_{n-1} = 1$. Use rule:

General Nonhomogeneous = General homogeneous + Particular Nonhomogeneous

There are little tricks for guessing particular nonhomogeneous solutions. For example, when the RHS is constant, the guess should also be a constant.[1]

So guess a particular solution of the form $b_n=C$.

Plug into the original recursion:

$1 = b_n - 2b_{n-1} = C - 2C = -C$. Therefore $C = -1$.

General solution: $a_n = A\cdot 2^n - 1$.

The Nonhomogeneous Case

Finally, use initial conditions to get closed solution. In the case of the Towers of Hanoi recursion, initial condition is:

$a_1 = 1$

Using general solution $an = A \cdot 2^n - 1$ we get:

$1 = a_1 = A \cdot 2^1 - 1 = 2A - 1$.

Therefore, $2 = 2A$, so $A = 1$.

Final answer: $a_n = 2^n - 1$

More Complicated

EG: Find the general solution to recurrence from the bit strings example:

$$a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$$

1) Rewrite as $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$ and solve homogeneous part:

Characteristic equation: $r^3 - 2r + 1 = 0$.

Guess root $r = \pm 1$ as integer roots divide.

$r = 1$ works, so divide out by $(r - 1)$ to get

More Complicated

$r^3 - 2r + 1 = (r - 1)(r^2 + r - 1)$.

Quadratic formula on $r^2 + r - 1$:

$$r = (-1 \pm \sqrt{5})/2$$

So $r_1 = 1$, $r_2 = (-1+\sqrt{5})/2$, $r_3 = (-1-\sqrt{5})/2$

General homogeneous solution:

$$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n$$

More Complicated

2) Nonhomogeneous particular solution to $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$

Guess the form $b_n = k\,2^n$. Plug guess in:

$$k\,2^n - 2k\,2^{n-1} + k\,2^{n-3} = 2^{n-3}$$

Simplifies to: $k = 1$.

So particular solution is $b_n = 2^n$

General Nonhomogeneous = General homogeneous + Particular Nonhomogeneous

Final answer:

$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n + 2^n$